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Five Spheres in Mutual Contact

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Abstract. With respect to a sphere having centre O and radius r, two points P and P' are said to be inverse if P' lies on the diameter OP and

$$OP \times OP' = r^2.$$

The transformation $P \to P'$ is called inversion. To be able to say that every point has an inverse, we add to the ordinary Euclidean space an extra point O' called *the point at infinity*. We thus create *inversive space*. Since circles and spheres through O invert into lines and planes, lines and planes are simply circles and spheres through the point at infinity. Two tangent spheres have just one common point: their point of contact. If this point is O, the two spheres, having no other common point, invert into two parallel planes. Consider 5 mutually tangent spheres having $\binom{5}{2} = 10$ distinct points of contact. If O is one of these ten points, we obtain by inversion two parallel planes with three ordinary spheres sandwiched between them. Since these three are congruent and mutually tangent, their centres are the vertices of an equilateral triangle. Analogously, if four congruent spheres are mutually tangent, their centres are the vertices of a regular tetrahedron. A fifth sphere, tangent to all these four, may be either a larger sphere enveloping them or a small one in the middle of the tetrahedral cluster. In this article it will be shown that here are fifteen spheres, each passing through six of the ten points of contact of the five mutually tangent spheres.

Key Words: inversive space, orthogonality, point at infinity, regular octahedron and tetrahedron, and triangular prism.

MSC 2000: 51M04, 51B10

1. Inversive Geometry

One of the most intriguing transformations of the Euclidean plane (or space) into itself is inversion in a circle (or sphere). Although inversion is usually defined in terms of distance (Coxeter [1], Chapter 6) there is an essentially simpler definition in the spirit of inversive

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geometry. This is the geometry of points and circles (and spheres) in which lines (and planes) are regarded as special cases of circles (and spheres). Then, instead of saying that circles invert into circles or lines, and lines into lines or circles, we say simply that circles invert into circles or that, from the standpoint of inversive geometry all circles are alike (and analogously all spheres are alike). On the other hand, *pairs* of circles (or spheres) may be of three kinds: *intersecting* or *tangent* or *disjoint*.

In the first case we have two circles (or spheres) with two common points (or a common circle), intersecting at various angles. If the angle is a right angle the circles (or spheres) are said to be *orthogonal*. For instance, a circle and a diameter are a pair of orthogonal "circles".

In the second case we have two tangent circles (or spheres) with just one common point. We observe that three mutually tangent circles (or spheres) are orthogonal to the circle through the three points of contact, as in Fig. 1. This observation provides a non-metrical definition for orthogonality. Two circles (or spheres) are orthogonal if one of them belongs to a triad of mutually tangent circles (or spheres) while the other passes through the three points of mutual contact (Ewald [6]).

In the third case we have two disjoint circles (or spheres) having no common point, separated by various *inversive distances* (Coxeter, Greitzer [4], pp. 123–131). For instance, two concentric circles (or spheres), with radii r_1 and r_2, are separated by the inversive distance $|\log \frac{r_1}{r_2}|$. This use of the word "distance" is justified by the observation that inversive distance, so defined, is "additive". The radii r_1, r_2, r_3 of three concentric circles (or spheres), nested so that $r_1 > r_2 > r_3$, are related by the identity

$$\log \frac{r_1}{r_2} + \log \frac{r_2}{r_3} = \log \frac{r_1}{r_3}.$$

Since any two disjoint circles can be inverted into concentric circles (Coxeter [1], p. 87), the inversive distance between them can reasonably be defined as the natural logarithm of the ratio of the radii (the longer to the shorter) of any two concentric circles into which the given circles can be inverted. This also applies to spheres. Finally, there is a non-metrical definition for inversion. With respect to a circle (or sphere) ω, two points P and P' are *inverse* if they both lie on two circles (or three spheres) orthogonal to ω.

We see now that inversive geometry deals with incidence of points and circles and spheres, angles of intersection, tangency and inversive distance, without necessarily making any mention of Euclidean distance or length. In this spirit, any set of mutually tangent circles is just like any other such set; thus Figures 1 and 2 are inversively equivalent. The following is a typical theorem of the inversive plane.

