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On the Bisectors of Weakly Separable Sets

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Abstract. A bisector of two sets is the set of points equidistant to them. Bisectors arise naturally in several areas of computational geometry. We show that bisectors of weakly linearly separable sets in $\mathbb{E}^d$ share many properties with separating lines. Among these, the bisector of a restricted class of linearly separated sets is a homeomorphic image of the linear separator. We also give necessary and sufficient conditions for the existence of a particular continuous map from a portion of any linear separator to the bisector.

Key Words: Bisector, symmetric axis, linearly separable sets

MSC 1994: 68U05, 51M05

1. Introduction

Bisectors which are defined as the set of points equidistant to two given sets (see Fig. 10), arise naturally in computing the symmetric axis transform [3] – [6]. A common means of computing the symmetric axis is the following recursive method from computational geometry[1]: Compute separately the symmetric axes of two subproblems and then merge them together. During this merge the bisector of the two subproblems is used to trim the symmetric axes of the subproblems. Certain properties of the bisector are the key to an efficient merge process. For example: One algorithm for computing symmetric axis of point sites in $\mathbb{E}^2$ using a divide-and-conquer strategy partitions the points into two almost equal-sized sets separated by a line. The bisector between these two sets is connected and is a single-valued map of the dividing line. In the case of computing the symmetric axis of multiply connected polygonal domains [6], certain bisectors are simple, closed curves. In each of these two examples, properties of the bisector allow a linear-time merge. Moreover, knowing the topology of the bisector helps in choosing an appropriate data structure for the bisector. For example: If the bisector of two

[1] A descriptive geometry approach to the construction of bisectors is presented in [1] and [7]

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sets in $\mathbb{E}^3$ is known to be a 2-manifold, then the QuadEdge data structure [2] can be used to represent and manipulate it.

In this research we show that bisectors of linearly separable sets have many properties of interest. The results presented here are for general sets in $\mathbb{E}^d$. There are several reasons for this. First, many of the results and proofs are simpler when the details of a particular class of sets do not intrude. More importantly, we want to broaden the study of bisectors beyond their use in algorithms for computing the symmetric axis of point sets in $\mathbb{E}^2$ because we believe that proximity properties of more general geometric elements in higher dimensions have important applications.

2. Linearly Separable Sets

We denote the *closure* of a set S by $\operatorname{cl} S$, the *interior* by $\operatorname{int} S$, the *boundary* by ∂S, and the closure of the *convex hull* of S by $\operatorname{ch}(S)$. Uppercase characters denote points in $\mathbb{E}^d$ and p_i denotes the i^{th} coordinate of the point P.

The *Euclidean distance* between two points P and Q is denoted by $d(P,Q)$. The distance between a point P and a nonempty set S is $d(P,S) = \inf\{d(P,Q) \mid Q \in S\}$. The *nearness* of S_1 and S_2 is $d(S_1, S_2) = \inf\{d(P,Q) \mid P \in S_1, Q \in S_2\}$.

Let $\operatorname{sl}(\pi_1, \pi_2)$ denote the *open slab* between the two distinct parallel hyperplanes π_1 and π_2. The sets S_1 and S_2 are *separated* by a slab $\operatorname{sl}(\pi_1, \pi_2)$ if S_1 and S_2 lie in different components of $\mathbb{E}^d \setminus \operatorname{sl}(\pi_1, \pi_2)$. S_1 and S_2 are *strongly linearly separable* if there exists an open slab that separates $\operatorname{cl} S_1$ and $\operatorname{cl} S_2$. A hyperplane contained in such an open slab is called a *strong linear separator*. S_1 and S_2 are *linearly separable* if there exists a hyperplane π, called a *linear separator*, such that $\operatorname{cl} S_1$ and $\operatorname{cl} S_2$ are in different components of $\mathbb{E}^d \setminus \pi$. Similarly, S_1 and S_2 are *weakly separable* if there exists a hyperplane π_w called a *weak linear separator*, such that $\operatorname{cl} S_1$ and $\operatorname{cl} S_2$ lie in the closures of different components of $\mathbb{E}^d \setminus \pi_w$. Fig. 1 shows some examples of separable sets and separators in $\mathbb{E}^2$.

Note that strongly linearly separable sets are also linearly separable and linearly separable sets are also weakly linearly separable. Likewise, a strong linear separator is also a linear separator and a linear separator is also a weak linear separator. Most results presented in this research work are formulated in terms of weakly linearly separable sets and weak linear separators.

A practically useful characterization of strong linear separability is given in

Theorem 1. *If at least one of the sets S_1 and S_2 is bounded, then S_1 and S_2 are strongly linearly separable if and only if* $\operatorname{ch}(S_1) \cap \operatorname{ch}(S_2) = \{\}$.

Proof: Since at least one of S_1 and S_2 is bounded, $\operatorname{ch}(S_1) \cap \operatorname{ch}(S_2) = \{\}$ implies that $d\left((\operatorname{ch}(S_1), \operatorname{ch}(S_2)\right) > 0$. Since the nearness is positive, there exists a slab that separates $\operatorname{ch}(S_1)$ and $\operatorname{ch}(S_2)$. Therefore S_1 and S_2 are strongly linearly separable. The converse is trivial. □

3. Partitions

In this section we investigate properties of the bisector of two weakly linearly separable sets S_1 and S_2. These are of use in Section 4 in studying the topology of the bisector. Hereafter, we assume that S_1 and S_2 are nonempty sets in $\mathbb{E}^d$ such that $\operatorname{cl} S_1 \cap \operatorname{cl} S_2 = \{\}$.

