Journal for Geometry and Graphics
Volume 3 (1999), No. 2, 183–191

Some Moebius-Geometric Theorems Connected to Euclidean Kinematics

Gunter Weiss, Karla Nestler, Gert Meinl

Institute for Geometry, Dresden University of Technology
Zellescher Weg 12–14, D-01062 Dresden, Germany
email: weiss@math.tu-dresden.de

Abstract. To four positions of an object in the Euclidean plane there exists an infinite set of four-bar linkages interpolating these given positions. The set contains an interpolating slider-crank as a special case.

The design of such a mechanism is based on geometric reasoning and the use of elementary geometric theorems. Usually such theorems and geometric mappings are proved by kinematic arguments. But they are also interesting for their own, independently from the kinematic point of view. There occur e.g. configurations of circles and lines related to Miquel's configuration in a (real) Moebius plane. Beginning with their kinematic aspects, some 'elementary' geometric theorems are discussed and generalized.

Key Words: Kinematics in the Euclidean plane, four-bar linkages, isogonal relation with respect to a triangle, Wallace's theorem, Moebius-geometry

MSC 1994: 53A17, 51M04, 51B10.

1. Introduction

Human synovial joints are more or less 'shaky' kinematic structures with gliding surfaces and fixing, resp. steering tendon apparatus. Elasticity and far reaching possibilities for other joints to compensate lacks of mobility allow joint prostheses, which extremely simplify the natural joint's kinematics. Starting point of our investigations was the analysis of the kinematics of the lower ankle joint, based on surgical experiments (cf. Acknowledgement). Sensors implanted into the heel bone produce data-files of positions in space. The resulting three-dimensional scattered point sets are approximated by planar forced motions, modeling the tip-over sideways movement of the lower ankle joint. By choosing a set of reasonable positions of the heel bone, one can finally approximate this tip-over movement by a four-bar mechanism in a seemingly satisfying manner. Problems of this kind have already been widely discussed (cf. [2], [1], [4], [6], [7], [8], [11], and many others). While Bottema and Roth (1979) prefer an analytic approach, Lichtenheld (1959) treats such problems in a graphic-constructive

ISSN 1433-8157/$ 2.50

way. Thereby, geometric reasoning is based on elementary geometric theorems, e.g. on the theorem concerning the angle at the center and at the circumference of a circle. Another important tool is the following well known Theorem of Wallace (cf. e.g. [9] or [3]):

Lemma 1: *The reflection of the orthocenter of a triangle T in the sides gives three points on the circumcircle c of the triangle T. For points X on c of T, and only for these points, the reflection in the sides produces collinear mirror images. The line through these images passes through the orthocenter of the triangle and is called Wallace line of X for the given triangle.*

As the poles for n given positions of an object in the plane show surprising incidences and configurations which forge links between kinematics and Euclidean circle-geometries, it seems to be worthwhile to treat such facts for their own sake and extend them e.g. to Moebius geometry or to iterative processes.

2. Three positions, the isogonal relation

Let $S_i = \{A_i, B_i, \dots, X_i\}$ be a set of points in the Euclidean plane, whereby the index i denotes an arbitrary position of this set. In general, the displacement defined by two such positions S_i, S_j is a rotation about the 'pole' P_{ij}. Let three positions S_1, S_2, S_3 be given such that the mutual displacements between any two define poles forming a triangle $P_{12}P_{23}P_{31}$. For any point $X_1 \in S_1$ we then receive the homologous positions X_2 and X_3 by factorizing the rotations into reflections in two pairs of sides of the pole triangle. So there exists a common mirror image X for any triplet of homologous points X_1, X_2, X_3, which turns out to be isogonally related to the circumcenter M_X of $X_1X_2X_3$ (cf. Fig. 1 and [1], p. 228 or [8]). That is, the directed angles e.g. $\sphericalangle XP_{12}P_{31}$ and $\sphericalangle M_XP_{12}P_{23}$ are opposite. Hence it follows from Lemma 1 that the orthocenter and the circumcenter of a triangle are a pair of isogonal points. Another consequence is (cf. [5])

