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Reconstruction of the Satellite Orbit via Orientation Angles

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Abstract. This paper presents an efficient geometric method to find the mathematical model for the normal orbit of a moving satellite observed from a given station on the earth. The method relies on getting a sufficient number of observations oriented from the earth station to the satellite which moves on its predictable orbit on the celestial space. The concurrence of the revolution of the earth and the motion of the satellite is utilized to orient the calculated normal orbit in its fixed plane. Rather than deriving the geometric model for the case of a known orbital plane, we reformulate the method of solution to study the case of an unknown orbital plane. Since the earth station rotates with the earth and the satellite moves, the lines of observation are generatrices of a ruled surface with the elliptic orbit as one directrix . In this paper we assume that the satellite obeys the Keplerian laws and that the true anomaly of the orbit is the only time-dependent Kepler element.

Key Words: Applied Geometry, Celestial Mechanics, Satellite Orbits

MSC 1994: 51N05, 70F15

1. Introduction

Since the initial operating capability of the Global Positioning System (GPS) has been achieved, the problem of using observations in order to predict the position of satellites is still of interest for scientists. The importance of predicting the actual orbit of a satellite becomes evident when a large number of navigation satellites are designed to operate simultaneously.

In fact, there are many disturbing forces that cause the satellite to deviate from the simple normal orbit [1]. The disturbance of the orbit is caused primarily by the nonspherecity of the

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gravitational potential, by the attraction of sun and moon and by the solar radiation pressure. Satellites closer to the earth are affected by additional forces such as a residual atmospheric drag. One can assess the effect of disturbance to the normal orbit if the actual orbit of the disturbed satellite is detected by the observation. This actual orbit is the result of an orbital motion restricted by the earth gravitational attraction and a number of other forces acting on the satellite [1, 2]. The attraction is also caused by the sun, moon and impulses on the satellite caused by impacting solar radiation particles. Mathematically, the motion equations of satellites are differential equations that are solved under initial conditions by means of numerical integration. The computed positions of the satellite can be compared with those obtained from observations and possible discrepancies are used to improve the force functions, the initial conditions, or the position of the observer. In this article we will assume that the disturbed satellite still moves on an ellipse with the center of the earth as one focus.

Tomas Soler et al. [3] have introduced a geometric treatment of a certain orientation problem which arises when the spherical model of the earth is replaced by a spheroid. In their paper, the orientation angles of a dish antenna to a geostationary satellite are achieved for a spherical earth first and then transformed approximately to be suitable for an ellipsoid.

The main objective of the present paper is to reconstruct the satellite orbit from measured orientation angles. These measurements are only the azimuth and the vertical angle of the observation line from the earth station to the position of the moving satellite. The motion of a satellite, as it is seen from an observer on earth, is usually called the topocentric motion of the satellite. Because of the revolution of the earth around its polar axis, the position of the earth station is time-dependent. For the satellite the time of flight is used herein not only for determining the position of the satellite in the celestial reference system but also for the rotational position of the earth. The earth is considered as spheroid (biaxial ellipsoid) and the deferences of heights with respect to the geoid (actual datum) are taken in consideration to actualize the modeled earth.

2. Geocentric and geodetic coordinate systems

The geodetic position of the earth station is specified on the ellipsoidal surface by the geodetic latitude $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and the geodetic longitude $\lambda \in (-\pi, \pi)$ while the celestial position of the satellite is described by six Keplerian elements. The local geodetic coordinate system (u, n, w) shown in Fig. 1 plays an important role in the development of the geometric model of the present problem. The axes n and u span the local geodetic horizon (tangent plane of the ellipsoid at point P). The axis n is tangent to the meridian at P and oriented towards the north pole. The axis w coincides with the normal to the surface at P pointing outwards. So, the axis u completes the right handed system (u, n, w). The spatial orientation with respect to the local geodetic coordinate system is completely specified by the positions of both the earth station and the satellite.

