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Visualisation of Configuration Spaces of Polygonal Linkages

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Abstract. Using *Morse theory*, the configuration space of a 4-gonal (respectively 5-gonal) linkage in the plane or on the unit-sphere can be visualised as a curve in $[0, 2\pi] \times [0, 2\pi]$ (respectively as a surface in $[0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$).

Key Words: Configuration space, polygonal linkages, Morse theory

MSC 2000: 53A17

1. Introduction and theory

Configuration spaces of mechanisms (mechanical linkages, bar mechanisms, robot arms) have been studied for centuries as one of the basic topics in kinematics. Therefore a lot of exciting results are known today (see the References for further information). Configuration spaces are often rather difficult to handle. Visualising them is an essential tool to study their geometry.

A mechanism in some metric space M that is built up from rigid bars joined along flexible links in a cyclic way is called a *polygonal linkage* $\mathcal{P}$. An n-gonal linkage $\mathcal{P}$ is defined by the n-tuple $(l_1, \ldots, l_n)$ of side-lengths. A *realisation* ξ of $\mathcal{P}$ in M is an n-tuple of points $(P_1, \ldots, P_n) \in M \times \cdots \times M$ such that $\operatorname{dist}(P_{i+1}, P_i) = l_i$ with all indices modulo n.

The configuration space of the n-gonal linkage $\mathcal{P}$ is the totality of all its admissible realisations in the metric space M, modulo the group $\mathrm{Iso}^+(M)$ which consists of all proper isometries of M, i.e.

$$[\mathcal{P}]_M = \{\xi \text{ realisation of } \mathcal{P} \text{ in } M\}/\mathrm{Iso}^+(M).$$

The configuration space is considered with the topology induced by the metric space M. The most efficient method to visualise the configuration space of 4- and 5-gonal linkages uses *Morse theory*.

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1.1. What is Morse theory?

Morse theory is a powerful tool in the study of smooth compact manifolds, which will be denoted here by V. We analyse level sets f^{-1} of a function $f : V \to \mathbb{R}$ having only non-degenerate critical points. Such a function is called a *Morse function*. The decomposition of V into these level sets contains an amazing amount of information about the topology of V. At each critical point a special kind of chart is constructed, making a Morse function look like a non-degenerate quadratic form. The index of this form is called the index of the critical point. If the set $f^{-1}[a,b]$ does not contain a critical point then it is shown under mild restrictions that $f^{-1}[a,b]$ is homeomorphic to $f^{-1}(a) \times [a,b]$. Otherwise, suppose that $f^{-1}[a,b]$ contains exactly one critical point, of index k. Then it turns out that up to a homotopy equivalence, $f^{-1}[a,b]$ is obtained from $f^{-1}(a)$ by attaching a k-cell. This leads directly to the construction of a CW-complex homotopy equivalent to V which has one k-cell for each critical point of index k. This technique is called *Morse surgery*. For more precise information the reader should consult [9].

1.2. Configuration spaces of polygonal linkages in the plane

The configuration space of a polygonal linkage $\mathcal{P}$ in the Euclidean plane $\mathbb{R}^2$ can be considered as

$$[\mathcal{P}]_{\mathbb{R}^2} = \{\xi \text{ realisation of } \mathcal{P} \text{ in } \mathbb{R}^2 \mid P_1 = (0,0) \text{ and } P_2 = (l_1, 0)\}.$$

The fixing of one bar in the plane avoids the quotient in the definition of the configuration space. We parameterise the last vertex P_n as follows, cf. Fig. 1,

$$P_n(\alpha_2, \ldots, \alpha_{n-1}) = \begin{pmatrix} l_1 + l_2 \cos\alpha_2 + \cdots + l_{n-1}\cos\alpha_{n-1} \\ l_2 \sin\alpha_2 + \cdots + l_{n-1}\sin\alpha_{n-1} \end{pmatrix}.$$

The function $D_{\mathbb{R}^2} : T^{n-2} \longrightarrow \mathbb{R}$ given by

$$D_{\mathbb{R}^2}(\alpha_2, \ldots, \alpha_{n-1}) = -\mathrm{dist}(P_1, P_n)^2 = -|P_1 - P_n|^2$$

is then a *Morse function*. To prove this we consider the *Jacobian* of $D_{\mathbb{R}^2}$ and find that the critical points have an easy geometric interpretation. All the angles α_i for $i \in \{2, \ldots, n-1\}$ must be 0 or π. In other words the critical points correspond to the degenerate realisations of the polygonal linkage, i.e. the realisations aligned on the x-axes.