2. Beecroft's Theorem

Any four mutually tangent circles that have six distinct points of contact determine another such tetrad of mutually tangent circles having the same six points of contact. Each circle of either tetrad is orthogonal to three circles of the other.

For each one of the four given circles, the remaining three, as we have seen, determine an orthogonal circle through the three points of contact. Considering the four circles in turn, we obtain four new circles which are in mutual contact at the same four points. This state of affairs is well illustrated by the thick and thin parts of Fig. 1.

Another way to prove Beecroft's theorem is to invert the mutually tangent circles in a circle whose centre O is any one of the six points of contact. Since the inversion takes O to

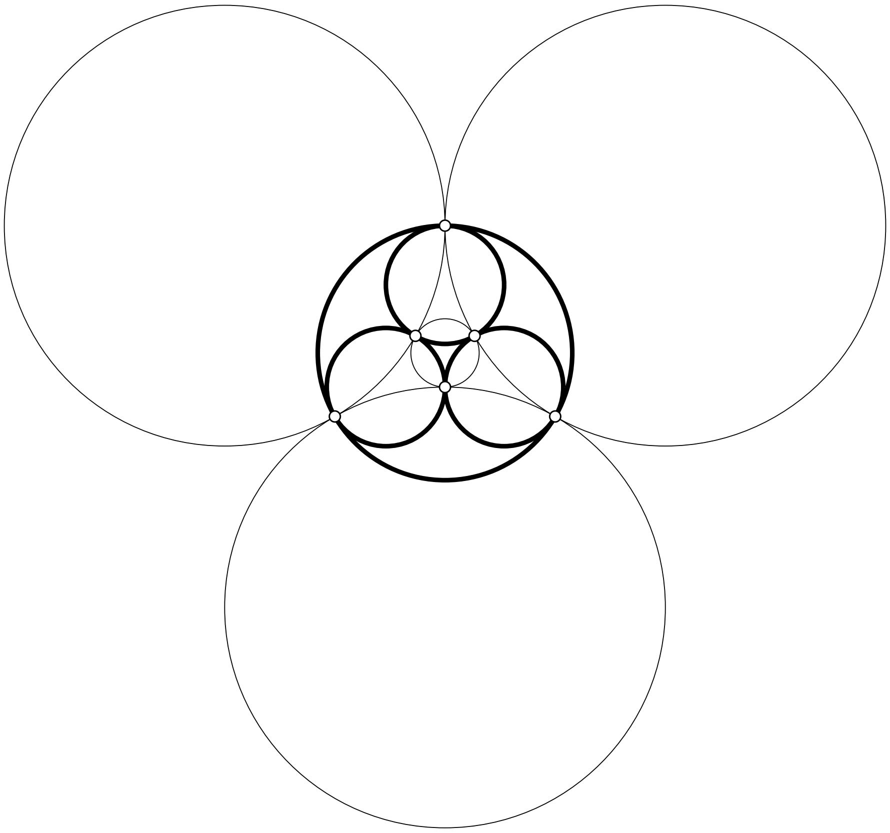

Figure 1: A trigonal view of BEECROFT's theorem

the point at infinity, yielding two parallel lines, the remaining two circles of the tetrad become congruent circles sandwiched between the parallel lines, as in the thick or thin part of Fig. 2. Here we see two pairs of parallel lines extending the sides of a square, and four circles having the sides as diameters. The six points of contact consist of the four vertices of the square, its centre, and the point at infinity.

BEECROFT's theorem plays an essential role in one of the neatest proofs for the Descartes circle theorem:

$$2\sum \varepsilon_i^2 = \left(\sum \varepsilon_i\right)^2,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are the reciprocals of the radii of four mutually tangent circles (COXETER [1], pp. vii, 14–16).

3. Stereographic Projection

Stereographic projection is a geographic mapping of a sphere σ' (the globe) from the north pole N onto the tangent plane σ at the south pole S, using straight lines through N (COXETER [1], p. 93). Each point P' on σ' is projected from N into a point P on the tangent plane at S. Thus P' is the second intersection of the line NP with the sphere σ'. This kind of projection may be regarded as inversion in the sphere ω with centre N and Radius NS. For ω inverts the plane σ into the sphere σ' (which passes through the centre of ω).