For point $P(\lambda, \varphi)$ on the ellipsoidal surface the position vector $\vec{R}$ with respect to the geocentric reference system $(\vec{i}, \vec{j}, \vec{k})$ is

$$\vec{R} = \vec{R}(\lambda, \varphi) = N(\varphi) \cos\varphi \cos\lambda\, \vec{i} + N(\varphi) \cos\varphi \sin\lambda\, \vec{j} + N(\varphi)(1 - e^2) \sin\varphi\, \vec{k} \tag{1}$$

where

$$N(\varphi) = \frac{a}{\sqrt{1 - e^2 \sin^2\varphi}}, \qquad e = \sqrt{1 - \frac{b^2}{a^2}}. \tag{2}$$

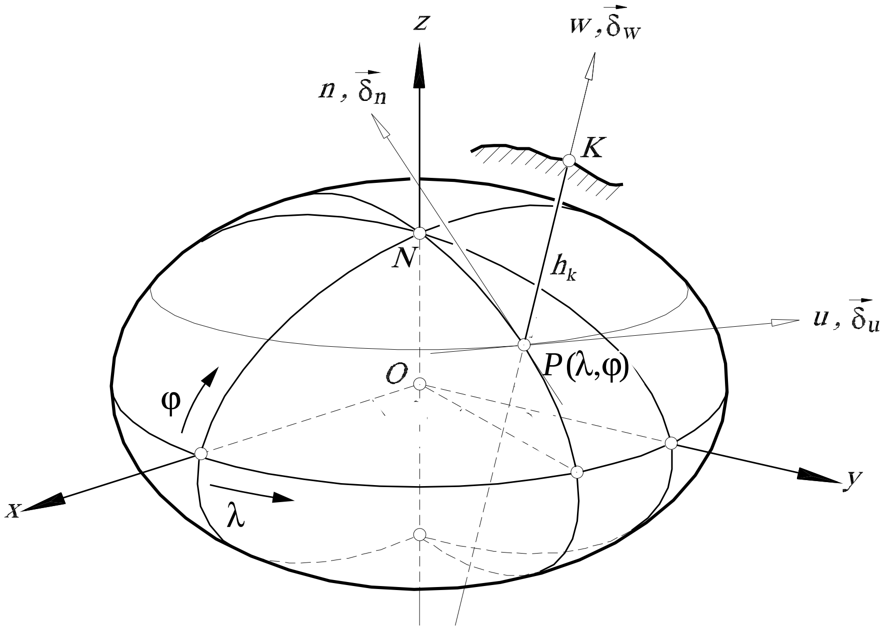

Figure 1:

The magnitudes a and b are the major and minor semi-axes and e is the eccentricity of the ellipsoid. Let the unit vectors of the geodetic axes u, n and w be denoted respectively by $\vec{\delta}_u, \vec{\delta}_n$ and $\vec{\delta}_w$. As known in differential geometry [4, 5], these vectors can be expressed as

$$\vec{\delta}_u = \frac{\partial\vec{R}/\partial\lambda}{|\partial\vec{R}/\partial\lambda|}, \quad \vec{\delta}_n = \frac{\partial\vec{R}/\partial\varphi}{|\partial\vec{R}/\partial\varphi|}, \quad \vec{\delta}_w = \vec{\delta}_u \times \vec{\delta}_n. \tag{3}$$

Differentiation of equation (1) and substitution in equations (3) yields

$$\vec{\delta}_n = -\sin\varphi\cos\lambda\,\vec{i} - \sin\varphi\sin\lambda\,\vec{j} + \cos\varphi\,\vec{k}, \tag{4}$$

$$\vec{\delta}_u = -\sin\lambda\,\vec{i} + \cos\lambda\,\vec{j} + 0\,\vec{k}, \tag{5}$$

$$\vec{\delta}_w = \cos\varphi\cos\lambda\,\vec{i} + \cos\varphi\sin\lambda\,\vec{j} + \sin\varphi\,\vec{k}. \tag{6}$$

Hence, the relationship between the geocentric coordinate system and the local coordinate system is

$$\begin{pmatrix} n \\ u \\ w \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}, \tag{7}$$

where

$$\Delta x = X - X_K, \qquad \Delta y = Y - Y_K, \qquad \Delta z = Z - Z_K. \tag{8}$$

Here (X_K, Y_K, Z_K) are the geocentric coordinates of the origin K of the local geodetic system. It has to be noted that K is a point belonging to the geoid and specified by the geodetic height h_K with respect to the ellipsoidal point $P(\lambda, \varphi)$. The position vector of K is

$$\vec{\kappa} = \vec{R} + h_k\,\vec{\delta}_w\,. \tag{9}$$

The inverse of the transformation (7) reads

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cos\lambda & -\sin\lambda & \cos\varphi\cos\lambda \\ -\sin\varphi\sin\lambda & \cos\lambda & \cos\varphi\sin\lambda \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix} \begin{pmatrix} n \\ u \\ w \end{pmatrix}. \tag{10}$$