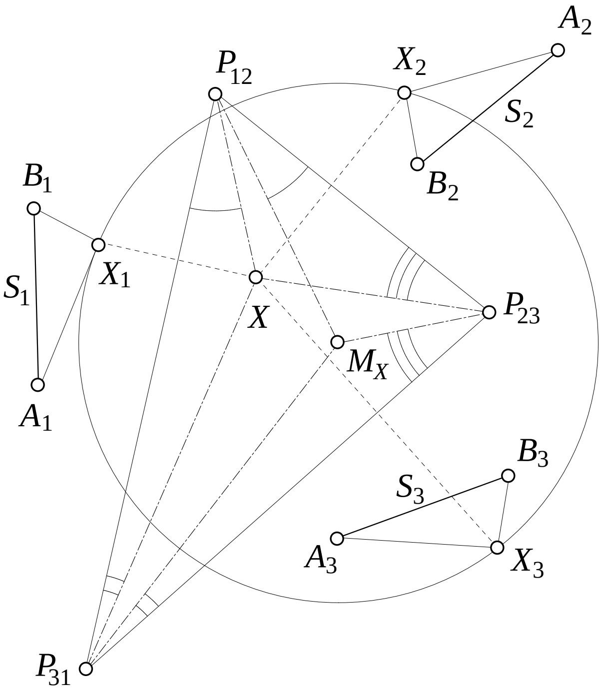

Figure 1: Three-position theory and the isogonal relation

Lemma 2: *If and only if X is a point on the circumcircle of a triangle, its isogonal point C with respect to that triangle is an ideal point in orthogonal direction to the Wallace line of X.*

The isogonal relation turns out to be an involutory quadratic transformation with the pole triangle as the singularity set. It is well known that each pair of isogonally related points represents a pair of focal points of a conic touching the sides of the pole triangle. By using the theorem on the angle of circumference of a circle, one can furthermore prove the following (cf. Fig. 2)

Theorem 1: *The circumcircles of the four triangles $X_1P_{12}P_{31}$, $X_2P_{23}P_{12}$, $X_3P_{31}P_{23}$, and $X_1X_2X_3$ pass through a common point F_X.*

This Theorem 1 is a counterpart to the following obvious extension of a theorem in [10], p. 258:

Theorem 2: *The circumcircles of the four triangles $X_1X_2P_{12}$, $X_2X_3P_{23}$, $X_3X_1P_{31}$, and $P_{12}P_{23}$ P_{31} have a point G_X in common.*

Theorems 1 and 2 are also related to a well-known theorem of Miquel (cf. e.g. [9]), which states that the four circumcircles of the sub-triangles of a complete quadrilateral pass through one point. This relationship suggests that Theorem 1 and 2 can be 'transferred' into the Moebius plane. (This transference is based on the generic term 'Moebius circle' for both, 'circle' and 'line', and is left to the reader.)

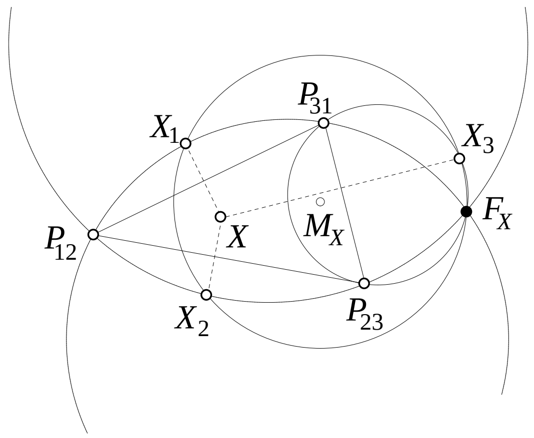

Figure 2: Concurrent circumcircles

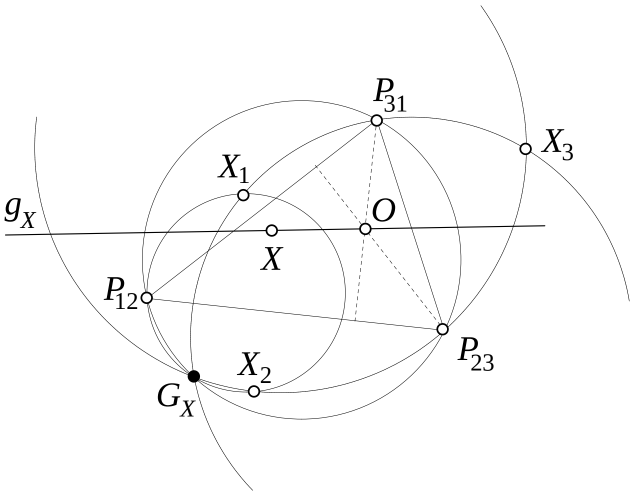

Figure 3: Wallace line of the common point of four circumcircles

Applying Lemma 2 one can say more about the point G_X of Theorem 2 (cf. Fig. 3):