In the spirit of inversive geometry this projection or inversion is a one-to-one correspondence because the image of N is the point at infinity on σ.

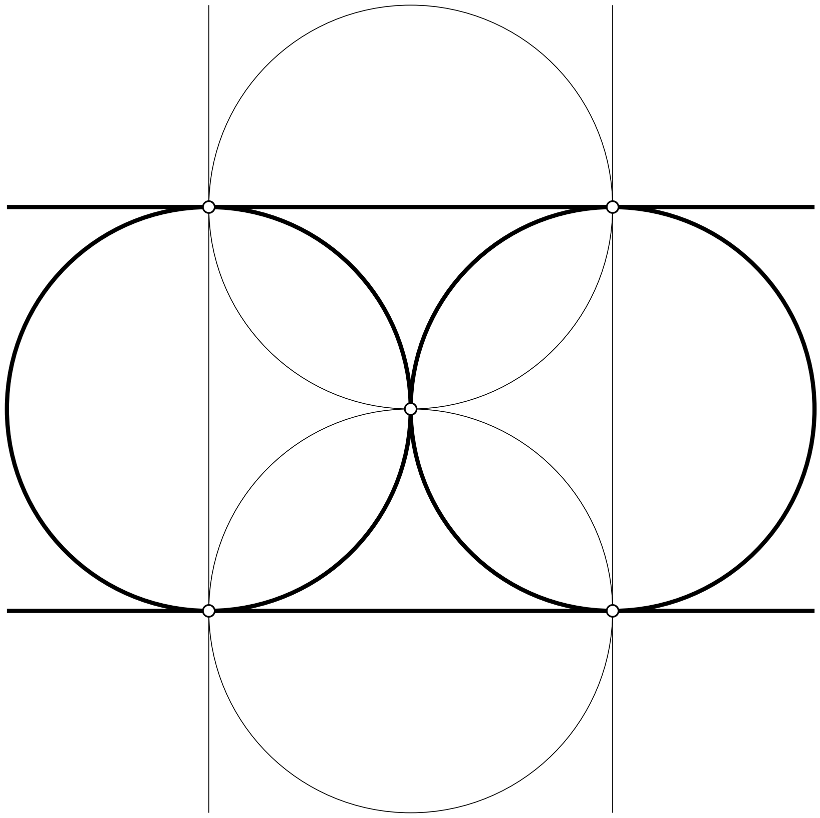

Figure 2: A tetragonal view of BEECROFT's theorem

The sphere σ' can be decomposed into eight octants (trirectangular spherical triangles which are spherical versions of the eight faces of a regular octahedron) by two perpendicular planes through NS and the plane through O parallel to σ, that is, by the three Cartesian coordinate planes. Stereographic projection from N into σ yields Fig. 2. If the three perpendicular planes through O are rotated to new positions, so that N becomes the centre of one octant, we obtain instead Fig. 1. (The thick and thin circles and lines come from the circumcircles of alternate faces of the octahedron.)

4. ROBINSON's ***Firmament***

A few years ago, the head of Toronto's Fields Institute introduced me to the Australian artist John Robinson who presented to the Institute a stainless steel version of his abstract sculpture *Intuition*, consisting of three "hollow triangles", interlocked in the manner of Boromean rings. During lunch I asked him to consider making a new abstract sculpture consisting of five balls whose diameters are in geometric progression with ratio

$$r = \tfrac{1}{2}\left(\sqrt{2} + 1 + \sqrt{2\sqrt{2} - 1}\right) \approx 1.88$$

because this particular ratio would allow them to be mutually tangent (COXETER [3]). He agreed to think about it. A year later he gave me, for my ninetieth birthday, the sculpture *Firmament* in which the four largest balls are made of wood while the smallest is of steel, barely visible in the middle (see Fig. 3).