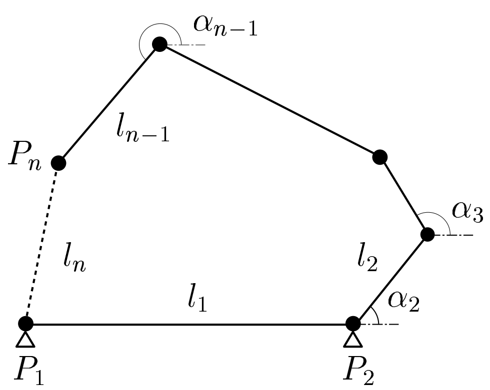


Figure 1: n-gonal linkage in the plane

To calculate the indices of the Morse function we consider the *Hessian* of $D_{\mathbb{R}^2}$. In our situation they have a very simple geometric interpretation: the index of a critical point is the number of bars which "point to the left", i.e. every aligned realisation is characterised by an $(n-1)$-tuple $(\varepsilon_2, \ldots, \varepsilon_n)$ with $\varepsilon_i \in \{-1, 1\}$, such that $l_1 + \varepsilon_2 l_2 + \cdots + \varepsilon_n l_n = 0$, and then the index of the function $D_{\mathbb{R}^2}$ is exactly the number of -1 in the set $\{\varepsilon_2, \ldots, \varepsilon_{n-1}\}$. The configuration space $[\mathcal{P}]_{\mathbb{R}^2}$ is the level surface $D_{\mathbb{R}^2}^{-1}(-l_n^2)$ of the Morse function $D_{\mathbb{R}^2}$, cf. [5]. Using *Morse surgery*, cf. [9], it is possible to determine the configuration space of a given polygonal linkage $\mathcal{P} = \mathcal{P}(l_1, \ldots, l_n)$: We forget the condition $|P_1 - P_n| = l_n$ and consider the function $D_{\mathbb{R}^2} = -|P_1 - P_n|^2$ on T^{n-2}. The configuration space $D_{\mathbb{R}^2}^{-1}(-(l_1 + \cdots + l_{n-1})^2)$ consists of a single point. The index of $D_{\mathbb{R}^2}$ at this point is 0. One then decreases the distance between P_1 and P_n up to $|P_1 - P_n| = l_n$. If no critical points appear, then the type of the configuration space does not change. On the other hand, if we pass through a critical point, the type of the configuration space changes depending on the index of the critical point. We have to apply a reconstruction technique, *Morse surgery*. This leads in the following result which can be found in [5] and [6]:

Theorem 1.1 *The configuration space of an n-gonal linkage $\mathcal{P}$ is a smooth oriented compact manifold if and only if no aligned realisation of $\mathcal{P}$ exists. Otherwise the neighbourhood of an aligned realisation $\xi \in [\mathcal{P}]_{\mathbb{R}^2}$ is a cone defined by $\sum_{i=2}^{n-1} \varepsilon_i x_i^2$, where $(\varepsilon_2, \ldots, \varepsilon_{n-1})$ characterises the aligned realisation ξ.*

This result allows us to determine the configuration space explicitly in the case of 4- and 5-gonal linkages. The case of 4-gonal linkages is easy, since there are two possible types of non-singular configuration spaces: let l (respectively s) denote the length of the longest (respectively shortest) bar, and p and q the lengths of the intermediate bars. The configuration space is a circle S^1 if $s + l > p + q$, and the disjoint union of two circles if $s + l < p + q$. In case of equality the configuration space is singular. These inequalities are known as the *Grashof conditions*, cf. [2]. For 5-gonal linkages the following surfaces can occur as non-singular configuration space: the sphere S^2, the torus T^2, the disjoint union of two tori, and the surfaces of genus 2, 3 and 4.

Remark 1.1 *In the higher dimensional case the surgery technique does not give a precise description of $[\mathcal{P}]_{\mathbb{R}^2}$ because we do not know where the reconstruction has to be done. Even for 5-gonal linkages different reconstructions of the same index can produce different manifolds.*

1.3. Configuration spaces of polygonal linkage on the unit-sphere

The idea is completely analogue to the flat case. A bar of length $l \leq \pi$ on the unit-sphere centred at the origin in $\mathbb{R}^3$ is a part of a great circle with length equal to l. Fix the first bar of the polygonal linkage on the equator such that $P_1 = (1, 0, 0)$ and $P_2 = (\cos l_1, \sin l_1, 0)$. Parameterise P_n in function of the angles $\alpha_2, \ldots, \alpha_{n-1}$ where α_i is the angle between the great circles which pass through P_i. The function $D_{\mathbb{S}^2} : T^{n-2} \longrightarrow \mathbb{R}$ given by

$$D_{\mathbb{S}^2}(\alpha_2, \ldots, \alpha_{n-1}) = \cos \operatorname{dist}(P_1, P_n) = \langle P_1, P_n \rangle$$

is then a Morse function. Its critical points correspond to the degenerate realisations of the polygonal linkage, i.e. the realisations aligned on the equator of the unit-sphere.

