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Flexible Cross-Polytopes in the Euclidean 4-Space

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Abstract. It is shown that the examples presented 1998 by A. Walz are special cases of a more general class of flexible cross-polytopes in $\mathbb{E}^4$. The proof is given by means of 4D descriptive geometry. Further, a parameterization of the one-parameter self-motions of Walz's polytopes is presented.

Key Words: Flexibility, polyhedra, 4D geometry.

MSC 2000: 52C25, 51N05, 53A17

1. Introduction

There is a basic and important question concerning the geometry of structures: Is a given structure rigid or is it not? In the engineering world there is a vigorous interest in rigidity, as bridges, buildings, mechanical gadgets and countless other things have to be built. This has been the background for interesting mathematical theories. And there is still a wide field of open problems left.

A long-standing problem is to prove if a smooth closed surface can *continuously flex*, i.e., one can find a continuous family of smooth surfaces each of which is isometric (in the intrinsic metric) to any other one and is not obtained from the initial surface by a rigid motion. A first piece-wise linear flexible embedding of the 2-sphere into the Euclidean 3-space was constructed 1978 by R. Connelly [2]. Two years later a simplified *"flexing sphere"* was presented by K. Steffen (see [4]). Both flexible polyhedra are based on Bricard's octahedra ([1], compare [6]).

A milestone in the theory of flexible polyhedra was recently the progress with the "Bellows Conjecture". This conjecture stated by R. Connelly says that any continuous flexion that preserves the edge lengths of a closed triangulated polyhedron preserves its volume. A first proof in $\mathbb{E}^3$ was given 1995 by I. Sabitov. A second proof by R. Connelly et al. ([3]) followed two years later.

If a polyhedron admits a continuous flexion then it admits also an analytical flexion, i.e., for each vertex the trajectory under the flexion can be expressed as an analytic function of

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the time t. One can weaken the continuous flexibility by limiting the Taylor series, i.e., by requiring that the edge lengths stay constant up to a given order of t, only. In this sense, *flexibility of first order* means that to each vertex a velocity vector can be assigned such that these are compatible with constant edge lengths. Additionally one must demand that these velocity vectors do not originate from a motion of the whole structure like a rigid body. When also compatible acceleration vectors can be assigned to each vertex then we get *second order flexibility*, and so on. Geometric characterizations of octahedra which are infinitesimally flexible of the orders 1 or 2 are given in [8].

2. A. Walz's flexible cross-polytopes in $\mathbb{E}^4$

In the Euclidean n-space $\mathbb{E}^n$ the analoga of octahedra are called *cross-polytopes* $\mathcal{C}_n$: These polytopes have $2n$ vertices coupled into pairs $(\mathbf{p}_1^i, \mathbf{p}_2^i)$ for $i = 1, \ldots, n$. The $4\binom{n}{2} = 2n(n-1)$ edges of $\mathcal{C}_n$ are $\mathbf{p}_{j_1}^i \mathbf{p}_{j_2}^k$ for $i \neq k$ and $j_1, j_2 \in \{1, 2\}$. The 2^n hyperfaces of $\mathcal{C}_n$ are the simplices $\mathbf{p}_{j_1}^1 \mathbf{p}_{j_2}^2 \ldots \mathbf{p}_{j_n}^n$ for any $j_1, \ldots, j_n \in \{1, 2\}$.

2.1. A descriptive geometry approach

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{d}_1, \mathbf{d}_2$ be the eight vertices of a four-dimensional cross-polytope $\mathcal{C}_4$. We partition the set of 24 edges into the edges of the quadrangles $(= \mathcal{C}_2)$

$$\mathcal{Q} := \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2, \quad \overline{\mathcal{Q}} := \mathbf{c}_1\mathbf{d}_1\mathbf{c}_2\mathbf{d}_2,$$

and the bipartite framework

$$\mathcal{F} := \{\, \mathbf{p}\,\overline{\mathbf{p}} \mid \mathbf{p} \in \mathcal{Q},\ \overline{\mathbf{p}} \in \overline{\mathcal{Q}}\,\}.$$

In 1998 at a conference in Canada[1] A. Walz presented a class of continuously flexible cross-polytopes in $\mathbb{E}^4$. Following Walz, we visualize these polyhedra using two complementary orthogonal projections of $\mathbb{E}^4$ onto planes: Each point $\mathbf{x} = (x, y, z, t) \in \mathbb{E}^4$ is mapped onto its *"top view"* $\mathbf{x}' = (x, y)$ and the *"front view"* $\mathbf{x}'' = (z, t)$, thus representing $\mathbb{E}^4$ as $\mathbb{E}^2 \times \mathbb{E}^2$ (compare [7]). Obviously, for any two points $\mathbf{x}, \mathbf{y} \in \mathbb{E}^4$ the distance is given by

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}' - \mathbf{y}'\|^2 + \|\mathbf{x}'' - \mathbf{y}''\|^2. \tag{1}$$

At Walz's example the quadrangles $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are located in two totally-orthogonal planes, say, parallel to the xy-plane and the zt-plane, respectively. Therefore we obtain the true size of $\mathcal{Q}$ in the top view, the true size of $\overline{\mathcal{Q}}$ in the front view, and we have $\mathbf{o}' := \mathbf{c}_1' = \ldots = \mathbf{d}_2'$ and $\mathbf{o}'' := \mathbf{a}_1'' = \ldots = \mathbf{b}_2''$ (see Fig. 1). The quadrangles $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are *antiparallelograms*[2] with the common circumcenter $\mathbf{o}$. Let $\rho, \overline{\rho}$ denote the radii of the circumcircles. Then due to (1) all edges of $\mathcal{F}$ have the same length $r := \sqrt{\rho^2 + \overline{\rho}^2}$.