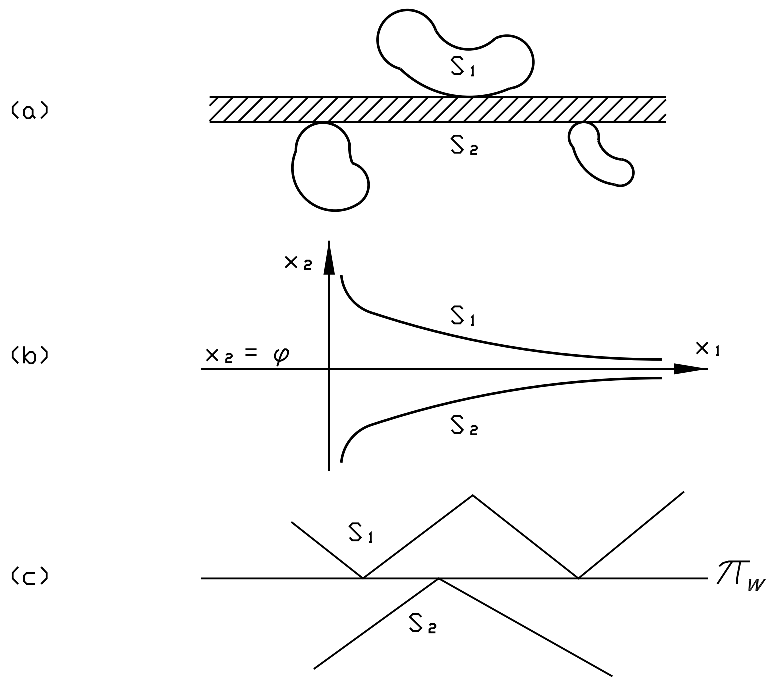

Figure 1: Three types of linear separability:

(a) S_1 and S_2 are strongly linearly separable since they are separated by the cross-hatched slab.

(b) $S_1 : x_2 = x_1^{-1}$ and $S_2 : x_2 = -x_1^{-1}$, $x_1 > 0$, are linearly separable (but not strongly separable) and $x_2 = 0$ is the linear separator.

(c) S_1 and S_2 are weakly linearly separable (but not linearly separable) and π_w is the weak linear separator.

The *bisector* $\mathrm{B}(S_1, S_2)$ of two sets S_1 and S_2 is the set of points equidistant to S_1 and S_2, i.e., $\mathrm{B}(S_1, S_2) = \{P \in \mathbb{E}^d \mid d(P, S_1) = d(P, S_2)\}$. Assume that Q is a point of $\mathrm{B}(S_1, S_2)$ and we define the *maximal ball* β to be the open ball centered at Q with radius $r = d(Q, S_1) = d(Q, S_2)$. We also define the *maximal sphere* $\sigma := \partial\beta$.

We observe that by definition β does not contain points of S_1 or S_2 and that σ contains at least one point P_1 from $\mathrm{cl}\, S_1$ and at least one point $P_2 \in \mathrm{cl}\, S_2$. The points P_1 and P_2 are called *touching points* of σ with S_1 and S_2, respectively, and σ is said to *touch* S_1 and S_2. Notice that $P_1 \neq P_2$ as $\mathrm{cl}\, S_1 \cap \mathrm{cl}\, S_2 = \{\}$. One maximal ball cannot contain another on because the included maximal sphere would lack touching points.

Without loss of generality, we assume that the hyperplane $x_d = 0$ is a weak linear separator π_w and that S_1 is contained in the closure of the open half-space $\pi_w^1 : x_d > 0$, and that S_2 is contained in the closure of the open half-space $\pi_w^2 : x_d < 0$. In the sequel we consider the mapping which takes a point on a weak linear separator orthogonally to π_w onto a point of the bisector. We first show in Section 3.1 that if a line perpendicular to π_w intersects $\mathrm{B}(S_1, S_2)$, then the intersection is connected. Then, in Section 3.2, we establish the necessary and sufficient conditions under which all lines perpendicular to π_w intersect $\mathrm{B}(S_1, S_2)$. Finally, in Section 3.3 we give necessary and sufficient conditions for a specific line perpendicular to π_w to intersect $\mathrm{B}(S_1, S_2)$.

3.1. Connectedness of the intersection with a perpendicular

Theorem 2. *If a line ℓ perpendicular to a weak linear separator π_w of S_1 and S_2 intersects* $\mathrm{B}(S_1,S_2)$*, then it does so in a connected subset of ℓ. Moreover, if a point Q is in the relative interior of $\ell \cap \mathrm{B}(S_1,S_2)$, then all the touching points of the maximal sphere centered at Q are in π_w.*

Proof: To show that the intersection is connected, assume that there are two distinct points Q_a and Q_b in $\ell \cap \mathrm{B}(S_1,S_2)$. Assume that B_a, S_a, B_b, and S_b are the corresponding maximal balls and spheres. Since Q_a and Q_b are distinct, $B_a \neq B_b$. Since one maximal ball cannot be contained in another, there are three remaining cases:

Case 1, S_a and S_b are disjoint:
Since Q_a and Q_b are in ℓ, there exists a hyperplane λ perpendicular to ℓ which is a strong linear separator of S_a and S_b (see Fig. 2a). Since λ is parallel to π_w, S_a cannot touch S_2 and/or S_b cannot touch S_1 — a contradiction.

Case 2, S_a and S_b intersect at one point P:
Assume that λ is the hyperplane perpendicular to ℓ through P (see Fig. 2b). S_a and S_b can touch both S_1 and S_2 only if $P \in \mathrm{cl}\, S_1 \cap \mathrm{cl}\, S_2$, which violates the assumption stated at the beginning of Section 3 that $\mathrm{cl}\, S_1 \cap \mathrm{cl}\, S_2 = \{\}$.