The configuration space $[\mathcal{P}]_{\mathbb{S}^2}$ of an n-gonal linkage $\mathcal{P}$ is the level surface $D_{\mathbb{S}^2}^{-1}(\cos(l_n))$ of the Morse function $D_{\mathbb{S}^2}$, cf. [7].

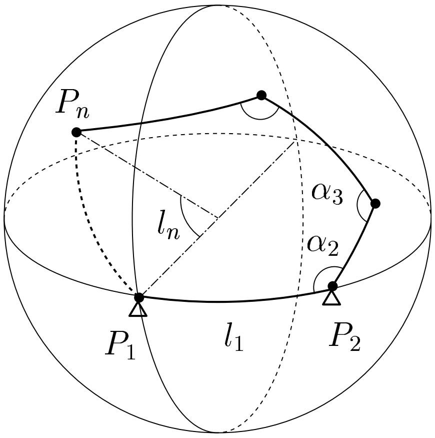


Figure 2: n-gonal linkage on the unit-sphere

If the perimeter $\sum_{i=1}^{n} l_i$ of the polygonal linkage $\mathcal{P} = \mathcal{P}(l_1, \ldots, l_n)$ is smaller than 2π (which is the perimeter of the unit-sphere), then the configuration space of a spherical polygonal linkage is diffeomorphic to the corresponding Euclidean configuration space, i.e. $[\mathcal{P}]_{\mathbb{S}^2} \simeq [\mathcal{P}]_{\mathbb{R}^2}$, cf. [7].

2. Visualisation of configuration spaces

Since $(\alpha_2, \ldots, \alpha_{n-1}) \in [0, 2\pi] \times \cdots \times [0, 2\pi] \subset \mathbb{R}^{n-2}$, the level surface D^{-1} is a subset in the fundamental domain of the $(n-2)$-torus. Hence the configuration space of a 4-gonal (respectively 5-gonal) linkage can be visualised as a curve in $[0, 2\pi] \times [0, 2\pi]$ (respectively as a surface in $[0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$).

In the planar case, we consider examples of 4- and 5-gonal linkages with one bar of variable length. For each of these examples, we give illustrations of sequences of the configuration spaces obtained by increasing the value of the variable length. This gives a way to understand how the configuration space changes if the length of one bar increases and passes through critical points.

In the spherical case, we show sequences of configuration spaces for some examples of 4- and 5-gonal linkages obtained by increasing the perimeter, whilst keeping the ratios of the lengths constant. Like this we can visualise what happens when the perimeter passes through the critical value 2π.

The visualisation is done using *Maple V Release 5.* For $n = 4$ (respectively $n = 5$) we use the command `implicitplot` (respectively `implicitplot3d`) to plot the configuration space in the fundamental domain of the $(n-2)$-torus, cf. Table 1.

```
> with(plots):
> L1 := 1: L2 := 1: L3 := 1: L4 := 1: L5 := 1:
> implicitplot3d({ (L1+L2*cos(x)+L3*cos(y)+L4*cos(z))^2+
                   (   L2*sin(x)+L3*sin(y)+L4*sin(z))^2-L5^2 = 0 },
                   x = 0..2*Pi, y = 0..2*Pi, z = 0..2*Pi,
                   axes = BOXED, scaling = CONSTRAINED,
                   orientation = [35,80], grid = [20,20,20],
                   style = PATCHNOGRID, lightmodel = light3);
```

Table 1: *Maple* code used to plot Fig. 7 (e)

2.1. 4-gonal linkage in the plane

Let us consider two examples of sequences of configuration spaces for planar 4-gonal linkages as shown in Fig. 3.

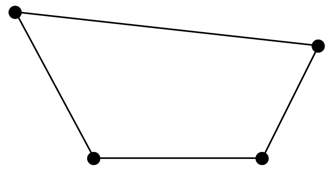


Figure 3: 4-gonal linkage

2.1.1. 4-gonal linkage $\mathcal{P}(1,1,1,1)$

Starting with $l_4 = 3$, in which case the configuration space is just a point, we decrease l_4 from 3 to 1.5 and 1.1. Since $l_4 = 3$ is a critical value of index zero, we get an S^1 in Fig. 4 (a) and (b). The interesting and most complex configuration space is that the equilateral 4-gonal linkage, which has exactly three singular points: they correspond to the three aligned configurations. At each branching point, essentially four different configuration classes can be reached. The space $[\mathcal{P}(1,1,1,1)]_{\mathbb{R}^2}$ consists of three S^1 each of which is pasted to the two others at a point. It is visualised in the middle of the sequence of Fig. 4. One of the two intersecting lines correspond to the rotation of the folded links about one base point. As we decrease l_4 from 1 to 0.9 and 0.5, the three singular points disappear simultaneously, since no aligned realisation is possible any more. We therefore obtain two disjoint S^1 which get smaller as l_4 gets shorter. This sequence ends in two distinct points, which is the configuration space of an equilateral triangle.