Suppose that both antiparallelograms $\mathcal{Q}, \overline{\mathcal{Q}}$ flex simultaneously like four-bar linkages in their planes such that the common center $\mathbf{o}$ of the circumcircles remains fixed and the radii $\rho, \overline{\rho}$ obey the condition

$$r^2 = \rho^2 + \overline{\rho}^2 = \text{const.} \tag{2}$$

[1] "Canadian Mathematical Society Winter 1998 Meeting" held at Queen's University and the Royal Military College, December 13-15, 1998. See `http://www.cms.math.ca/Events/winter98/w98-abs/node20.html`.

[2] These are nonconvex quadrangles with opposite sides of equal lengths. Antiparallelograms have always a line of symmetry. If the four vertices are not aligned, there is a circumcircle.

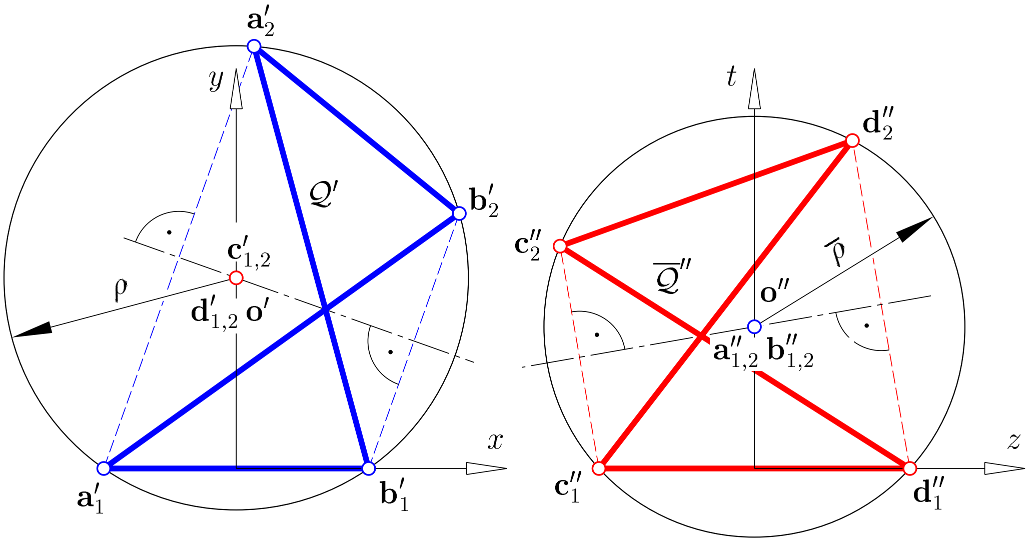

Figure 1: A. WALZ's four-dimensional flexible cross-polytope $\mathcal{C}_4$ represented in top view and front view

Then all edges of $\mathcal{C}_4$ preserve their lengths. Since all 2-faces of $\mathcal{C}_4$ are triangles, the planar motions define a continuous selfmotion of the cross-polytope.[3]

2.2. Analytic representation of the flexion

For obtaining an analytic representation of this flexion, we use a coordinate frame such that

$$\mathbf{a}_1 = (-\alpha, 0, 0, \tau), \quad \mathbf{b}_1 = (\alpha, 0, 0, \tau), \quad \mathbf{c}_1 = (0, \eta, -\gamma, 0), \quad \mathbf{d}_1 = (0, \eta, \gamma, 0) \tag{3}$$

with $\alpha, \gamma > 0$. We keep the top views of $\mathbf{a}_1$ and $\mathbf{b}_1$ fixed as well as the front views of $\mathbf{c}_1$ and $\mathbf{d}_1$. Hence α and γ are constant[4] while the coordinates η and τ vary. This induces translations of the planes spanned by $\mathcal{Q}$ and $\overline{\mathcal{Q}}$, respectively.

Let

$$\begin{array}{rcl} 2\beta := & \|\mathbf{b}_2 - \mathbf{a}_1\| = \|\mathbf{b}_1 - \mathbf{a}_2\| & > \; 2\alpha = \|\mathbf{b}_1 - \mathbf{a}_1\| = \|\mathbf{b}_2 - \mathbf{a}_2\|, \\ 2\delta := & \|\mathbf{d}_2 - \mathbf{c}_1\| = \|\mathbf{d}_1 - \mathbf{c}_2\| & > \; 2\gamma = \|\mathbf{d}_1 - \mathbf{c}_1\| = \|\mathbf{d}_2 - \mathbf{c}_2\|. \end{array}$$

It is well known (e.g. [9]) that in any position of the four-bar linkage $\mathcal{Q}'$ in the xy-plane the coupler $\mathbf{a}_2'\mathbf{b}_2'$ is the image of the frame link $\mathbf{a}_1'\mathbf{b}_1'$ under the reflection in any tangent line l of the ellipse e (=fixed polode) with focal points $\mathbf{a}_1', \mathbf{b}_1'$ and semi-axes β and $\sqrt{\beta^2 - \alpha^2}$ (see Fig. 2).