Case 3, S_a and S_b intersect in a nondegenerate lower-dimensional sphere:
Assume that λ is the hyperplane through the sphere of intersection. Note that λ is parallel to π_w. If $\lambda \neq \pi_w$, the sets of touching points of S_a and S_b lie in opposite closed half-spaces bounded by λ (see Fig. 2c). Therefore, since λ is parallel to π_w, S_a cannot touch S_2 and/or S_b cannot touch S_1, which is a contradiction. Thus we have $\lambda = \pi_w$. S_a and S_b can touch both S_1 and S_2 only if one of the touching points on S_1 and one of the touching points on S_2 are in π_w. These touching points are contained in $S_a \cap S_b$. Assume that Q is strictly between Q_a and Q_b, and that S_Q is the sphere centered at Q and passing through $S_a \cap S_b$ (see Fig. 2d). Clearly the open ball B defined by S_Q is contained in $B_a \cap B_b$ and hence is free of points of $S_1 \cup S_2$. Thus S_Q is a maximal sphere touching both S_1 and S_2 in π_w, and $Q \in \mathrm{B}(S_1,S_2)$. The theorem follows directly. □

3.2. Intersection with every perpendicular

In Section 3.1 we proved that if the intersection between the bisector and a line perpendicular to a weak linear separator exists, then it is connected. In this section we give necessary and sufficient conditions for every line perpendicular to the weak linear separator to intersect the bisector.

Assume that

$$I := \left\{ X = (x_1, \ldots, x_{d-1}) \in \mathbb{E}^{d-1} \mid \alpha_i \leq x_i \leq \beta_i \ \text{ for } \ i = 1, \ldots, d-1 \right\},$$

for real numbers $\alpha_i \leq \beta_i$ is a closed $(d-1)$-cell in π_w.

Theorem 3. *Every line perpendicular to the weak linear separator π_w intersects* $\mathrm{B}(S_1,S_2)$ *if and only if*

(1) $S_1, S_2 \not\subset \pi_w$*, or*

(2) $S_2 \subset \pi_w$ *and* $\mathrm{cl}\, S_1 \cap \pi_w = \{\}$*, or vice versa.*

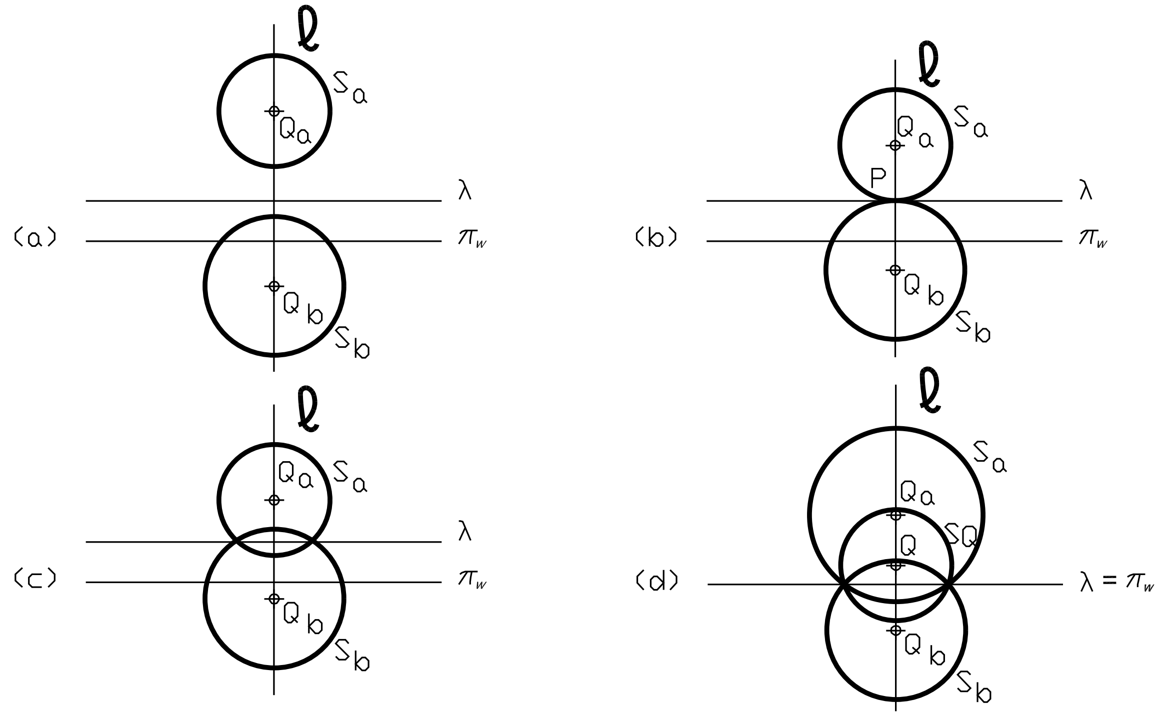

Figure 2: Illustration of various cases for proof of Theorem 2.

Moreover, if every line perpendicular to π_w *intersects* $\mathrm{B}(S_1, S_2)$*, then for any* $(d-1)$*-cell* I *in* π_w *the set* $\mathrm{B}(S_1, S_2) \cap (I \times \mathbb{R})$ *is bounded.*

We prove Theorem 3 by showing the sufficient conditions in Lemma 1 and 2 and the necessary condition in Lemma 3.