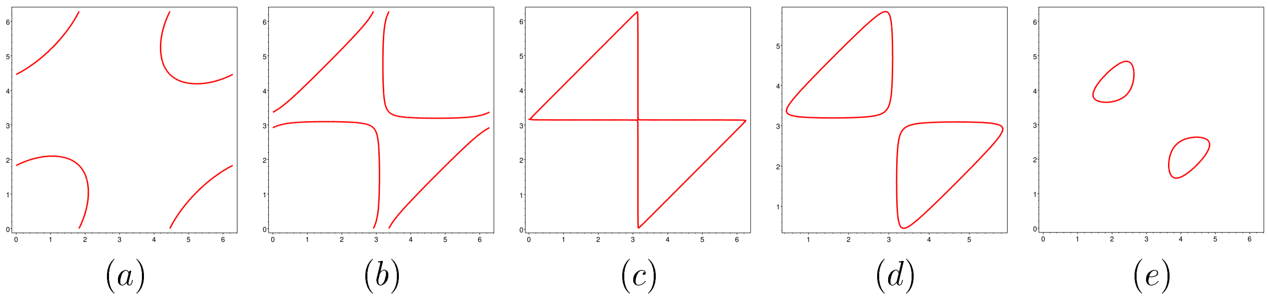
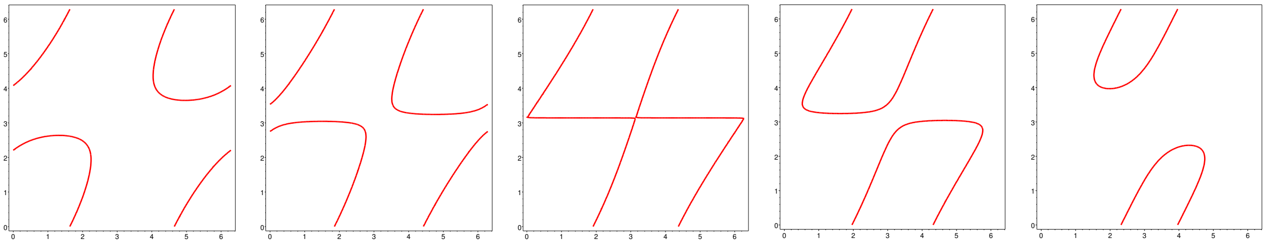


Figure 4: 4-gonal linkage $\mathcal{P}(1,1,1,l_4)$ with $l_4 \in \{1.5, 1.1, 1, 0.9, 0.5\}$

2.1.2. 4-gonal linkage $\mathcal{P}(1,3,1,3)$

In the next sequence we decrease l_4 from 5 to 3.5 and 3.1. We begin with a single point as the configuration space, which blows up to an S^1, cf. Fig. 5 (a) and (b). Then for $l_4 = 3$ the quadrilateral can be aligned in two different ways, corresponding to the two singular points visible in Fig. 5 (c). The space $[\mathcal{P}(1,3,1,3)]_{\mathbb{R}^2}$ consists of two S^1 pasted together at two points. At each branching point, essentially four different configuration classes can be reached. As we decrease l_4 from 3 to 2.9 and 2.5, the two singular points disappear simultaneously. We again get a single S^1 which gets smaller as l_4 gets shorter. This sequence ends in a single point, when $l_4 = 1$, and is the configuration space of the degenerate quadrilateral of side-length one, three, one and one.



(a) (b) (c) (d) (e)

Figure 5: 4-gonal linkage $\mathcal{P}(1, 3, 1, l_4)$ with $l_4 \in \{3.5, 3.1, 3, 2.9, 2.2\}$

2.2. 5-gonal linkage in the plane

Let us consider two examples of sequences of configuration spaces for planar 5-gonal linkages, as shown in Fig. 6.