3. Description of the orbit

The pioneer KEPLER states in his theory of orbits, definitely in his first law, that the trajectory of the satellite under the conditions of a normal orbit is an ellipse. As shown in Fig. 2, the six Keplerian elements of the orbital position of a satellite are $(\hat{\Omega}, \hat{\omega}, \hat{\imath}, \hat{a}, \hat{e}, \hat{\Theta})$, where $\hat{\Omega}$ is the right ascension of the ascending node of the orbit. $\hat{\omega}$ is the argument of the perigee, $\hat{\imath}$ the inclination of the orbital plane with respect to the earth's equatorial plane, $\hat{a}$ the major semi-axis of the elliptic orbit, $\hat{e}$ the numerical eccentricity of the orbit and $\hat{\Theta}$ the true anomaly.

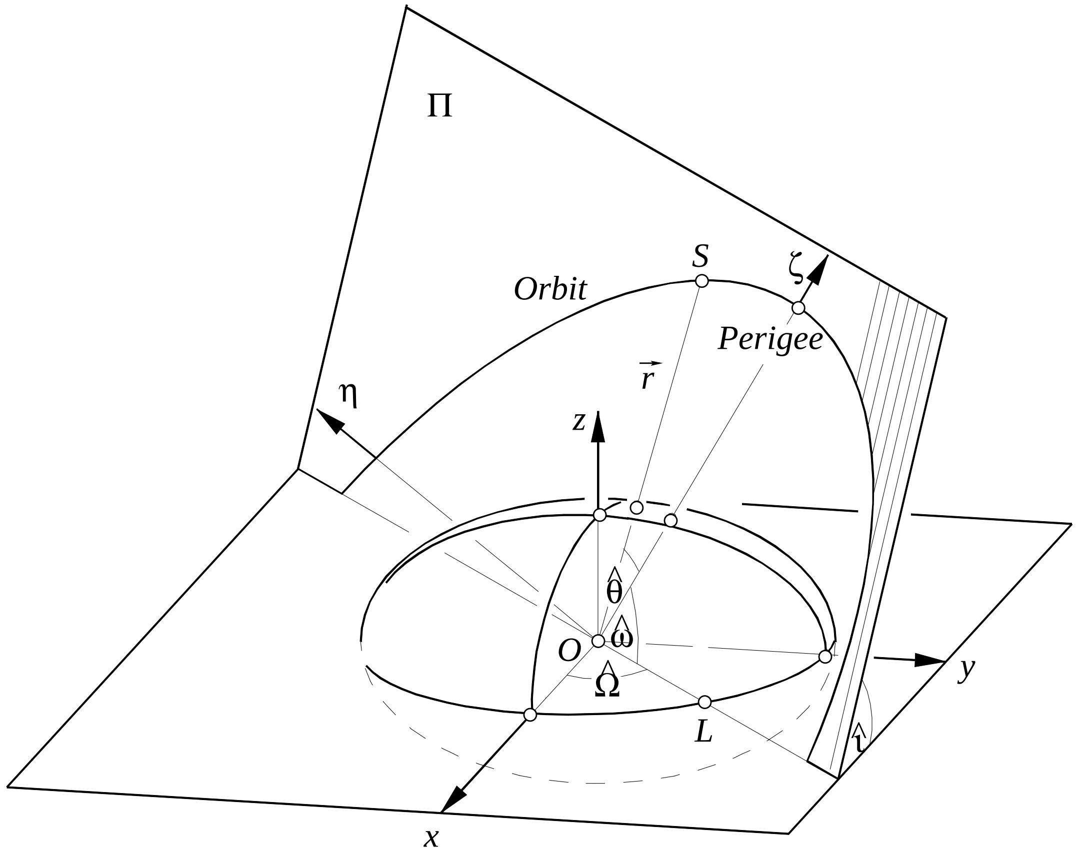

Figure 2:

The first three elements $(\hat{\Omega}, \hat{\omega}, \hat{\imath})$ are used to define the position of the orbital plane Π and to fix the orbit on the true celestial system. The other three elements $(\hat{a}, \hat{e}, \hat{\Theta})$ are used to define the size of the orbit and to fix the satellite on its orbit at a certain epoch. For Kepler's orbit one focus called the occupied focus (or prime focus) is located at the center O of the earth. This center is used as the origin for both the coplanar orbit axes ζ, η in Π and the geocentric reference axes x, y, z. The perigee $\hat{\omega}$ is necessary to define the positive direction of the major axis of the ellipse. On the other hand, the true anomaly $\hat{\Theta}$, as shown in Fig. 2, is the polar angle for the position vector $\vec{r} = \vec{r}(\hat{\Theta})$ of the satellite S with respect to the origin O and the major axis ζ. The norm $r := |\vec{r}|$ is the polar distance from O to S, and the parametric equation of the ellipse in the orbital plane is

$$r = \frac{\rho}{1 + \hat{e}\cos\hat{\Theta}} \tag{11}$$

where ρ is a semi-parameter of the ellipse called (semi-latus rectum) [2].

The two parameters ρ and $\hat{e}$ are specified from the major and minor semi-axes $\hat{a}$ and $\hat{b}$ of the elliptic orbit such that

$$\hat{e} = \sqrt{1 - \frac{\hat{b}^2}{\hat{a}^2}}, \qquad \rho = \frac{\hat{b}^2}{a}. \tag{12}$$

KEPLER's second law means that the geocentric vector $\vec{r}$ sweeps equal areas during equal times [2]. This leads to the possibility of expressing the time of flight of the satellite as a function of the satellite's true anomaly $\hat{\Theta}$ such as [2]

$$t = t(\hat{\Theta}) = \sqrt{\frac{\rho^3}{\mu(1-\hat{e}^2)^3}} \left(\sin^{-1} \frac{\sqrt{1-\hat{e}^2}\,\sin\hat{\Theta}}{1+\hat{e}\cos\hat{\Theta}} - \hat{e}\,\frac{\sqrt{1-\hat{e}^2}\,\sin\hat{\Theta}}{1+\hat{e}\cos\hat{\Theta}} \right) + T \tag{13}$$

where $\mu = 3986005(10)^8$ m^3/sec^2 is the earth gravitational constant.

The magnitude t is the time of the perigee passage; T is the time of the periapsis passage of satellite. Unfortunately, the true anomaly $\hat{\Theta}$ can't be expressed explicitly as a function of time. Therefore, an iterative solution will be imposed for evaluating the proper value of $\hat{\Theta}$.

The planar coordinate system $[O, \zeta, \eta]$ in Π is used herein to specify the position of the satellite in the orbital plane. As mentioned before in Fig. 2, the axes ζ, η are the principal axes of the ellipse. The positive direction of the major axis ζ goes from the center O to the orbit perigee making the angle $\hat{\omega}$ with the ascending line OL.

The relation between the geocentric coordinates X_S, Y_S, Z_S and the coplanar coordinates ζ_S, η_S of the satellite position is [1]

$$\begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \mathrm{a}_{13} \\ \mathrm{a}_{21} & \mathrm{a}_{22} & \mathrm{a}_{23} \\ \mathrm{a}_{31} & \mathrm{a}_{32} & \mathrm{a}_{33} \end{pmatrix}}_{A} \begin{pmatrix} \zeta_S \\ \eta_S \\ 0 \end{pmatrix} \tag{14}$$

where

$$A = \begin{pmatrix} \cos\hat{\Omega}\cos\hat{\omega} - \sin\hat{\Omega}\cos\hat{\imath}\sin\hat{\omega} & -\cos\hat{\Omega}\sin\hat{\omega} - \sin\hat{\Omega}\cos\hat{\imath}\cos\hat{\omega} & \sin\hat{\Omega}\sin\hat{\imath} \\ \sin\hat{\Omega}\cos\hat{\omega} + \cos\hat{\Omega}\cos\hat{\imath}\sin\hat{\omega} & -\sin\hat{\Omega}\sin\hat{\omega} + \cos\hat{\Omega}\cos\hat{\imath}\cos\hat{\omega} & -\cos\hat{\Omega}\sin\hat{\imath} \\ \sin\hat{\imath}\sin\hat{\omega} & \sin\hat{\imath}\cos\hat{\omega} & \cos\hat{\imath} \end{pmatrix}. \tag{15}$$