Theorem 3: *The reflection of G_X in the sides of the pole triangle $P_{12}P_{23}P_{31}$ produces three collinear points on the Wallace line g_X of G_X, which passes through the orthocenter O of the triangle and the common mirror-image X of the points X_1, X_2, X_3.*

In a 'dual' generalization of Theorem 1 a line x_0 is reflected in the sides of a triangle $P_{12}P_{23}P_{31}$ (cf. Fig. 4):

Theorem 4: *The reflection of a line x_0 in the sides of a triangle $P_{12}P_{23}P_{31}$ produces a trilateral $x_1x_2x_3$, which is perspective to the original triangle. The center of this perspectivity is the incenter I of $x_1x_2x_3$, and it is also a point of the circumcircle of $P_{12}P_{23}P_{31}$. The radius of the incircle of $x_1x_2x_3$ equals the distance of x_0 to the orthocenter O of $P_{12}P_{23}P_{31}$.*

A simple proof is based on the reflection of a line y_0 parallel to x_0, passing through the orthocenter O of $P_{12}P_{23}P_{31}$, in the sides of this triangle (cf. Fig. 4).

Let us return to Theorem 1 and discuss the relative position of the point F_X to X and the pole triangle $P_{12}P_{23}P_{31}$ (cf. Fig. 5):

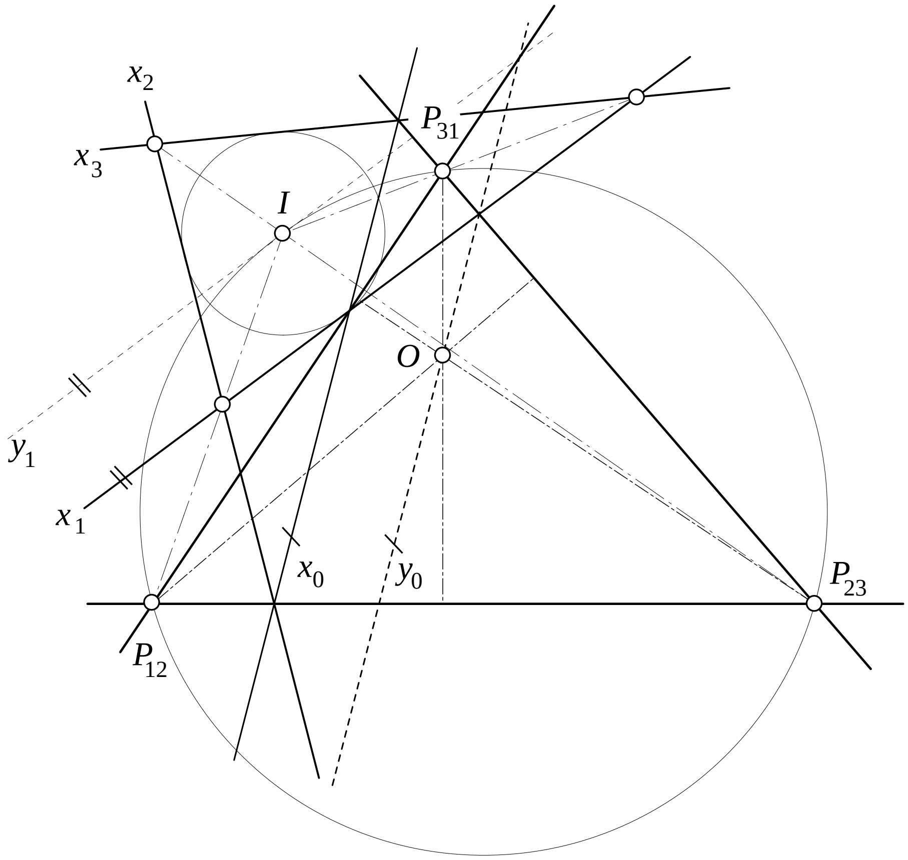

Figure 4: Reflecting a line in the sides of a triangle

Theorem 5: *Let C_1, C_2, C_3, and M_X denote the circumcenter of the triangles $X_1P_{12}P_{31}$, $X_2P_{23}P_{12}$, $X_3P_{31}P_{23}$, and $X_1X_2X_3$, respectively, and let F_X be the common point of the circumcircles of these triangles. The reflection of F_X in the sides of $C_1C_2C_3$ gives the poles P_{12}, P_{23}, P_{31}. The reflection of C_i in the 'new' sides $P_{ij}P_{ki}$ $(i, j, k = 1, 2, 3)$ produces three points M_i, which are the circumcenters of triangles $XP_{ij}P_{ki}$. A final reflection of P_{ij} in the sides M_iM_j of this new triangle leads back to point X as a common mirror image.*