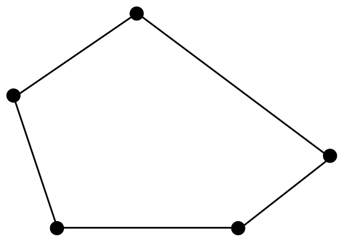


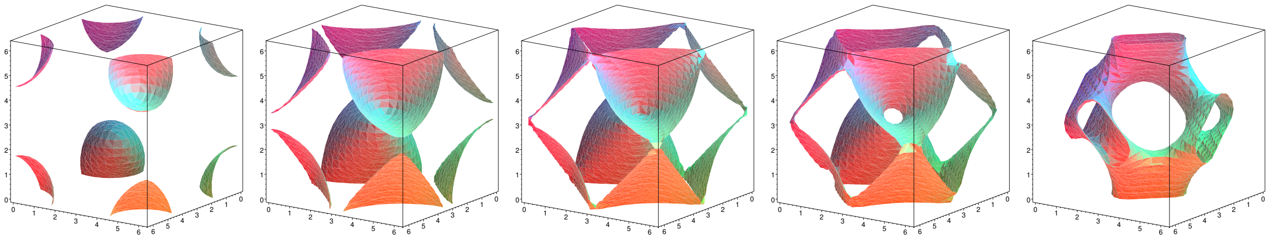
Figure 6: 5-gonal linkage

2.2.1. 5-gonal linkage $\mathcal{P}(1, 1, 1, 1, 1)$

Starting with $l_5 = 4$, in which case the configuration space is just a point, we decrease l_5 from 4 to 3. Since $l_5 = 4$ is a critical value of index zero, we get an S^2, cf. Fig. 7 (a) and (b). Further decreasing l_5 from 2.1 to 1.9, we pass the critical value $l_5 = 2$. The configuration space $[\mathcal{P}(1, 1, 1, 1, 2)]_{\mathbb{R}^2}$ is a sphere with four pairs of distinct points pairwise identified, i.e. it has locally four double-cones. Decreasing l_5 from 2 to 1.9, the four double-cones blow up to cylinders. The configuration space of the equilateral 5-gonal linkage is a surface of genus four, visualised in Fig. 7 (e). It is non-singular because the equilateral 5-gonal linkage can not be aligned. Further decreasing l_5 from 1 to 0 we do not pass any more critical values and so the type of configuration space does not change. At the limiting value $l_5 = 0$, it collapses to the configuration space of the planar equilateral 4-gonal linkage, cf. Fig. 4 (c).

2.2.2. 5-gonal linkage $\mathcal{P}(3, 1, 3, 1, 3)$

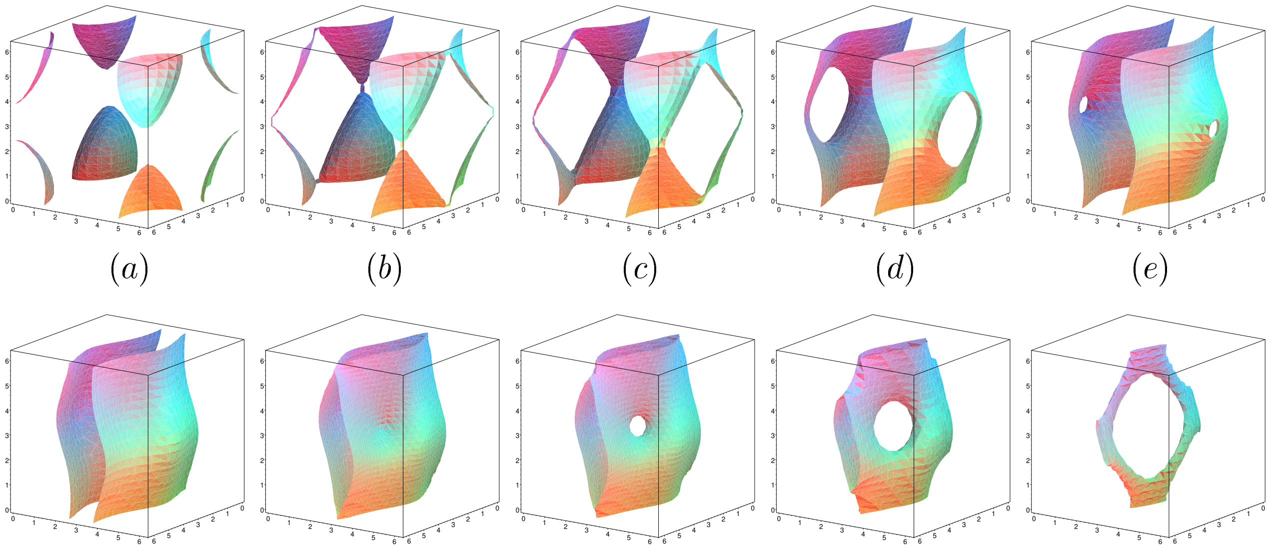
Starting with $l_5 = 8$, in which case the configuration space is a point, we decrease l_5 to 6, which is the next critical value. At first the corresponding configuration space is an S^2, cf. Fig. 8 (a), and when $l_5 = 6$ the configuration space is a sphere which has locally two double-cones, cf. Fig. 8 (b). By decreasing l_5 from 6 to 5.9 we blow up the two double-cones to cylinders, obtaining a surface of genus two, cf. Figs. 8 (c), (d) and (e). Further decreasing l_5 to 3 we