Let the tangent line l touch the ellipse e at the instantaneous pole

$$\left(\beta \sin\varphi, \sqrt{\beta^2 - \alpha^2}\,\cos\varphi\right). \tag{4}$$

Then l intersects the minor axis ($x = 0$) at the point $(0, \eta)$ with

$$\eta = \sqrt{\beta^2 - \alpha^2}/\cos\varphi. \tag{5}$$

[3] When we replace the condition (2) either by $\cos\rho\,\cos\overline{\rho} = $ const. or by $\cosh\rho\,\cosh\overline{\rho} = $ const., we obtain flexible cross-polytopes in the *elliptic* or *hyperbolic* 4-space, respectively.

[4] Under these conditions the hyperface $\mathcal{S}_1 := \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\mathbf{d}_1$ of $\mathcal{C}_4$ is still movable in $\mathbb{E}^4$. It performs an elliptic motion parallel to the yt-plane. The trajectories of the vertices $\mathbf{a}_1, \ldots, \mathbf{d}_1$ are located on straight lines (note (7) and (5)).

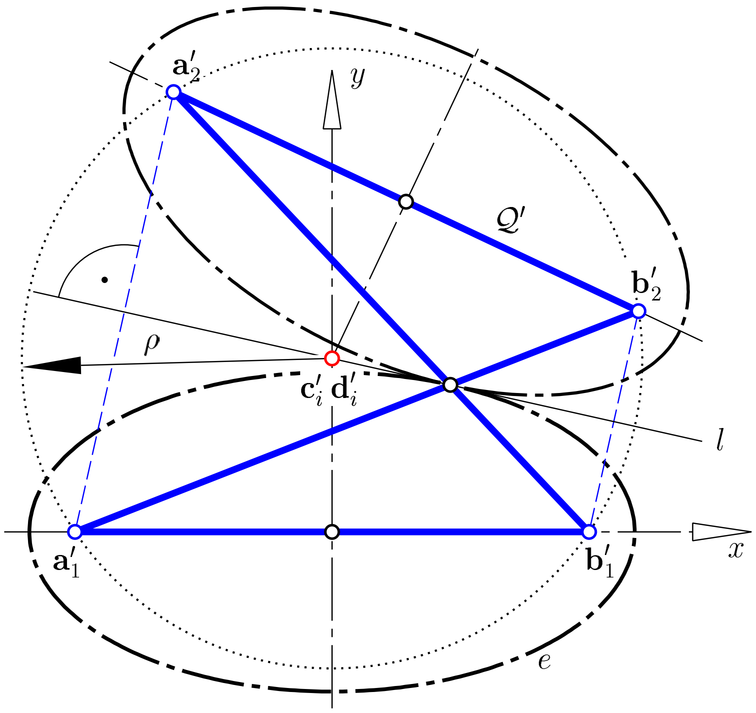

Figure 2: The antiparallelogram-motion as a symmetric rolling of ellipses

This point is the center $\mathbf{o}'$ of the circumcircle of $\mathcal{Q}'$. Therefore the radius obeys

$$\rho^2 = \alpha^2 + \eta^2 = \beta^2 + (\beta^2 - \alpha^2)\ \tan^2\varphi \geq \beta^2.$$

In the same way the flexes of $\overline{\mathcal{Q}}''$ in the front view are obtained under a symmetric rolling of ellipses with semi-axes δ and $\sqrt{\delta^2 - \gamma^2}$. We set for the instantaneous pole of this motion in the zt-plane

$$\left(\delta \sin\psi, \sqrt{\delta^2 - \gamma^2}\ \cos\psi\right). \tag{6}$$

Hence the circumcircle of $\overline{\mathcal{Q}}''$ has the center $\mathbf{o}'' = (0, \tau)$ with

$$\tau = \sqrt{\delta^2 - \gamma^2}/\cos\psi, \tag{7}$$

and its radius obeys

$$\overline{\rho}^2 = \gamma^2 + \tau^2 = \delta^2 + (\delta^2 - \gamma^2)\ \tan^2\psi \geq \delta^2.$$

The necessary condition (2) implies

$$r^2 - \beta^2 - \delta^2 = (\beta^2 - \alpha^2)\tan^2\varphi + (\delta^2 - \gamma^2)\tan^2\psi \geq 0\,,$$

i.e.,

$$\frac{\beta^2 - \alpha^2}{r^2 - \beta^2 - \delta^2}\ \tan^2\varphi + \frac{\delta^2 - \gamma^2}{r^2 - \beta^2 - \delta^2}\ \tan^2\psi = 1\,. \tag{8}$$