Theorem 5 shows that a certain process of three consecutive reflections defines a mapping of points X via triangles to points F_X. It seems to be expedient to begin with the arbitrarily given point X together with the triangle $M_1M_2M_3$ and by reflecting X in the first triangle receiving a second triangle $P_{12}P_{23}P_{31}$. The 'reflection' — roughly speaking — of the first triangle in the second defines a third triangle $C_1C_2C_3$. Finally, by reflecting the second triangle in the third we end up with the point F_X.

A variation of the mentioned iterative reflection process is the following one: Let a triangle T_1 and an interior point X be given. The reflection of X in the sides of T_1 produces a triangle T_2, reflecting X in T_2 gives T_3, reflecting X in T_i gives T_{i+1}, $i \in \mathbb{N}$ (cf. Fig. 6). It turns out that the set of triangles T_i splits up into (in general) three subsets of *similar* triangles.

In Fig. 6 it is shown how the angles of T_1 are split up by transversals from X to the vertices of T_1 and how the parts are rearranged in T_2 and T_3 until they finally are recomposed in T_4. Thus $\sigma : T_1 \mapsto T_4$ is (in general) a similarity and X is the fixed point of σ. So the iteration process $I: T_1 \mapsto T_2 \mapsto \ldots \mapsto T_i$ is attractive or repulsive according to the similarity factor of σ which can be $s < 1$ or $s > 1$. (For acute T_1 and an inner point X the process is

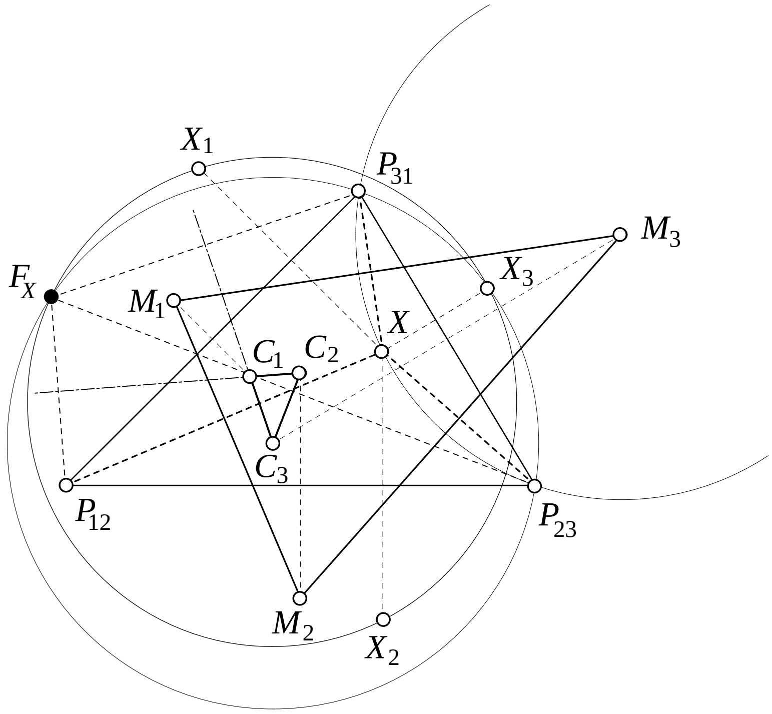

Figure 5: Reflection in the triangle of circumcenters

attractive with one exception, namely T_1 being equilateral and X its center.) If X coincides with the orthocenter of T_1, there occur only two subsets of similar triangles, and the triangles of each subset are X-perspective.