Applying relation (14) to the position vector $\vec{r} = \vec{r}(X_S, Y_S, Z_S)$ of the satellite and by the inverse relation, one can express the local satellite's coordinates ζ_S, η_S such as

$$\zeta_S = \vec{\Phi}_1 \cdot \vec{r}, \qquad \eta_S = \vec{\Phi}_2 \cdot \vec{r} \tag{16}$$

where $\vec{\Phi}_1$ and $\vec{\Phi}_2$ are the following transformation vectors

$$\begin{aligned} \vec{\Phi}_1 = \; & (\cos\hat{\Omega}\cos\hat{\omega} - \sin\hat{\Omega}\cos\hat{\imath}\sin\hat{\omega})\,\vec{i} + \\ & +(\sin\hat{\Omega}\cos\hat{\omega} + \cos\hat{\Omega}\cos\hat{\imath}\sin\hat{\omega})\,\vec{j} + \sin\hat{\imath}\sin\hat{\omega}\,\vec{k} \end{aligned} \tag{17}$$

and

$$\begin{aligned} \vec{\Phi}_2 = \; & (-\cos\hat{\Omega}\sin\hat{\omega} - \sin\hat{\Omega}\cos\hat{\imath}\cos\hat{\omega})\,\vec{i} + \\ & +(-\sin\hat{\Omega}\sin\hat{\omega} + \cos\hat{\Omega}\cos\hat{\imath}\cos\hat{\omega})\,\vec{j} + \sin\hat{\imath}\cos\hat{\omega}\,\vec{k}. \end{aligned} \tag{18}$$

4. Orientation from the earth station to a satellite

When a satellite S is observed from an earth station K, then due to the rotation of the earth about its principal vertical axis and due to the orbital motion of the satellite the observation lines are generatrices of a ruled surface whose directrix is the elliptic orbit. This surface (see Fig. 3) intersects the orbital plane along the orbit.

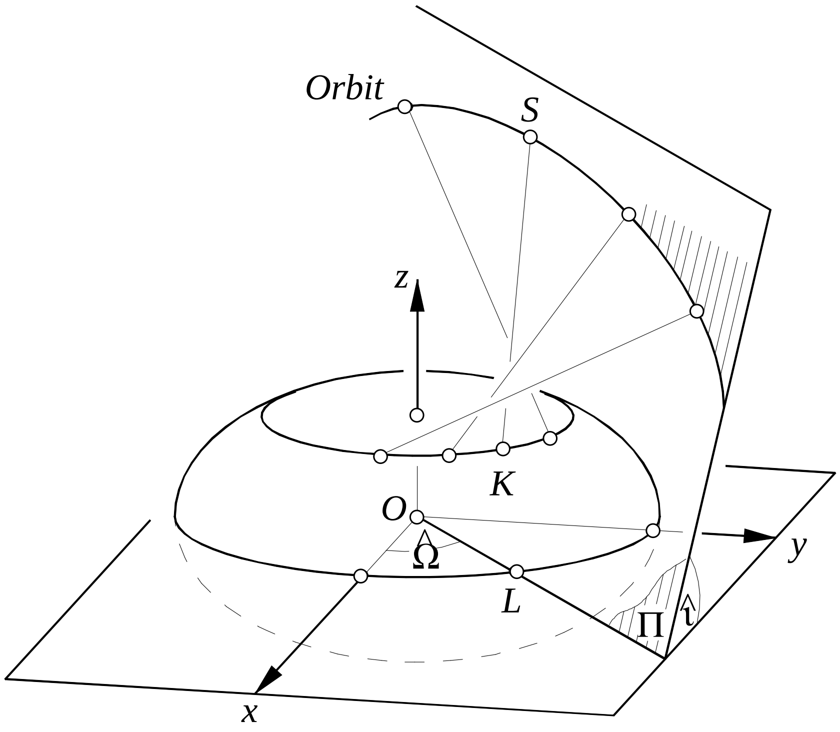

Figure 3:

The generatrix will be specified, as shown in Fig. 4, by the actual position of earth station K and the angles of orientation (azimuth β and zenith γ) from the geodetic earth station K to the satellite position S. For each position of the satellite the two angles β and γ of orientation are practically measured at the earth station. Therefore, the unit vector $\vec{\chi}$ of the observation line can be specified with respect to the local geodetic coordinate system (u, n, w) (see also Fig. 3) as

$$\vec{\chi} = (\cos\gamma\sin\beta,\ \cos\gamma\cos\beta,\ \sin\gamma). \tag{19}$$