Lemma 1. *If* $S_1, S_2 \not\subset \pi_w$*, then every line perpendicular to* π_w *intersects* $\mathrm{B}(S_1, S_2)$*. Moreover,* $\mathrm{B}(S_1, S_2) \cap (I \times \mathbb{R})$ *is bounded.*

Proof: Assume that Q is a point on the line

$$\ell := \{X = (x_1, \dots, x_d) \mid x_i = q_i,\ i = 1, \dots, d-1\}$$

that is perpendicular to π_w. We define $f(Q) = d^2(Q, S_1) - d^2(Q, S_2)$, and we observe that $Q \in \mathrm{B}(S_1, S_2)$ if and only if $f(Q) = 0$. First we assume that $q_d > 0$. Let $P \in S_1$, $p_d > 0$. Such a point must exist since $S_1 \not\subset \pi_w$ (see Fig. 3).

Since $d^2(Q, S_1) \leq d^2(Q, P) = \sum_{i=1}^{d} (p_i - q_i)^2$ and $d^2(Q, S_2) \geq d^2(Q, \pi_w) = q_d^2$,

$$f(Q) \leq \sum_{i=1}^{d} (p_i - q_i)^2 - q_d^2 = \sum_{i=1}^{d-1} (p_i - q_i)^2 + p_d^2 - 2p_d q_d.$$

For sufficiently large q_d we get $f(Q) < 0$. By a symmetry argument, for sufficiently small $q_d < 0$ we get $f(Q) > 0$. But $d(Q, S_i)$ is a continuous function of P (as assumed in Theorem 3). Therefore, since f changes sign, it must have at least one zero, which implies that the line ℓ intersects $\mathrm{B}(S_1, S_2)$.

To prove that $\mathrm{B}(S_1, S_2) \cap (I \times \mathbb{R})$ is bounded, we observe that for

$$q_d > \left(\sum_{i=1}^{d} (p_i - q_i)^2 + p_d^2 \right) / 2p_d$$

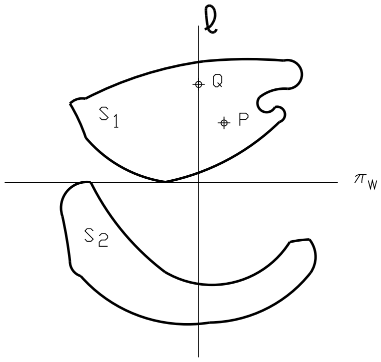

Figure 3: Selecting points P and Q for the proof of Lemma 1.

we get $f(Q) < 0$. Thus the intersection of $\mathrm{B}(S_1, S_2)$ with ℓ is bounded in the positive x_d-direction by a continuous function of $q_1, \ldots, q_d$. It is similarly bounded in the negative x_d-direction. In particular, as I is compact, $\mathrm{B}(S_1, S_2) \cap (I \times \mathbb{R})$ is bounded. □

Lemma 2. *If* $S_2 \subset \pi_w$ *and* $\mathrm{cl}\, S_1 \cap \pi_w = \{\}$*, then every line perpendicular to* π_w *intersects* $\mathrm{B}(S_1, S_2)$*. Moreover,* $\mathrm{B}(S_1, S_2) \cap (I \times \mathbb{R})$ *is bounded.*

Proof: Let Q be a point on a line ℓ perpendicular to π_w and vary q_d so that Q moves along ℓ. By the arguments used in the proof of Lemma 1 we have $f(Q) = d^2(Q, S_1) - d^2(Q, S_2) < 0$ for sufficiently large $q_d > 0$.

We now show that there exists a $q_d < 0$ such that $f(Q) \geq 0$. Assume that $U := \ell \cap \pi_w$ and P is a point of $\mathrm{cl}\, S_2$ closest to U. If $P = U$, then $d^2(Q, S_2) = q_d^2$. Therefore, since $d^2(Q, S_1) > d^2(Q, \pi_w) = q_d^2$, we get $q_d \leq 0$.

If $P \neq U$, then we consider an open ball B_u of radius $d(U, P)$ centered at U. If $\mathrm{cl}\, S_1 \cap \mathrm{cl}\, B_u = \{\}$, then $d^2(U, S_1) > d^2(U, S_2)$, which leads to $f(U) > 0$. Otherwise, since $\mathrm{cl}\, S_1 \cap \mathrm{cl}\, B_u$ is compact, there exists a point $T \in (\mathrm{cl}\, S_1 \cap \mathrm{cl}\, B_u)$ with smallest x_d-coordinate. Moreover, $t_d > 0$ because $\mathrm{cl}\, S_1 \cap \pi_w = \{\}$ (see Fig. 4).

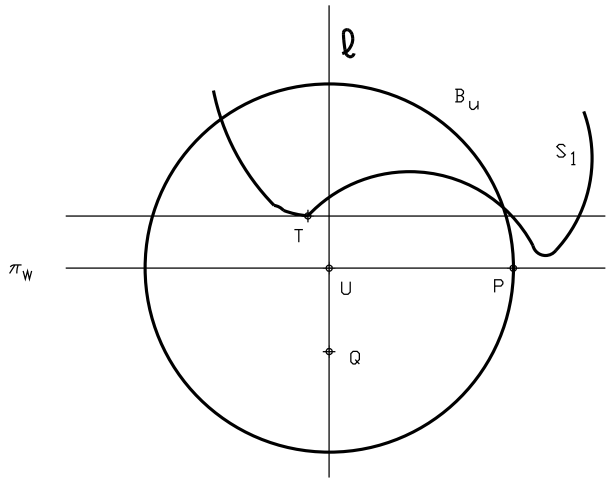

Figure 4: Construction for proof of Lemma 2.