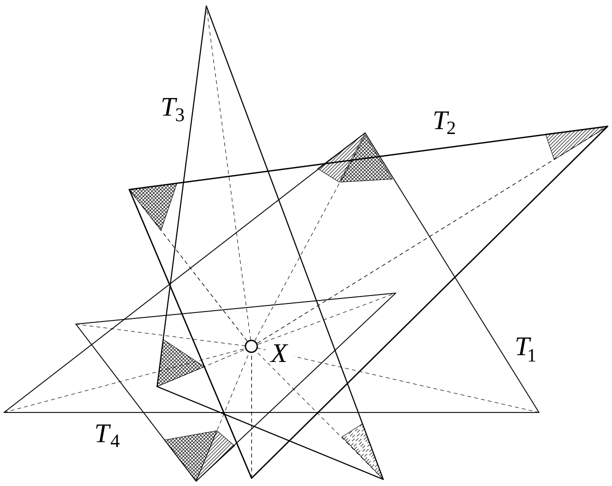

Figure 6: Reflection triangles T_i of a point X in triangles T_i, $(i = 1, 2, \ldots)$

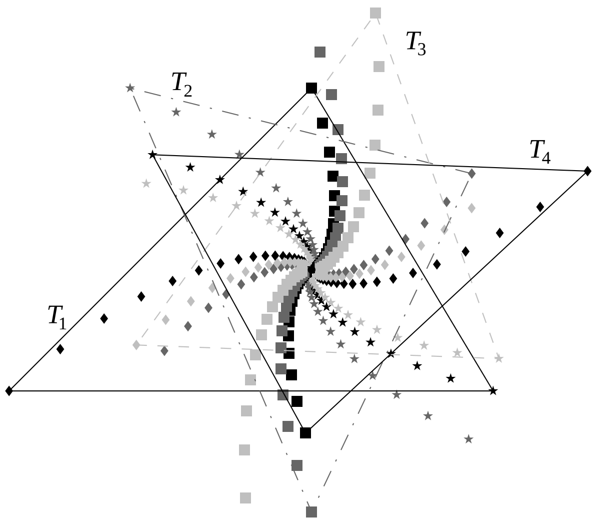

Figure 7: Orbits of vertices of T_1 under iterative reflection of X in T_i

Fig. 7 shows the path polygons of the vertices of T_1 under the iteration process I. The orbits of the orthocenter of T_1 and the point F_X associated to X according to Theorem 1 are presented in Fig. 8 and Fig. 9, respectively, while Fig. 10 and Fig. 11 show the orbit of the circumcenter of T_1 resp. the diagram of the radii of T_i.