This equation couples the parameters φ and ψ and gives rise to a *closed one-parameter flexion of* $\mathcal{C}_4$: We set for $0 \leq t < 2\pi$

$$\varphi = \arctan\left(\sqrt{\frac{r^2 - \beta^2 - \delta^2}{\beta^2 - \alpha^2}}\ \cos t\right), \quad \psi = \arctan\left(\sqrt{\frac{r^2 - \beta^2 - \delta^2}{\delta^2 - \gamma^2}}\ \sin t\right). \tag{9}$$

Then by reflecting $\mathbf{a}_1, \mathbf{b}_1$ in the tangent line l of the ellipse e at the pole (4) we obtain

$$\begin{aligned}\mathbf{a}_2 &= \left(-\alpha + \frac{2(\beta^2-\alpha^2)\sin\varphi}{\beta-\alpha\sin\varphi},\ \frac{2\beta\sqrt{\beta^2-\alpha^2}\,\cos\varphi}{\beta-\alpha\sin\varphi},\ 0,\ \tau\right),\\ \mathbf{b}_2 &= \left(\ \ \alpha + \frac{2(\beta^2-\alpha^2)\sin\varphi}{\beta+\alpha\sin\varphi},\ \frac{2\beta\sqrt{\beta^2-\alpha^2}\,\cos\varphi}{\beta+\alpha\sin\varphi},\ 0,\ \tau\right).\end{aligned} \tag{10}$$

In the same way (6) results in

$$\begin{aligned}\mathbf{c}_2 &= \left(0,\ \eta, -\gamma + \frac{2(\delta^2-\gamma^2)\sin\psi}{\delta-\gamma\sin\psi},\ \frac{2\delta\sqrt{\delta^2-\gamma^2}\,\cos\psi}{\delta-\gamma\sin\psi}\right),\\ \mathbf{d}_2 &= \left(0,\ \eta,\ \ \ \gamma + \frac{2(\delta^2-\gamma^2)\sin\psi}{\delta+\gamma\sin\psi},\ \frac{2\delta\sqrt{\delta^2-\gamma^2}\,\cos\psi}{\delta+\gamma\sin\psi}\right).\end{aligned} \tag{11}$$

The reflection of the top view in the tangent line l can be extended to a reflection of the 4-space in a hyperplane L being orthogonal to the xy-plane and passing through l. As l contains the top views $\mathbf{c}_i' = \mathbf{d}_i'$ for $i = 1, 2$, the 4D-reflection maps

$$\mathbf{a}_1 \mapsto \mathbf{a}_2,\ \mathbf{b}_1 \mapsto \mathbf{b}_2,\ \mathbf{c}_i \mapsto \mathbf{c}_i,\ \mathbf{d}_i \mapsto \mathbf{d}_i.$$

In the same way the reflection of the front view leads to a reflection of $\mathbb{E}^4$ in a hyperplane $\overline{\mathrm{L}}$ mapping

$$\mathbf{a}_i \mapsto \mathbf{a}_i,\ \mathbf{b}_i \mapsto \mathbf{b}_i,\ \mathbf{c}_1 \mapsto \mathbf{c}_2,\ \mathbf{d}_1 \mapsto \mathbf{d}_2.$$

$\overline{\mathrm{L}}$ ist orthogonal to L. Hence in any position of the flexing cross-polytope the two complementary hyperfaces $\mathcal{S}_1 = \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\mathbf{d}_1$ and $\mathcal{S}_2 = \mathbf{a}_2\mathbf{b}_2\mathbf{c}_2\mathbf{d}_2$ of $\mathcal{C}_4$ are mirror images with respect to a 2-plane $\mathrm{L} \cap \overline{\mathrm{L}}$.

3. A new class of flexible cross-polytopes in $\mathbb{E}^4$

It turns out that Walz's polytopes are special cases in a larger class of flexible cross-polytopes:

Theorem 1: *Let $\mathcal{C}_4$ be a cross-polytope with the quadrangle $\mathcal{Q} = \mathbf{a}_1 \dots \mathbf{b}_2$ in the hyperplane $z = 0$ and symmetric with respect to $x = 0$, and the complementary quadrangle $\overline{\mathcal{Q}} = \mathbf{c}_1 \dots \mathbf{d}_2$ in $x = 0$ and symmetric with respect to $z = 0$, i.e. (see Fig. 3),*

$$\mathbf{a}_{1,2} = (\pm\alpha_1, \alpha_2, 0, \alpha_4),\ \ \mathbf{b}_{1,2} = (\pm\beta_1, \beta_2, 0, \beta_4),\ \ \mathbf{c}_{1,2} = (0, \gamma_2, \pm\gamma_3, \gamma_4),\ \ \mathbf{d}_{1,2} = (0, \delta_2, \pm\delta_3, \delta_4)$$

for $\alpha_1, \beta_1, \gamma_3, \delta_3 > 0$, $|\beta_2 - \alpha_2| + |\beta_4 - \alpha_4| \neq 0$ and $|\gamma_2 - \delta_2| + |\gamma_4 - \delta_4| \neq 0$.
Then $\mathcal{C}_4$ can flex while the quadrangles $\mathcal{Q}$, $\overline{\mathcal{Q}}$ remain in their hyperplanes and the symmetries are preserved.