For $q_d \leq 0$, $d^2(Q, S_1) \geq (t_d - q_d)^2$. Therefore, since $d^2(Q, S_2) = \sum_{i=1}^{d}(p_i - q_i)^2$,

$$f(Q) \geq t_d^2 - 2t_d q_d - \sum_{i=1}^{d-1}(p_i - q_i)^2$$

if $q_d \leq 0$. For sufficiently small $q_d \leq 0$ is $f(Q) \geq 0$. Thus $f(Q)$ has a zero and the intersection result follows. Boundedness is guaranteed, since T is confined to a bounded set as long as $U \in I$. □

To complete proof of Theorem 3, it remains to prove the necessary condition.

Lemma 3. *If $S_2 \subset \pi_w$ and $\mathrm{cl}\, S_1 \cap \pi_w \neq \{\}$, then there exist lines perpendicular to π_w that do not intersect* $\mathrm{B}(S_1, S_2)$.

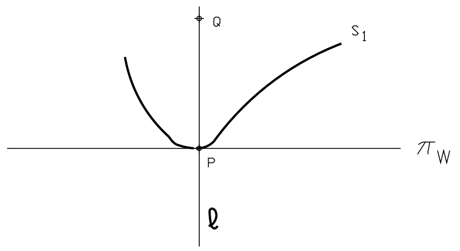

Figure 5: Construction for proof of Lemma 3.

Proof: Consider the line ℓ perpendicular to π_w through a point $P \in \mathrm{cl}\, S_1 \cap \pi_w$ (see Fig. 5). For any $Q \in \ell$ we obtain $d^2(Q, S_1) \leq d^2(Q, P) = q_d^2$. Since $P \notin \mathrm{cl}\, S_2$ and $S_2 \in \pi_w$, $d^2(Q, S_2) > q_d^2$. Therefore $d^2(Q, S_1) < d^2(Q, S_2)$ for all $Q \in \ell$, so ℓ cannot intersect $\mathrm{B}(S_1, S_2)$. □

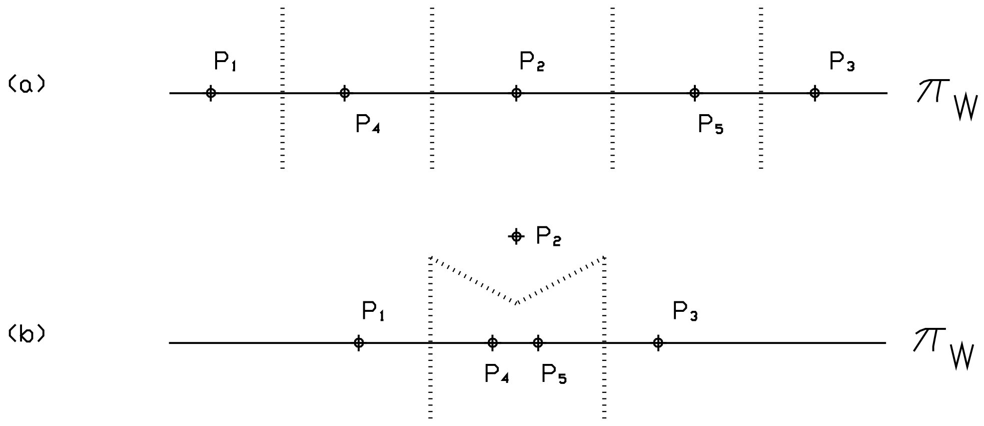

Figure 6: Sets illustrating the condition of Lemma 3.

Fig. 6 shows two examples as an illustration of Lemma 3. Let $S_1 := \{P1, P2, P3\}$ and $S_2 := \{P4, P5\}$. We have:

1. Both S_1 and S_2 are completely contained in their weak linear separator π_w. $\mathrm{B}(S_1, S_2)$ consists of four lines (shown dashed), all of which are perpendicular to π_w.
2. S_1 is only partially contained in π_w. In both cases there exist lines perpendicular to π_w that do not intersect $\mathrm{B}(S_1, S_2)$.

Corollary 1. *If S_1 and S_2 are weakly linearly separated by π_w and $\operatorname{cl} S_1 \subset \pi_w^1$, then every line perpendicular to π_w intersects $\mathrm{B}(S_1, S_2)$ in a single point.*

Corollary 2. *If a hyperplane π is a linear separator of S_1 and S_2, then every line perpendicular to π intersects $\mathrm{B}(S_1, S_2)$ in a single point.*

3.3. Intersection with a specific perpendicular

Theorem 3 gave the necessary and sufficient conditions under which every line perpendicular to π_w intersects $\mathrm{B}(S_1, S_2)$. Even if such global conditions do not hold, it is still possible to obtain local results. This section gives necessary and sufficient conditions under which specific line perpendicular to π_w intersects $\mathrm{B}(S_1, S_2)$.

Lemma 4. *Let $S_1, S_2 \subset \pi_w$ and let U be a point in π_w. A line ℓ perpendicular to π_w and passing through U intersects $\mathrm{B}(S_1, S_2)$ if and only if $d(U, S_1) = d(U, S_2)$. Moreover, if ℓ intersects $\mathrm{B}(S_1, S_2)$, then $\ell \subset \mathrm{B}(S_1, S_2)$.*

Proof: Let $Q \in \ell$. Then $d^2(Q, S_i) = d^2(U, S_i) + q_d^2$ for $i = 1, 2$. Therefore, if $d(U, S_1) = d(U, S_2)$, then $d(Q, S_1) = d(Q, S_2)$, which implies that $\ell \subset \mathrm{B}(S_1, S_2)$. Conversely if $Q \subset \mathrm{B}(S_1, S_2)$, then $d^2(Q, S_1) = d^2(Q, S_2)$ and hence $d(U, S_1) = d(U, S_2)$. □