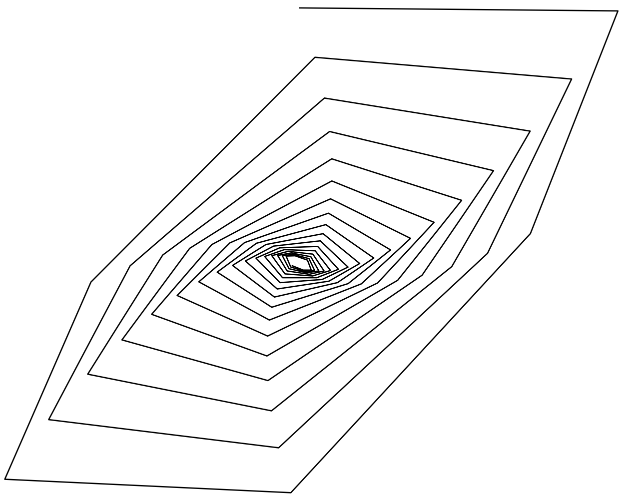

Figure 8: Orbit of orthocenter of T_1 under iteration I

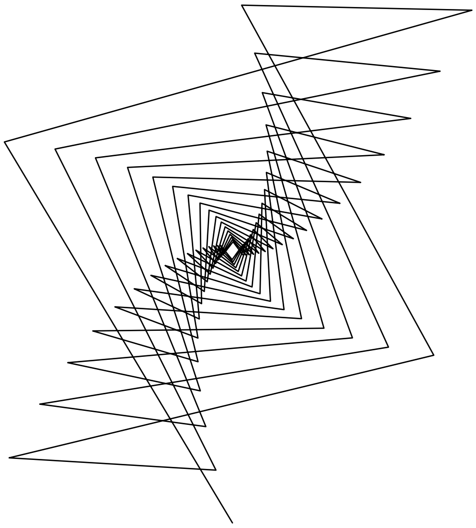

Figure 9: Orbit of F_X under iteration I

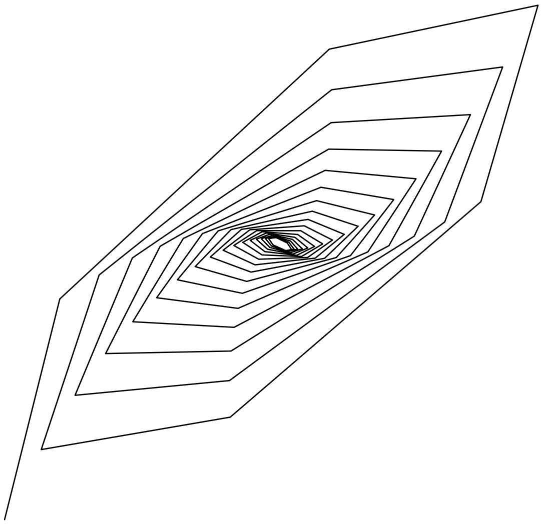

Figure 10: Orbit of circumcenter of T_1 under iteration I

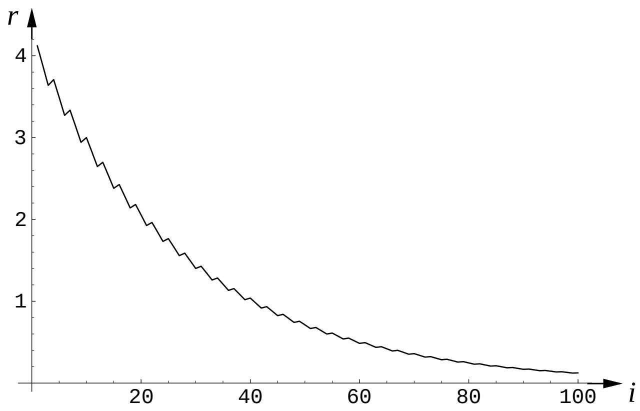

Figure 11: Diagram of radius lengths of the circumcircles of T_i

3. Four positions

In the following we start with four positions S_i of the set of points $S_i = \{A_i, B_i, \ldots X_i, \ldots\}$ in the Euclidean plane and want to interpolate these positions by a four-bar linkage or a slider crank. This problem is equivalent to the finding of (at least) two quadruples of homologous points $X_1, \ldots, X_4$ and $Y_1, \ldots, Y_4$ on a circle or on a line. This problem has e.g. been solved in [5] and in [1], p. 235.

The four consecutive positions S_i $(i = 1, \ldots, 4)$ are realized by three rotations with centers P_{12}, P_{23}, P_{34}. For the remaining pairs of positions, the poles P_{31}, P_{24} and P_{41} are added, and we receive the complete pole plan of the four given positions. Any three positions S_i, S_j, S_k give rise to one of four pole triangles (cf. chapter 2). Let $T^4 := P_{12}P_{23}P_{34}$ and $T^1 := P_{23}P_{34}P_{42}$ be two of these triangles. Then, because of Lemma 1 and 2, the line w connecting the orthocenters O^4 of T^4 and O^1 of T^1 is at the same time the Wallace line of a point W^4 on the circumcircle c^4 of T^4 and of point W^1 on the circumcircle c^1 of T^1. Hence, e.g. W^4 is isogonal to the ideal point U_0^4 which describes the direction orthogonal to w. (By Theorem 4 the point W^4 can be interpreted as the degenerated mirror image of w at T^4.) As a consequence, the mirror images of $X_0^4 := W^4$ with respect to T^4 are the (collinear) positions X_1, X_2, X_3 on w of a point X of the given point set S. Applied to $W^1 =: X_0^1$ the same considerations lead to the collinear positions X_2, X_3, X_4. Thus, all positions X_i are collinear with w. We will receive the same result if starting with the triangles $T^2 := P_{13}P_{34}P_{41}$ and $T^3 := P_{12}P_{24}P_{41}$.

Hence, the four triangles T^i have collinear orthocenters O^i (cf. Fig. 12 and [5]). By similar considerations one can prove that the four circumcircles c^i of T^i have a common point Y_0 and that the isogonal points W^i, together with this point Y_0, are concyclic (cf. Fig. 12).