(a) (b) (c) (d) (e)

Figure 7: 5-gonal linkage $\mathcal{P}(1,1,1,1,l_5)$ with $l_5 \in \{3, 2.1, 2, 1.9, 1\}$

pass the critical value 4. First the surface of genus two separates into two disjoint tori, cf. Fig. 8 (f). At the critical value $l_5 = 2$, the two tori join together at two points, cf. Fig. 8 (g), to become a surface of genus three, cf. Fig. 8 (h), (i) and (k). At the limiting value $l_5 = 0$, it collapses to the configuration space of the planar 4-gonal linkage $\mathcal{P}(3,1,3,1)$, shown in Fig. 5 (c).



(a) (b) (c) (d) (e)

(f) (g) (h) (i) (k)

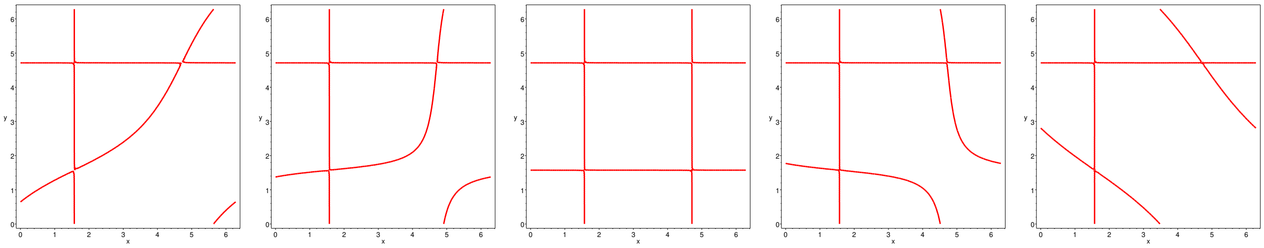
Figure 8: 5-gonal linkage $\mathcal{P}(3,1,3,1,l_5)$ with $l_5 \in \{6.3, 6, 5.9, 5, 4.1, 3, 2, 1.9, 1.5, 0.5\}$

2.3. 4-gonal linkage on the unit-sphere

If the perimeter of the polygonal linkage passes through the critical value 2π, there is a very interesting change of the configuration space. The following two examples visualise this change for 4-gonal linkages.

2.3.1. Equilateral 4-gonal linkage

In the first two pictures in the sequence of Fig. 9, we visualise the configuration space of an equilateral 4-gonal linkage with perimeter $\sum_{l=1}^{4} l_i$ much smaller than 2π, cf. Fig. 9 (a), and close to 2π, cf. Fig. 9 (b). As in the flat case, the configuration space has three singular points which correspond to the three aligned configurations. The configuration space consists of three S^1, each of which is pasted to the two others at a point. Increasing the perimeter, we pass through the critical perimeter $\sum_{l=1}^{4} l_i = 2\pi$; the configuration space now has four singular points and consists of four S^1, each of which is pasted to the two others at a point. It is visualised in the middle of the sequence of Fig. 9. The new (forth) branching point corresponds to the configuration where the equilateral 4-gonal linkage lies completely on the equator of the unit-sphere. Increasing the perimeter further, the singular point previously added is removed, since the configuration it represents is no longer possible. So again we get three S^1, each of which is pasted to the two others at a point, shown in the Figs. 9 (d) and (e).

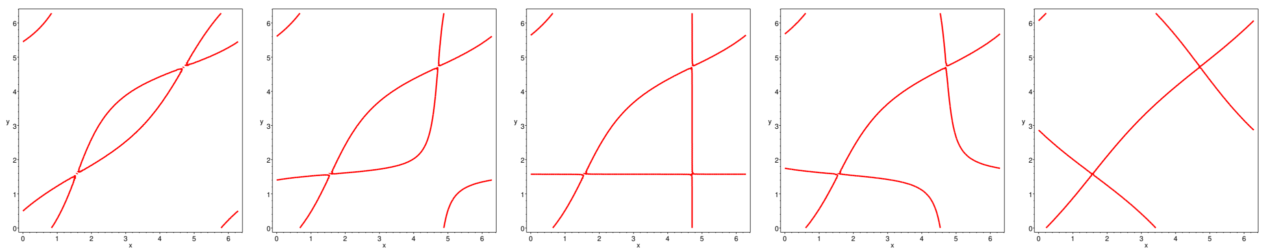


(a) (b) (c) (d) (e)