Lemma 5. *Let $S_2 \subset \pi_w$, $S_1 \not\subset \pi_w$, $\operatorname{cl} S_1 \cap \pi_w \neq \{\}$, and $U \in \pi_w$ such that $d(U, S_2) \neq d(U, \operatorname{cl} S_1 \cap \pi_w)$. A line ℓ perpendicular to π_w and passing through U intersects $\mathrm{B}(S_1, S_2)$ if and only if $d(U, S_2) < d(U, \operatorname{cl} S_1 \cap \pi_w)$. Moreover, if ℓ intersects $\mathrm{B}(S_1, S_2)$, then it does so in a single point.*

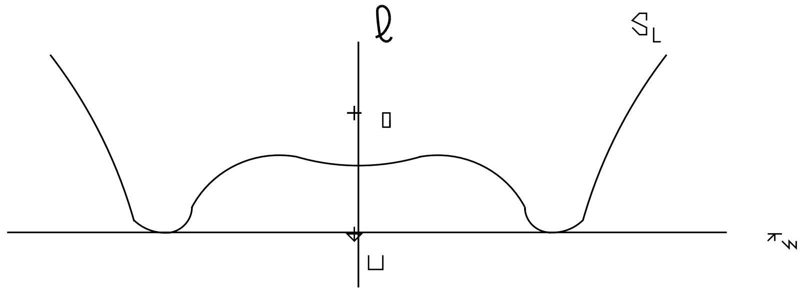

Figure 7: Construction for proof of Lemma 5.

Proof: Let Q be a point in ℓ (see Fig. 7). Sufficiency follows from arguments similar to those of the proof Lemma 2. To show the necessity, assume that $Q \in \mathrm{B}(S_1, S_2)$. Since $d^2(Q, S_1) = d^2(Q, S_2)$, $d^2(Q, S_2) = d^2(U, S_2) + q_d^2$, $d^2(Q, \operatorname{cl} S_1 \cap \pi_w) = d^2(U, \operatorname{cl} S_1 \cap \pi_w) + q_d^2$, and $d^2(Q, S_1) \le d^2(Q, \operatorname{cl} S_1 \cap \pi_w)$, we have

$$d^2(U, S_2) + q_d^2 \le d^2(U, \operatorname{cl} S_1 \cap \pi_w) + q_d^2.$$

Necessity follows since, by hypothesis, $d(U, S_2) \neq d(U, \operatorname{cl} S_1 \cap \pi_w)$.

To see that the intersection is a single point, assume the contrary. Then, by Theorem 2, there exists $Q \in \operatorname{int}[\ell \cap \mathrm{B}(S_1, S_2)]$ such that the touching points of the maximal sphere centered at Q are contained in π_w. This implies that the touching points in S_1 are in $\operatorname{cl} S_1 \cap \pi_w$ and that $d^2(Q, S_2) = d^2(Q, \operatorname{cl} S_1 \cap \pi_w)$ — a contradiction. □

Let B be an open $(d-1)$-dimensional ball and let V be a point on a line perpendicular to the hyperplane that contains B, and passing through the center of B. The *truncated semicone* $C(V,B)$ is defined to be $\operatorname{int}\operatorname{ch}(V \cup B)$.

Lemma 6. *Let $S_2 \subset \pi_w$, $S_1 \not\subset \pi_w$, $\operatorname{cl} S_1 \cap \pi_w \neq \{\}$ and $U \in \pi_w$ such that $d(U,S_2) = d(U, \operatorname{cl} S_1 \cap \pi_w)$. Let $B \subset \pi_w$ be the $(d-1)$-dimensional ball of radius $d(U,S_2)$ centered at U. A line ℓ perpendicular to π_w and passing through U intersects $\mathrm{B}(S_1,S_2)$ if and only if there exists a point $Q \in (\ell \cap \pi_w^1)$ such that $C(Q,B) \cap S_1 = \{\}$. Furthermore, if ℓ intersects $\mathrm{B}(S_1,S_2)$, then a half-line of ℓ is contained in $\mathrm{B}(S_1,S_2)$.*

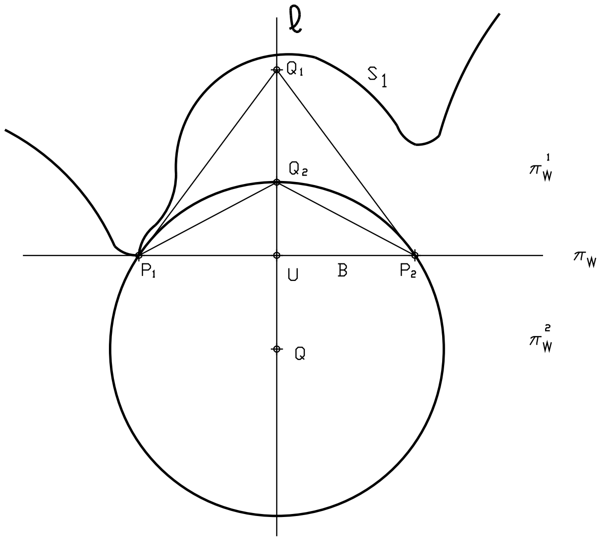

Figure 8: Construction for proof of Lemma 6.

Proof: To show sufficiency, assume that there is a point $Q_1 \in (\ell \cap \pi_w^1)$ such that $C(Q1,B) \cap S_1 = \{\}$ (see Fig. 8).