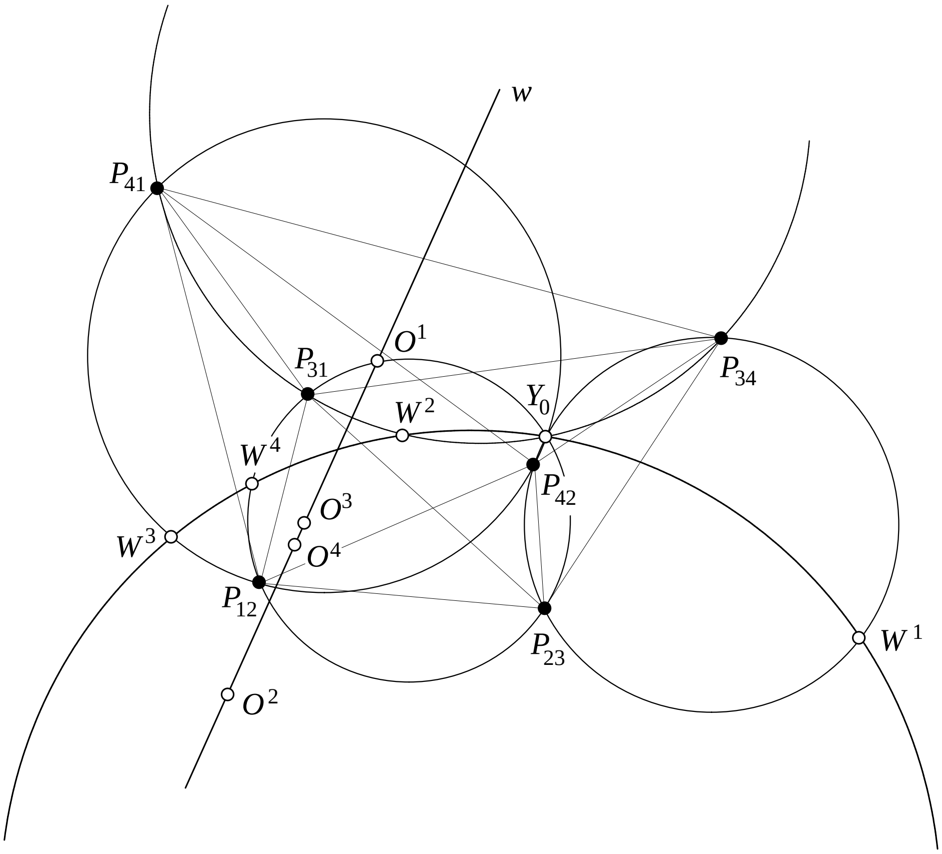

Figure 12: Configuration of the four pole triangles of four given positions

We summarize these remarkable facts in

Theorem 6: *In (planar) four-position theory the four pole triangles Y_0 have their orthocenters O^i on one line w, and their circumcircles have one point Y_0 in common. This point Y_0 and the (degenerated) mirror images W^i of w at T^i are located on the same circle.*

Let the poles P_{12}, P_{23} and P_{34} be given, as well as the angles of rotation $\varphi_{12} : S_1 \to S_2$, $\varphi_{23} : S_2 \to S_3$ and $\varphi_{34} : S_3 \to S_4$. Then it is possible to complete the configuration of the poles by taking into account that e.g. $\sphericalangle P_{23}P_{12}P_{31} = \varphi_{12}/2$. Therewith, an arbitrary point X_1 determines the positions X_2, X_3, X_4 as well as X_0^i. (Thereby each X_0^i is the mirror image of X_j, $(j = 1, \ldots, 4)$, with respect to the triangle T^i of the pole configuration.) Obviously, the circle with center P_{12} through X_1 also passes through X_2, X_0^3 and X_0^4. As there are six such circles and all together eight points, and each circle contains four of these eight points and each point belongs to three circles, this configuration is the well known configuration of Miquel (cf. Fig. 13).

Theorem 7: *In (planar) four-position theory any quadruple of homologous points X_i and their mirror images X_i in the pole triangles T^i of the four positions are in Miquel's configuration, that will say, they are — four by four — located on circles.*