Figure 9: Equilateral 4-gonal linkage with $\sum_{l=1}^{4} l_i \in \{\frac{4}{3}\pi, \frac{5}{3}\pi, 2\pi, \frac{7}{3}\pi, \frac{8}{3}\pi\}$

2.3.2. 4-gonal linkage with length ratio $[1, 3, 1, 3]$

In the next sequence we increase the perimeter of a 4-gonal linkage with length ratio $[1, 3, 1, 3]$. If $\sum_{l=1}^{4} l_i$ is less than 2π then the configuration space is diffeomorphic to the configuration space of the planar 4-gonal linkage $\mathcal{P}(1, 3, 1, 3)$, which consists of two S^1 pasted together at two points, cf. Fig. 5 (c) and Figs. 10 (a) and (b). Increasing the perimeter, we pass through the critical perimeter $\sum_{l=1}^{4} l_i = 2\pi$; the configuration space now has three singular points and consists of three S^1, each of which is pasted to the two others at a point, cf. Fig. 9 (c). The new (third) branching point corresponds to the configuration where the 4-gonal linkage lies completely on the equator of the unit-sphere. Further increasing the perimeter, the singular point previously added is removed. Again we have two S^1 pasted together at two points, shown in Fig. 10 (d) and (e).



(a) (b) (c) (d) (e)

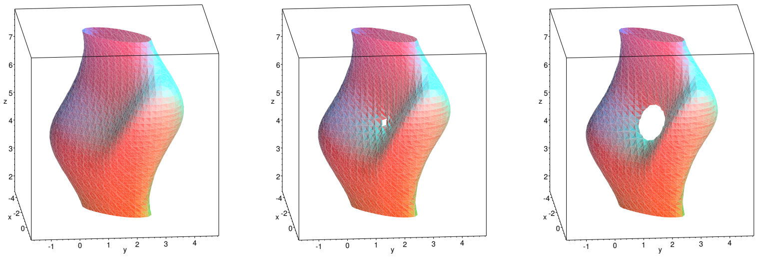
Figure 10: Length ratio is $[1, 3, 1, 3]$ and $\sum_{l=1}^{4} l_i \in \{\frac{4}{3}\pi, \frac{5}{3}\pi, 2\pi, \frac{7}{3}\pi, \frac{8}{3}\pi\}$

2.4. 5-gonal linkage on the unit-sphere

In the case of 5-gonal linkages on the unit-sphere the situation is different: passing through the critical perimeter 2π adds a new handle.

2.4.1. 5-gonal linkage with length ratio $[3, 2, 5, 1, 2]$

In the following sequence we start with a 5-gonal linkage which, if the perimeter is less than 2π, has a torus as its configuration space, cf. Fig. 11 (a). Increasing the perimeter adds a singular point which is locally a double-cone, cf. Fig. 11 (b), and then this double-cone is blown up to a cylinder. The configuration space therefore becomes a surface of genus two, cf. Fig. 11 (c).

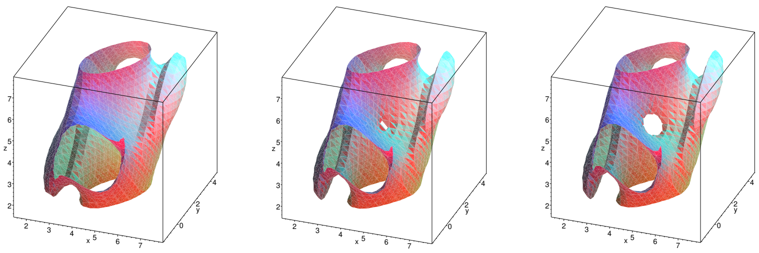


(a) (b) (c)

Figure 11: The length ratio is $[3, 2, 5, 1, 2]$ and $\sum_{l=1}^{5} l_i \in \{\frac{5}{3}\pi, 2\pi, \frac{7}{3}\pi\}$

2.4.2. 5-gonal linkage with length ratio $[2, 1, 3, 1, 2]$

The same procedure can be seen in the following sequence, cf. Fig. 12 (a): first there is a surface of genus two, then increasing the perimeter adds a singular point which is locally a double-cone, cf. Fig. 12 (b), then this double-cone is blown up to a cylinder. The configuration space therefore becomes a surface of genus three, cf. Fig. 12 (c).