According to relation (10), the unit vector $\vec{\chi}$ can be transformed into a new vector $\vec{\varepsilon}$ related to the geocentric coordinate reference system (x, y, z) such that

$$\vec{\varepsilon} = (\varepsilon_X\,\vec{i} + \varepsilon_Y\,\vec{j} + \varepsilon_Z\,\vec{k}) \tag{20}$$

where

$$\begin{pmatrix} \varepsilon_X \\ \varepsilon_Y \\ \varepsilon_Z \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cos\lambda\cos\gamma\sin\beta - \sin\lambda\cos\gamma\cos\beta + \cos\varphi\cos\lambda\sin\gamma \\ -\sin\varphi\sin\lambda\cos\gamma\sin\beta + \cos\lambda\cos\gamma\cos\beta + \cos\varphi\sin\lambda\sin\gamma \\ \cos\varphi\cos\gamma\sin\beta + \sin\varphi\sin\gamma \end{pmatrix}. \tag{21}$$

Then the observation line g can be expressed with respect to the geocentric coordinate system as

$$\vec{g}(q) = \vec{\kappa} + q\,\vec{\varepsilon}, \tag{22}$$

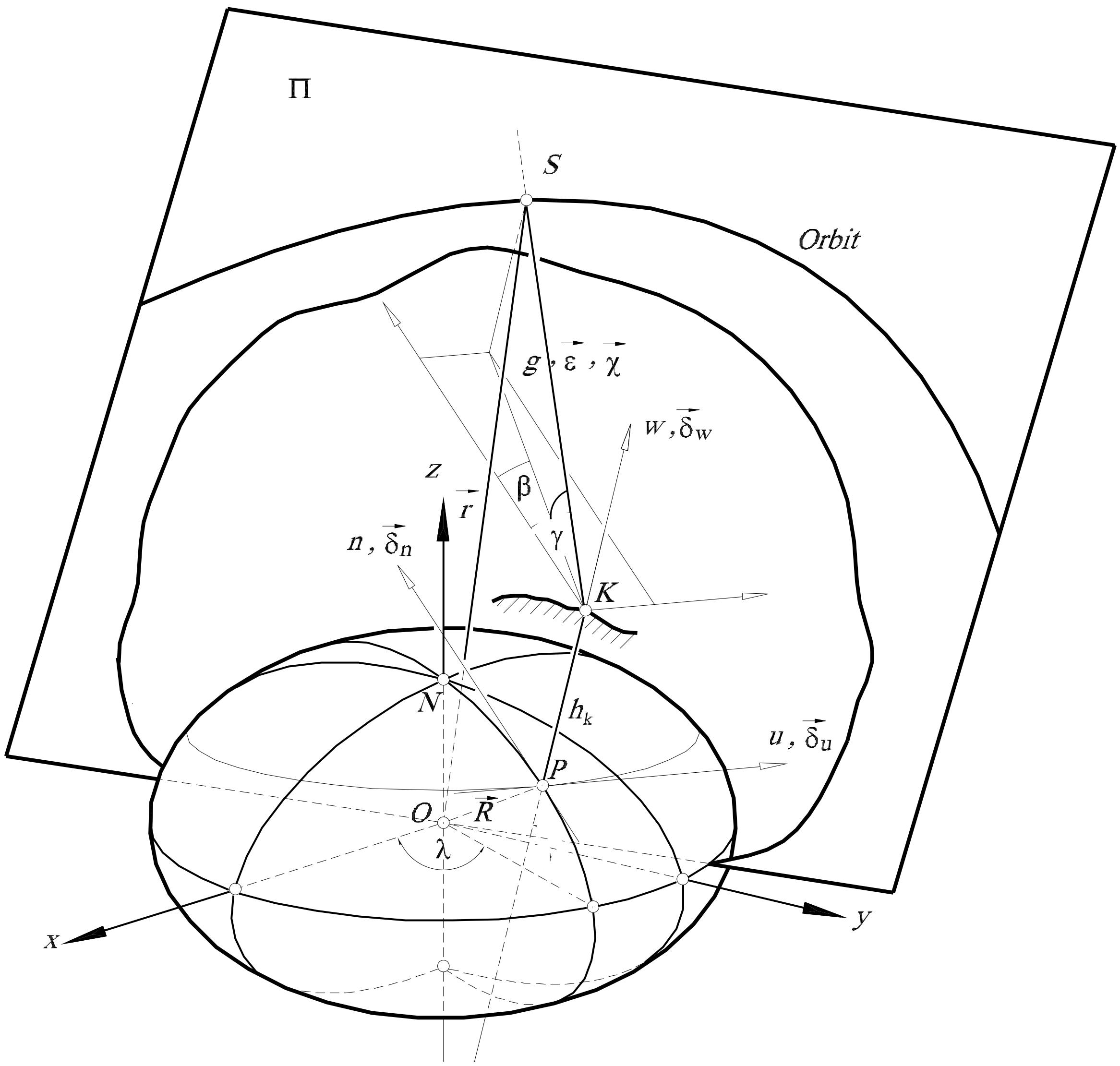

Figure 4:

where q is a scalar expressing the length along the observation line g from the earth station K to any current point on g.

Substitution from equation (9) in equation (22) gives

$$\vec{g}(q) = \vec{R} + h_k \vec{\delta}_w + q\,\vec{\varepsilon}. \tag{23}$$

This vector equation (23) describes the observation line with respect to the geocentric reference system.

5. Geometric representation of satellite positions in the orbital plane

As previously shown in Fig. 2, the orbital plane Π makes an inclination angle $\hat{\imath}$ with the equatorial plane and intersects it along a trace joining the origin O and the ascending node L. This trace OL makes an angle $\hat{\Omega}$ with the positive x-axis. Therefore, the orbital plane Π

can be denoted by $\Pi[O, \hat{\imath}, \hat{\Omega}]$ and represented in the global reference system by the vector dot product

$$\vec{\sigma}_n \cdot \vec{\sigma}_t = 0 \tag{24}$$

where

$$\vec{\sigma}_n = \sin\hat{\Omega}\;\vec{\imath} - \cos\hat{\Omega}\;\vec{\jmath} + \cot\hat{\imath}\;\vec{k}. \tag{25}$$