Remark 1: The vertices $\mathbf{a}_1, \dots, \mathbf{b}_2$ in the 3-space $z = 0$ form a planar antiparallelogram $\mathcal{Q}$ because of the symmetry with respect to $x = 0$ (Fig. 4). After adapting the xyt-coordinates in this hyperplane, the affine span of $\mathcal{Q}$ can be defined as xy-plane. This implies $\alpha_4 = \beta_4 = 0$ and $\alpha_2 \neq \beta_2$ in Theorem 1.

Remark 2: Also $\overline{\mathcal{Q}} = \mathbf{c}_1\mathbf{d}_1\mathbf{c}_2\mathbf{d}_2$ is an antiparallelogram. Its affine span within $x = 0$ is orthogonal to the affine span of $\mathcal{Q}$ but needs not be totally orthogonal as it is the case at Walz's example. Total orthogonality is characterized by

$$(\beta_2 - \alpha_2)(\delta_2 - \gamma_2) + (\beta_4 - \alpha_4)(\delta_4 - \gamma_4) = 0.$$

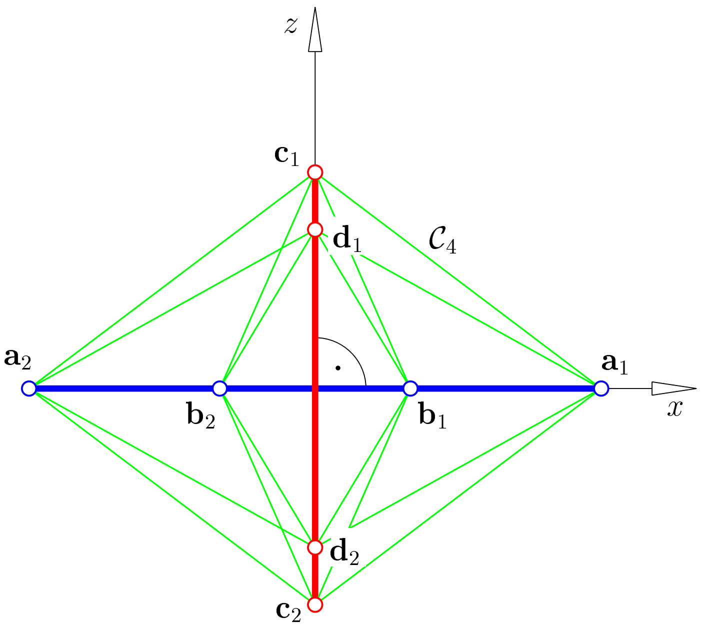

Figure 3: The flexible cross-polytope $\mathcal{C}_4$ of Theorem 1, orthogonally projected into the xz-plane

3.1. Proof of Theorem 1

We prefer a constructive proof based again on top view and front view. According to Remark 1 we specify $\alpha_4 = \beta_4 = 0$. This implies $\mathbf{a}_i'' = \mathbf{b}_k''$ for all $i, k \in \{1, 2\}$, and the top view shows $\mathcal{Q}$ in true size.

There are eight edge lengths to distinguish at $\mathcal{C}_4$ (see Fig. 3):

$$\begin{array}{ll} l_{ab} := \|\mathbf{a}_i - \mathbf{b}_i\|,\ \bar{l}_{ab} := \|\mathbf{a}_i - \mathbf{b}_j\|,\ l_{cd} := \|\mathbf{c}_i - \mathbf{d}_i\|,\ \bar{l}_{cd} := \|\mathbf{c}_i - \mathbf{d}_j\|, & i \neq j, \\ l_{ac} := \|\mathbf{a}_i - \mathbf{c}_k\|,\ l_{ad} := \|\mathbf{a}_i - \mathbf{d}_k\|,\ l_{bc} := \|\mathbf{b}_i - \mathbf{c}_k\|,\ l_{bd} := \|\mathbf{b}_i - \mathbf{d}_k\|, & i, k \in \{1, 2\}. \end{array} \tag{12}$$

In order to prove the continuous flexibility of $\mathcal{C}_4$, we look for a flex $\widetilde{\mathcal{C}}_4$ with vertices $\widetilde{\mathbf{a}}_1, \ldots, \widetilde{\mathbf{d}}_2$ and the same edge lengths (12), sufficiently near to the initial position $\mathcal{C}_4$, but incongruent.