Now we consider the one-parameter family of d-dimensional balls (and the associated boundary spheres) that intersect π_w in B and whose centers lie on ℓ. Some of these boundary spheres must intersect pi_w^1 within $C(Q_1,B)$. Since B intersects neither $\operatorname{cl} S_1$ nor $\operatorname{cl} S_2$, but the boundary of B touches both $\operatorname{cl} S_1$ and $\operatorname{cl} S_2$, such spheres must be maximal spheres. Furthermore, any member of the family whose center has smaller x_d-coordinate must also be maximal. Thus, a half-line of ℓ is contained in $\mathrm{B}(S_1,S_2)$.

To show necessity, we consider a maximal sphere σ (and the associated ball β) centered at $Q \in [\ell \cap \mathrm{B}(S_1,S_2)]$. Note that $U \notin \operatorname{cl}(S_1 \cup S_2)$ because $d(U,S_2) = d(U, \operatorname{cl} S_1 \cap \pi_w)$ and $\operatorname{cl} S_1 \cap \operatorname{cl} S_2 = \sigma$. Therefore exists $Q_2 \in (pi_w^1 \cap \ell \cap \sigma)$ (see Fig. 8). Since $C(Q2,B) \subset \beta$, $C(Q2,B) \cap S_2 = \{\}$. $\square$

An example where the conditions of Lemma 6 do not hold is shown in Fig. 9. S_1 is an arc of a circle, $S_2 \subset \pi_w$ and $d(U,S_2) = d(U, \operatorname{cl} S_1 \cap \pi_w)$. The line ℓ will not intersect $\mathrm{B}(S_1,S_2)$.

Necessary and sufficient conditions for any specific line perpendicular to π_w to intersect $\mathrm{B}(S_1,S_2)$ follow directly from Theorem 3 and Lemmas 4 - 6.

From the above study, we can conclude the following Theorem:

Theorem 4. *Assume that $U \in \pi_w$ and let ℓ be the line perpendicular to π_w passing through U. The line ℓ intersects $\mathrm{B}(S_1,S_2)$ if and only if (up to a switch of S_1 and S_2):*

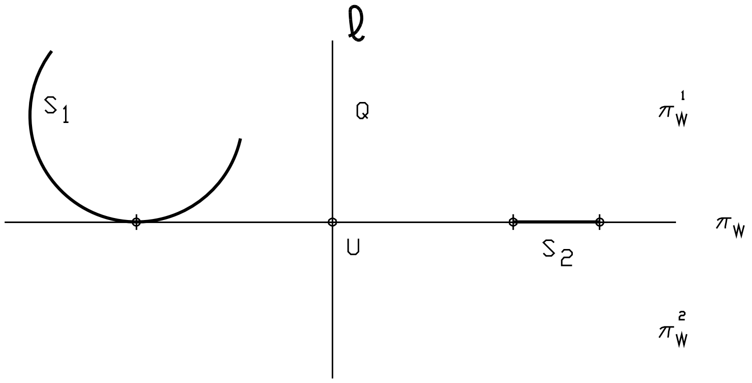

Figure 9: An example where the conditions of Lemma 6 do not hold.

(1) $S_1 \not\subset \pi_w$ *and* $S_2 \not\subset \pi_w$*, or*

(2) $S_2 \subset \pi_w$*, and* $\operatorname{cl} S_1 \cap \pi_w = \{\}$*, or*

(3) $S_1 \subset \pi_w$*,* $S_2 \subset \pi_w$ *and* $d(U, S_1) = d(U, S_2)$*, or*

(4) $S_2 \subset \pi_w$*,* $S_1 \not\subset \pi_w$*,* $\operatorname{cl} S_1 \cap \pi_w \neq \{\}$*, and* $d(U, S_2) < d(U, \operatorname{cl} S_1 \cap \pi_w)$*, or*

(5) $S_2 \subset \pi_w$*,* $S_1 \not\subset \pi_w$*,* $\operatorname{cl} S_1 \cap \pi_w \neq \{\}$*, and* $d(U, S_2) = d(U, \operatorname{cl} S_1 \cap \pi_w)$*, and there exists a point* $Q \in (\ell \cap pi^1_w)$ *such that* $C(Q, B) \cap S_1 = \{\}$*, where* $B \subset \pi_w$ *is the open* $(d-1)$*-dimensional ball of radius* $d(U, S_2)$ *centered at* U*.*

4. Continuous mapping from linear separator to bisector

We have thus far shown exactly when a line perpendicular to π_w intersects $\mathrm{B}(S_1, S_2)$ at a single point. This defines a mapping which lifts points of the separator up to the bisector. We now show that wherever such a mapping exists, it is continuous. Notice that this map will automatically be a homeomorphism, as its inverse is the orthogonal projection onto the separator a well-defined continuous map.

Theorem 5. *Let the hyperplane* $\pi_w : x_d = 0$ *be a weak linear separator of* S_1 *and* S_2 *and let* M *be a open subset of* π_w *such that for all* $P \in M$ *the line through* P *perpendicular to* π_w *intersects* $\mathrm{B}(S_1, S_2)$ *in a single point. Then the mapping* $b : M \to \mathbb{R}$ *defined by* $(x_1, \ldots, x_{d-1}, b(x_1, \ldots, x_{d-1})) \in \mathrm{B}(S_1, S_2)$ *is continuous.*

To prove this we use the known computational geometry principle: Let E and F be topological spaces. A function $g : E \to F$ is said to have a *closed graph* if its graph $\{(x, y) \mid y = g(x),\ x \in E\}$ in the product space $E \times F$ is a closed set. If g has a closed graph and F is compact, then g is continuous.