$\vec{\sigma}_t$ is the position vector of a current point (X, Y, Z) on the orbital plane Π and $\vec{\sigma}_n$ is a normal vector to this plane.

The mathematical solution of the two vector equations (24) and (25) gives a particular value q_S of the parameter q such as

$$q_S = -\frac{\vec{\sigma}_n \cdot (\vec{R} + h_k\vec{\delta}_w)}{\vec{\sigma}_n \cdot \vec{\varepsilon}}. \tag{26}$$

The satellite position $\vec{r}(X_S, Y_S, Z_S)$ is identical with the particular position $\vec{g}(q_S)$ on the observation line provided q_S is the distance from the earth station to the satellite position.

6. Calculation of the parameters of the orbit

The orbit of a satellite in the celestial space is assumed to be an ellipse lying in the orbital plane Π. The satellite's positions are the points of intersection between the orbital plane and the observation lines. As mentioned before, the observation lines are oriented generatrices of a ruled surface. Since one focus of the ellipse is located at the center O of the earth, less than five points will be sufficient to determine the equation of the elliptical orbit.

At first we will focus on the case in which both the orbital plane Π and the ellipsoidal point $P(\lambda, \varphi)$ of the earth station are given at a certain time t. In this case already two generatrices of the ruled surface (i.e., only two observation lines) will be sufficient to determine the parametric equation of the orbit.

Therefore, two pairs of the orientation angles (β_i, γ_i), $i = 1, 2$, have to be measured at times t_i from the time-dependent earth stations $P_i(\lambda_i, \varphi_i)$ to the time-dependent satellite positions $S_i(\zeta_{Si}, \eta_{Si})$. If we pay attention to the fact that the earth performs a full rotation every day, we can achieve the second position of the earth station with respect to the first position such that $\lambda_2 = \lambda_1 + \Delta\lambda$ and $\varphi_2 = \varphi_1$ where the difference $\Delta\lambda$ depends on the time difference $\Delta t = t_2 - t_1$. The time – position relation for a satellite is expressed by equation (13).