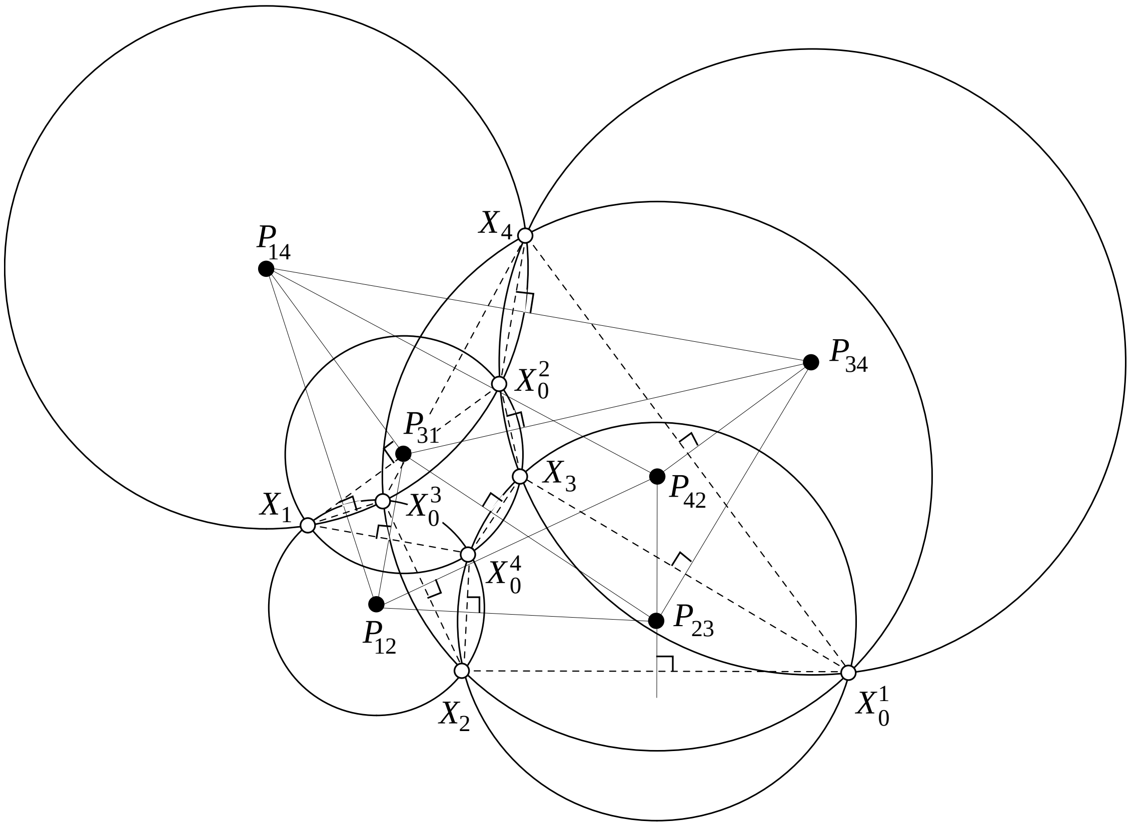

Figure 13: Miquel's configuration defined by the poles of four given position and any set of homologous points

4. Acknowledgements

The authors express their gratitude to Dipl. med. T. Illert and Dr. M. Amlang from the Surgical Department of the Dresden University of Technology ('Klinik für Unfall- und Wiederherstellungschirurgie') for providing them with data-material to the kinematics of the lower ankle joint and for having stimulated them to research on Theoretical Kinematics and Elementary Geometry.

References

[1] O. Bottema, B. Roth: *Theoretical Kinematics*. Dover Publications Inc., New York 1979.

[2] L. Burmester: *Lehrbuch der Kinematik*. Teubner, Leipzig 1888.

[3] O. Giering: *Affine and Projective Generalization of Wallace Lines*. J. Geom. Graph. **1**, 119–133 (1997).

[4] W. Jank: *Zur Kinematik des Kniegelenks*. Berichte der mathem.-statist. Sektion im Forschungszentrum Graz, Austria, Nr. **148**, 1–11 (1980).

[5] W. Lichtenheld: *Die Bedeutung der Geometrie bei der Lösung getriebetechnischer Probleme*. Sitzungsberichte d. Deutschen Akademie der Wissenschaften, Berlin, Jahrg. 1959, Nr. 3, 3–26 (1959).

[6] W. Lichtenheld: *Konstruktionslehre der Getriebe*. Akademie-Verlag, Berlin 1961.

[7] K. Luck, K.-H. Modler: *Getriebetechnik*. Akademie-Verlag, Berlin 1990.

[8] H. Stachel: *Zur Gruppe der ebenen Bewegungen*. ÖMG Didaktik-Reihe, Heft **9**, 186–200 (1982).

[9] G. Weiss: *Einige ebene Schließungssätze und mögliche Erweiterungen derselben*. Wissenschaftliche Zeitschrift der TU Dresden Nr. **45**(6), 18–22 (1996).

[10] D. Wells: *Dictionary of Curious and Interesting Geometry*. Penguin Books, London 1991.

[11] W. Wunderlich: *Ebene Kinematik*. B.I. Mannheim-Wien-Zürich 1970.

Received August 14, 1998; final form July 1, 1999