(a) (b) (c)

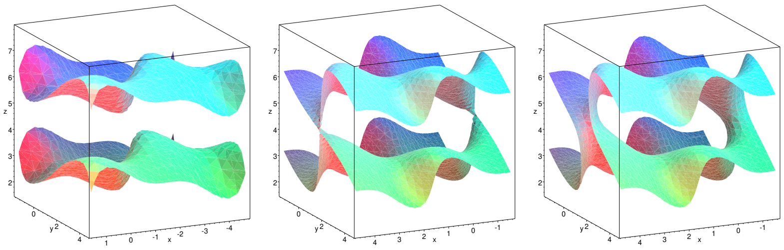
Figure 12: The length ratio is $[2, 1, 3, 1, 2]$ and $\sum_{l=1}^{5} l_i \in \{\frac{5}{3}\pi, 2\pi, \frac{7}{3}\pi\}$

2.4.3. 5-gonal linkage with length ratio $[16, 7, 16, 16, 7]$

A last interesting sequence is the union of two disjoint tori to a surface of genus two, cf. Fig. 13.

2.5. Equilateral 5-gonal linkage on the unit-sphere

Every picture show must have an end. The last configuration space we show is that of the equilateral 5-gonal linkage on the sphere with perimeter a little bit larger than 2π, cf. Fig. 14. It is in fact the prize example, since it is a surface of genus five, which is the maximal genus among all configuration spaces of 5-gonal linkages on the unit-sphere.



(a) (b) (c)

Figure 13: The length ratio is $[16, 7, 16, 16, 7]$ and $\sum_{l=1}^{5} l_i \in \{\frac{5}{3}\pi, 2\pi, \frac{7}{3}\pi\}$

3. Animated sequences of configuration spaces

Using the command structure of `animate` it is pretty easy to create *animated gifs* with *Maple V Release 5*. Slightly varied pictures of configuration spaces are displayed one after the other to obtain a film. As a time parameter t in the film, we take the variable last length for planar polygonal linkages, and the perimeter for spherical polygonal linkages. The *Maple* code in Table 2 is an example of an animated sequence of configuration spaces which contains the equilateral planar 5-gonal linkage $\mathcal{P}(1, 1, 1, 1, 1)$.

```
> with(plots):
> creation := proc(t)
  local L1, L2, L3, L4, L5, pic, dispic, film:
  L1 := 1: L2 := 1: L3 := 1: L4 := 1: L5 := t:
  pic := implicitplot3d({ (L1+L2*cos(x)+L3*cos(y)+L4*cos(z))^2+
                          (   L2*sin(x)+L3*sin(y)+L4*sin(z))^2-L5^2 = 0 },
                          x = 0..2*Pi, y = 0..2*Pi, z = 0..2*Pi,
                          axes = BOXED, scaling = CONSTRAINED,
                          orientation = [35,80], grid = [20,20,20],
                          style = PATCHNOGRID, lightmodel = light3):
  dispic := display(pic):
  RETURN(dispic):
  end:

  n  := 20;
  t0 := 0;
  dt := 0.2;
  film := array(1..n, [ ]):

  for i from 1 to n do
  t := t0+i*dt:
  dispic := creation(t):
  film[i] := dispic:
  od:

> display3d(seq([film[i]], i = 1..n), insequence = true);
```

Table 2: *Maple* code used to create an animated sequence of configuration spaces for the planar 5-gonal linkage $\mathcal{P}(1, 1, 1, 1, t)$: there are $n = 20$ pictures in the sequence and the step-size between two pictures is $dt = 0.2$, therefore the time t varies in the interval from $t_0 = 0$ to $t_1 = 4$.

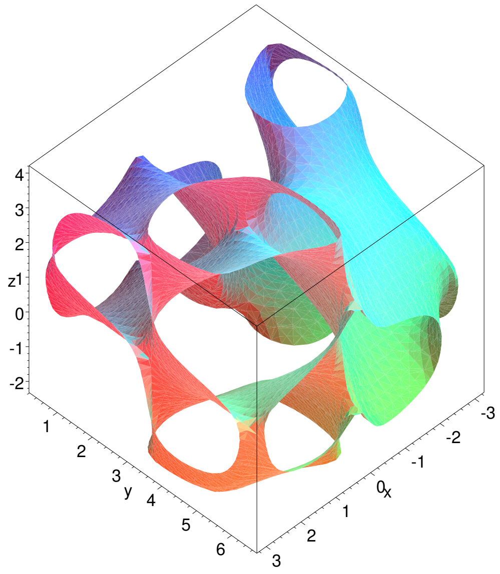


Figure 14: Equilateral 5-gonal linkage with $\sum_{i=1}^{5} l_i = \frac{7}{3}\pi$

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