In the orbital plane the angle $\hat{\omega}$ defines the perigee and therewith the positive direction of the major axis ζ of the satellite orbit. On the other hand, to represent the orbit in the orbital plane, the two parameters ρ, $\hat{e}$ of equation (11) have to be determined. These parameters can be expressed explicitly as functions of the independent variable $\hat{\omega}$. Therefore this problem can be transformed into an algebraic equation with only one unknown variable $\hat{\omega}$. The equations (1), (16), (19), and (26) hold for every observation line. Specifying them by the known variables $(\lambda_i, \varphi_i, h_{ki}, \beta_i, \gamma_i)$, $i = 1, 2$, one can calculate the two points $S_i(\zeta_{Si}, \eta_{Si})$. Then the two parameters ρ, $\hat{e}$ are received as functions of $\hat{\omega}$ according to

$$\rho = \frac{r_1\zeta_{S2} - r_2\zeta_{S1}}{\zeta_{S2} - \zeta_{S1}}, \qquad \hat{e} = \frac{r_1 - r_2}{\zeta_{S2} - \zeta_{S1}} \tag{27}$$

where

$$\zeta_{Si} = \vec{\Phi}_{1i} \cdot \vec{g}(q_{Si}), \quad \eta_{Si} = \vec{\Phi}_{2i} \cdot \vec{g}(q_{Si}), \quad r_i = \sqrt{\zeta_{Si}^2 + \eta_{Si}^2}\,. \tag{28}$$

The measured time difference Δt during the revolution of the earth, from the first to the second earth station, is equal to the time difference that can be calculated from equation (13) between the two positions of satellite. Thus

$$\begin{aligned}\Delta t = \sqrt{\frac{\rho^3}{\mu(1-\hat{e}^2)^3}} \Bigg[&\left(\sin^{-1} \frac{\sqrt{1-\hat{e}^2}\,\sin\hat{\Theta}_2}{1+\hat{e}\cos\hat{\Theta}_2} - \hat{e}\,\frac{\sqrt{1-\hat{e}^2}\,\sin\hat{\Theta}_2}{1+\hat{e}\cos\hat{\Theta}_2} \right) \\ &- \left(\sin^{-1} \frac{\sqrt{1-\hat{e}^2}\,\sin\hat{\Theta}_1}{1+\hat{e}\cos\hat{\Theta}_1} - \hat{e}\,\frac{\sqrt{1-\hat{e}^2}\,\sin\hat{\Theta}_1}{1+\hat{e}\cos\hat{\Theta}_1} \right) \Bigg]\end{aligned} \tag{29}$$

where

$$\hat{\Theta}_i = \cos^{-1} \frac{\zeta_{Si}}{r_i}\,. \tag{30}$$

Substitution of (27) and (28) in equation (29) gives an algebraic equation for one variable $\hat{\omega}$. By an iterative solution of this equation a proper value of $\hat{\omega}$ can be re-substituted in equations (27), (28) to get the required parameters ρ, $\hat{e}$ of the elliptical orbit.

The general case, when the orbital plane is unknown for the observer, leads to three essential unknowns, namely $\hat{\Omega}$, $\hat{i}$, $\hat{\omega}$. Thus the solution needs three simultaneous equations containing these unknowns. Evidently, one additional generatrix of the ruled surface will be necessary to make the problem numerically solvable. The three generatrices (i.e., three observation lines) intersect the orbital plane at three positions belonging to the presumed elliptical orbit. Due to the computability of these three positions on the orbit we can achieve a system of three algebraic equations consisting of equation (29) and of

$$\frac{r_1\zeta_{S2} - r_2\zeta_{S1}}{\zeta_{S2} - \zeta_{S1}} = \frac{r_2\zeta_{S3} - r_3\zeta_{S2}}{\zeta_{S3} - \zeta_{S2}}\,, \qquad \frac{r_1 - r_2}{\zeta_{S2} - \zeta_{S1}} = \frac{r_2 - r_3}{\zeta_{S3} - \zeta_{S2}}\,. \tag{31}$$

7. Conclusion

A geometric method for reconstructing the orbit of a moving satellite has been proposed herein. The path of the satellite is represented as a non prescribed directrix of a ruled surface generated by a set of observation lines looking from a time-dependent earth station to the moving satellite. Generatrices of this ruled surface are obtained from measuring orientation angles and the time of flight of the satellite. The time of revolution of the earth station is utilized concurrently with the time passage of the satellite to fix the orbital position of satellite with respect to the position of the earth station. The presented reconstruction of the normal orbit is valid not only in the case of a known orbital plane but also for an unknown orbital plane. This method is useful for predicting the actual position of a satellite affected by disturbing forces such as the attraction of the sun and moon and solar radiation impulses.

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