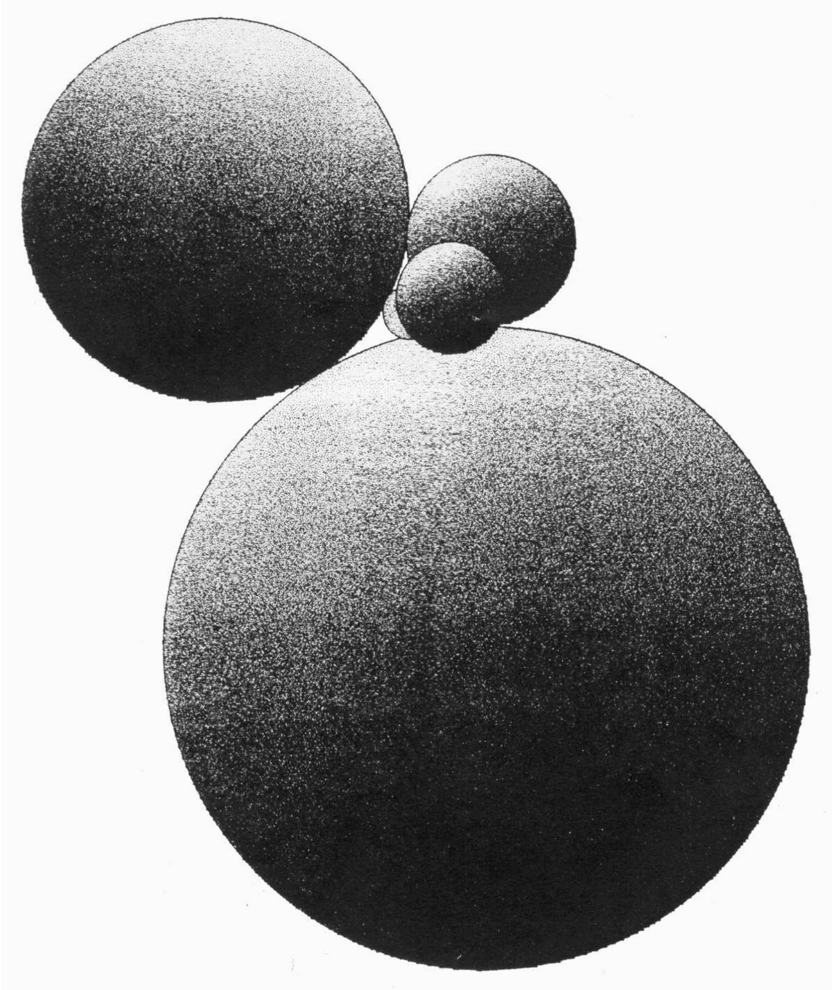

Figure 3: Five spheres with radii 1, r, r^2, r^3, r^4

This arrangement of five balls is interesting because it illustrates the inversively invariant concept of five mutually tangent spheres whose ten points of contact will be seen to lie by sixes on fifteen spheres, one which reduces to a plane when the diameters (or radii) are in geometric progression, as in *Firmament*.

Any five spheres in mutual contact can be inverted (using a sphere whose centre is any one of the ten points of contact) into two parallel planes with three congruent spheres sandwiched between them. Therefore all mutually tangent pentads of spheres are inversively equivalent to that special set of 2+3 (and hence to one another). Thus we lose no generality by concentrating on a special case.

4.1. The case when two of the five are parallel planes

Since the three spheres tangent to the parallel planes are also tangent to one another, their centres form an equilateral triangle and the ten points of mutual contact consist of three belonging to the congrent spheres, the one point at infinity, and the six points of contact of the three congruent spheres with the two parallel planes. These last six clearly lie on a sphere: they are the vertices of a uniform triangular prism. Hence

For each of the ten pairs of any five mutually tangent spheres, their six points of contact with *the remaining three spheres all lie on a sphere* (Coxeter [3], p. 46).

4.2. The case when four of the five are congrent

For any four congruent spheres in mutual contact, their centres are the vertices of a regular tetrahedron whose edges are bisected by the six points of mutual contact. These six all lie on a sphere, namely the *intersphere* (Cundy, Rollett [5], p. 79) which touches the six edges of the tetrahedron at their midpoints. Hence

For each of five mutually tangent spheres, there is a sphere passing through the six points of mutual contact of the remaining four.

4.3. Conclusion

Combining the results of sections 4.1. and 4.2., we obtain our final result:

For any five mutually tangent spheres, there are $10 + 5$ *spheres each passing through six of the ten points of mutual contact.*

Acknowledgement

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