We suppose that $\mathbf{a}_1$ and $\mathbf{b}_1$ are kept fixed, i.e., $\widetilde{\mathbf{a}}_1' = \mathbf{a}_1'$ and $\widetilde{\mathbf{b}}_1' = \mathbf{b}_1'$, and we still insist on $\widetilde{\mathbf{a}}_i'' = \widetilde{\mathbf{b}}_k''$ for all $i, k \in \{1, 2\}$. Now we specify a posture $\widetilde{\mathbf{a}}_1'\widetilde{\mathbf{b}}_1'\widetilde{\mathbf{a}}_2'\widetilde{\mathbf{b}}_2'$ of the antiparallelogram $\mathcal{Q}$ in the top view (Fig. 4). The equations

$$\|\widetilde{\mathbf{a}}_i' - \widetilde{\mathbf{c}}_k'\|^2 + \|\widetilde{\mathbf{a}}_i'' - \widetilde{\mathbf{c}}_k''\|^2 = l_{ac}^2\,, \quad \|\widetilde{\mathbf{b}}_i' - \widetilde{\mathbf{c}}_k'\|^2 + \|\widetilde{\mathbf{b}}_i'' - \widetilde{\mathbf{c}}_k''\|^2 = l_{bc}^2$$

imply together with $\widetilde{\mathbf{a}}_i'' = \widetilde{\mathbf{b}}_i''$

$$\|\widetilde{\mathbf{a}}_i' - \widetilde{\mathbf{c}}_k'\|^2 - \|\widetilde{\mathbf{b}}_i' - \widetilde{\mathbf{c}}_k'\|^2 = l_{ac}^2 - l_{bc}^2 = \text{const.} \tag{13}$$

Let $\mathbf{c}_0, \mathbf{d}_0$ denote the pedal points of $\mathbf{c}_i'$ and $\mathbf{d}_i'$ on the line $\mathbf{a}_1'\mathbf{b}_1'$ (see Fig. 4). Due to (1) and

$$\|\mathbf{a}_1' - \mathbf{c}_k'\|^2 - \|\mathbf{b}_1' - \mathbf{c}_k'\|^2 = \|\mathbf{a}_1' - \mathbf{c}_0\|^2 - \|\mathbf{b}_1' - \mathbf{c}_0\|^2$$

the points $\mathbf{c}_0$ and analogously $\mathbf{d}_0$ must be also the pedal points of $\widetilde{\mathbf{c}}_i'$ and $\widetilde{\mathbf{d}}_i'$, respectively.

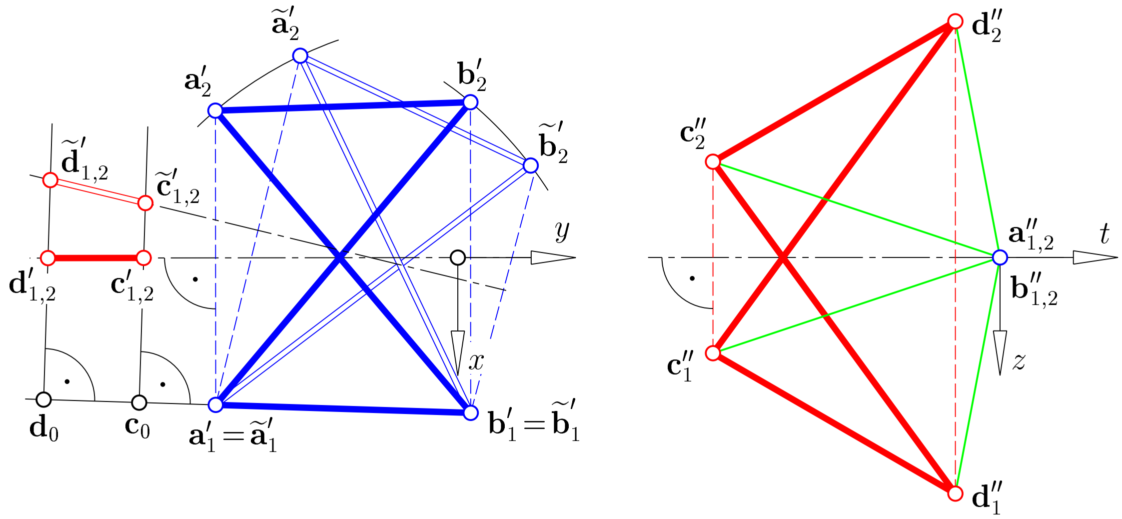

Figure 4: Generalized flexible cross-polytope $\mathcal{C}_4$ in top view and front view