Proof: The mapping b is a function by hypothesis. We show that b is continuous at an arbitrary $P \in M$:

Let $I := \{X \in \mathbb{E}^{d-1} \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, d-1\}$, for real numbers $\alpha_i < \beta_i$, $i = 1, \ldots, d-1$ be a closed $(d-1)$-cell in M containing P in its interior and let b_1 denote the restriction of b to I. We claim that b_1 has a compact graph and thus, by the previous principle, b is continuous at P.

The graph of b_1 is the intersection of $\mathrm{B}(S_1, S_2)$ and $I \times \mathbb{R}$. Referring to the proof of Lemma 1, $\mathrm{B}(S_1, S_2) = f^{-1}(0)$. Since the inverse image of a closed set under a continuous map is closed, $\mathrm{B}(S_1, S_2)$ is closed. Therefore, the graph of b_1 is closed.

It remains to show that the range of b is also bounded and, hence, compact. Since the line perpendicular to π_w at any point $P \in M$ intersects $\mathrm{B}(S_1, S_2)$, one of the five conditions of Theorem 4 must hold for each point in M. Furthermore, since by hypothesis, each such line intersects $\mathrm{B}(S_1, S_2)$ in a single point, Lemma 4 and 6 imply that, up to a switch of S_1 and S_2, one of the following must hold:

1. $S_1 \not\subset \pi_w$ and $S_2 \not\subset \pi_w$, or
2. $S_2 \subset \pi_w$ and $\mathrm{cl}\, S_1 \cap \pi_w = \{\}$, or
3. $S_2 \subset \pi_w$, $S_1 \not\subset \pi_w$, $\mathrm{cl}\, S_1 \cap \pi_w \neq \{\}$, and $d(U, S_2) < d(U, \mathrm{cl}\, S_1 \cap \pi_w)$ for all $U \in M$.

In the first two cases Theorem 3 establishes that the graph is bounded. In the third case Lemma 5 and the fact that I is compact establishes that the graph is bounded. Therefore, the graph of b_1 is compact, which implies that b is continuous at each point of M. □

Thus, when π is a linear separator, we assume that the hyperplane $\pi : x_d = 0$ be a linear separator of S_1 and S_2. If b is the mapping $b : \pi \to \mathbb{R}$ such that $(x_1, \ldots, x_{d-1}, b(x_1, \ldots, x_{d-1})) \in \mathrm{B}(S_1, S_2)$, then b is a continuous function. In fact, the perpendicular projection of $\mathrm{B}(S_1, S_2)$ onto π is a homeomorphism. This result describes that the bisector of linearly separated point sites in $\mathbb{E}^2$ is a monotone chain. More importantly, it shows that $\mathrm{B}(S_1, S_2)$ is a $(d-1)$-manifold in $\mathbb{E}^2$.

5. Conclusion

In this paper we have presented some general properties of bisectors of sets in $\mathbb{E}^d$ that are separated by hyperplanes. We have given necessary and sufficient conditions for the perpendicular projection of the bisector of two weakly linearly separated sets onto a separator to be a homeomorphism. This study needs to be expanded in two major directions:

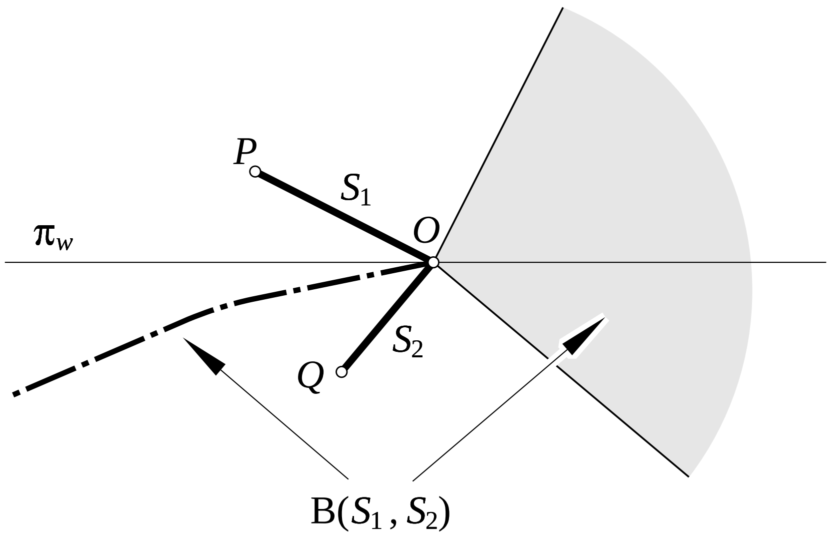

Figure 10: A nonhomogeneously two-dimensional bisector. $\mathrm{B}(S_1, S_2)$ is the bisector between two open line-segments $S_1 := OP$ and $S_2 := OQ$ that do not intersect. However, the closures of the line-segments share the endpoint O.

Throughout this paper we have required that $\mathrm{cl}\, S_1 \cap \pi_w = \{\}$. When the closures of the sets are not disjoint, the bisector need not be a manifold, as illustrated in Fig. 10. This issue has been addressed in the literature on symmetric axes of point sets and open line segments in $\mathbb{E}^2$ by defining the bisectors between individual elements so that they are always homogeneously one dimensional [3, 5, 6]. We think that this needs further investigation.

Another direction in which the theory could be generalized is to investigate the general conditions under which the bisector of two sets is a $(d-1)$-manifold that subdivides $\mathbb{E}^d$ into two disjoint regions. In this paper we have shown that if the two sets are linearly separable, then their bisector has this property. Also, some sufficient conditions for the bisector to be a simply closed curve were given in [4, 6] for sets in $\mathbb{E}^2$ that are not even weakly linearly separable.

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