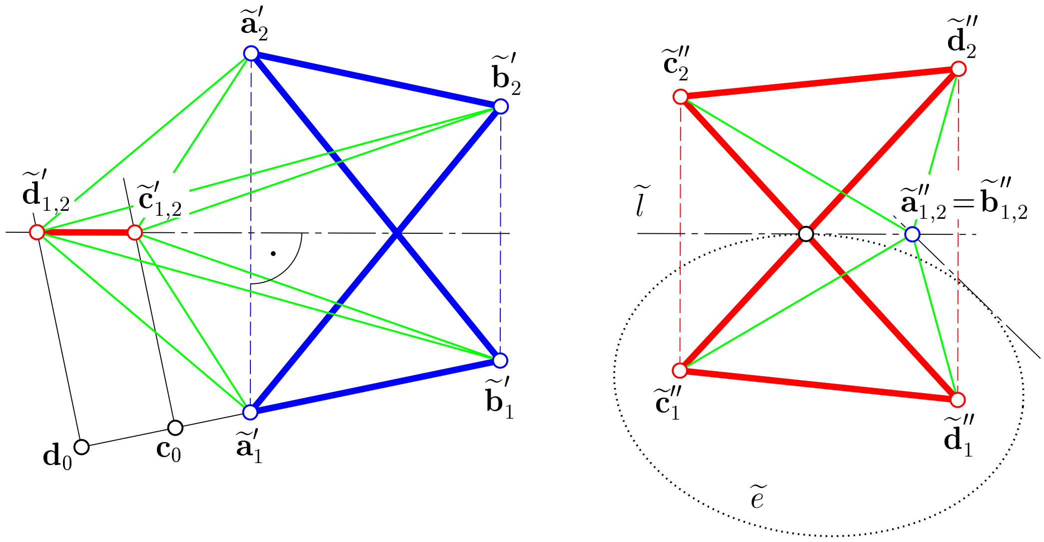

Figure 5: Flex $\widetilde{\mathcal{C}}_4$ of the cross-polytope $\mathcal{C}_4$ displayed in Fig. 4

When $\mathbf{c}'_i$ and $\mathbf{d}'_i$ happen to coincide, then this holds true for $\widetilde{\mathbf{c}}'_i$ and $\widetilde{\mathbf{d}}'_i$, too. This means that total orthogonality between the affine spans of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ in the initial position will be preserved under the flexion.

After the top view of $\widetilde{\mathcal{C}}_4$ has been fixed, in the front view the dimensions of the antiparallelogram $\widetilde{\mathbf{c}}''_1\widetilde{\mathbf{d}}''_1\widetilde{\mathbf{c}}''_2\widetilde{\mathbf{d}}''_2$ as well as the distances $\|\widetilde{\mathbf{a}}''_1 - \widetilde{\mathbf{c}}''_k\|$ and $\|\widetilde{\mathbf{a}}''_1 - \widetilde{\mathbf{d}}''_k\|$ are defined. Because of $\gamma_3\delta_3 > 0$ in Theorem 1 we have $l_{cd} < \overline{l}_{cd}$, hence $\|\mathbf{c}''_1 - \mathbf{d}''_1\| < \|\mathbf{c}''_1 - \mathbf{d}''_2\|$ which implies also $\|\widetilde{\mathbf{c}}''_1 - \widetilde{\mathbf{d}}''_1\| < \|\widetilde{\mathbf{c}}''_1 - \widetilde{\mathbf{d}}''_2\|$.

We specify $\widetilde{\mathbf{c}}''_1$ and $\widetilde{\mathbf{d}}''_1$ and determine $\widetilde{\mathbf{a}}''_1$. As $\widetilde{\mathbf{a}}''_1$ has to be located on the axis of symmetry of the antiparallelogram $\widetilde{\mathbf{c}}''_1\widetilde{\mathbf{d}}''_1\widetilde{\mathbf{c}}''_2\widetilde{\mathbf{d}}''_2$, we construct a line $\widetilde{l}$ through $\widetilde{\mathbf{a}}''_1$ tangent to the ellipse $\widetilde{e}$, the fixed polode of the antiparallelogram motion (compare Fig. 2). Continuity guarantees uniqueness. The reflection in the tangent line $\widetilde{l}$ gives $\widetilde{\mathbf{c}}''_2$ and $\widetilde{\mathbf{d}}''_2$ (Fig. 5).

The conditions $\widetilde{\mathbf{c}}'_1 = \widetilde{\mathbf{c}}'_2$ and $\widetilde{\mathbf{d}}'_1 = \widetilde{\mathbf{d}}'_2$ imply that the minor semi-axis of the ellipse $\widetilde{e}$ is constant, as due to (1)

$$\|\widetilde{\mathbf{c}}_1'' - \widetilde{\mathbf{d}}_2''\|^2 - \|\widetilde{\mathbf{c}}_1'' - \widetilde{\mathbf{d}}_1''\|^2 = \|\widetilde{\mathbf{c}}_1 - \widetilde{\mathbf{d}}_2\|^2 - \|\widetilde{\mathbf{c}}_1 - \widetilde{\mathbf{d}}_1\|^2 = \bar{l}_{cd}^2 - l_{cd}^2 = \text{const.} > 0\,. \qquad (14)$$

In the same way as for Walz's examples one can prove that in any position of the new flexing cross-polytope the two complementary hyperfaces $\mathcal{S}_1 = \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\mathbf{d}_1$ and $\mathcal{S}_2 = \mathbf{a}_2\mathbf{b}_2\mathbf{c}_2\mathbf{d}_2$ are mirror images with respect to a 2-plane.

3.2. Analytic representation of the flexion, an outline

Our construction of the posture $\widetilde{\mathcal{C}}_4$ is the basis for an analytic representation of the continuous flexion. However, we cannot present it in explicit form.

We replace the coordinate system given in Theorem 1 (Figs. 3 and 4) by that used in Subsection 2.2 (compare Fig. 1). Again, we fix $\mathbf{a}_1'$ and $\mathbf{b}_1'$ in the top view. The front views $\widetilde{\mathbf{c}}_1''$ and $\widetilde{\mathbf{d}}_1''$ remain in a symmetric position on the fixed z-axis. Starting from the given initial position, we parametrize by the angle φ which defines the instantaneous pole of the antiparallogram-motion in the top view (cf. (4)) and the reflected points $\widetilde{\mathbf{a}}_2'$ and $\widetilde{\mathbf{b}}_2'$ (see first two coordinates in (10)). For each φ sufficiently near to the initial value we compute the distances showing up in the front view, the position of $\widetilde{\mathbf{a}}_1''$ and the tangent line $\widetilde{l}$ which defines the parameter ψ for the ellipse $\widetilde{e}$ (with varying dimensions). Finally we get the points $\widetilde{\mathbf{c}}_2''$ and $\widetilde{\mathbf{d}}_2''$ by reflecting $\widetilde{\mathbf{c}}_1''$ and $\widetilde{\mathbf{d}}_1''$ in $\widetilde{l}$ (compare (11)).

The limits for the angle φ are much more complex than that of Walz's example. The following three conditions must hold, and it depends on the given dimensions of $\mathcal{C}_4$ which of these conditions are essential.

1. $\|\mathbf{a}_1' - \widetilde{\mathbf{c}}_i'\| \le l_{ac}$ and $\|\mathbf{a}_1' - \widetilde{\mathbf{d}}_i'\| \le l_{ad}$ defines limits for the points $\widetilde{\mathbf{c}}_i'$ and $\widetilde{\mathbf{d}}_i'$ on the lines through $\mathbf{c}_0$ and $\mathbf{d}_0$, respectively. Due to (13), these conditions imply the analogous conditions for $\mathbf{b}_1'$.
2. $\|\widetilde{\mathbf{c}}_i' - \widetilde{\mathbf{d}}_i'\| \le l_{cd}$ defines limits for the inclination of the tangent line l of e (compare Fig. 2). This gives $-\frac{\pi}{2} < -\varphi_l \le \varphi \le \varphi_l < \frac{\pi}{2}$ for a certain positive φ_l. In the limiting cases $\varphi = \pm\varphi_l$, hence $\widetilde{\mathbf{c}}_1'' = \widetilde{\mathbf{d}}_1''$ in Fig. 5, the ellipse $\widetilde{e}$ becomes a circle with radius $\frac{1}{2}\sqrt{\bar{l}_{cd}^2 - l_{cd}^2}$ according to (14).
3. In order to guarantee that $\widetilde{\mathbf{a}}_1''$ exists and is not located inside the ellipse $\widetilde{e}$, we have to obey the conditions

$$\|\widetilde{\mathbf{a}}_1'' - \widetilde{\mathbf{c}}_1''\| + \|\widetilde{\mathbf{a}}_1'' - \widetilde{\mathbf{d}}_1''\| \ge \|\widetilde{\mathbf{c}}_1'' - \widetilde{\mathbf{d}}_2''\| \quad \text{and} \quad \Big|\|\widetilde{\mathbf{a}}_1'' - \widetilde{\mathbf{c}}_1''\| - \|\widetilde{\mathbf{a}}_1'' - \widetilde{\mathbf{d}}_1''\|\Big| \le \|\widetilde{\mathbf{c}}_1'' - \widetilde{\mathbf{d}}_1''\|.$$

   In the limiting case of the first inequality $\widetilde{\mathbf{a}}_1''$ coincides with the point of intersection between the lines $\widetilde{\mathbf{c}}_1''\widetilde{\mathbf{d}}_2''$ and $\widetilde{\mathbf{d}}_1''\widetilde{\mathbf{c}}_2''$. When in the second inequality we have equality then $\widetilde{\mathbf{a}}_1''$ lies on the z-axis – outside the segment $\widetilde{\mathbf{c}}_1''\widetilde{\mathbf{d}}_1''$.

4. Conclusion

In this paper flexible cross-polytopes in $\mathbb{E}^4$ have been presented. There are many open problems left around this topic:

The characterization of first-order infinitesimal flexibility for cross-polytopes $\mathcal{C}_n$ in $\mathbb{E}^n$ seems to be similar to that in $\mathbb{E}^3$ (compare [8], Theorem 1): We decompose the vertex set of

$\mathcal{C}_n$ into two subsets which – in analogy to $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ in $\mathbb{E}^4$ – define two sub-cross-polytopes $\mathcal{P}$ and $\overline{\mathcal{P}}$ of types $\mathcal{C}_{n/2}$ for even n and of type $\mathcal{C}_{(n+1)/2}$ and $\mathcal{C}_{(n-1)/2}$ for odd n. Then infinitesimal flexibility of order 1 is given if and only if the two complementary substructures $\mathcal{P}$ and $\overline{\mathcal{P}}$ are located on the same quadric $Q \subset \mathbb{E}^n$, provided $\mathcal{P}$ is affinely independent.[5] However, a complete proof is open.

The cross-polytopes presented in Theorem 1 seem to be the only flexible cross-polytopes in $\mathbb{E}^4$, and no nontrivially flexible cross-polytopes are expected for higher dimensions. However, a proof of these conjectures is left for future research, too.

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[5]For a cross-polytopes $\mathcal{P}$ or $\overline{\mathcal{P}}$ of type $\mathcal{C}_k$ this means that the affine spans of all its $(k-1)$-dimensional faces are subsets of this quadric Q.