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On Curves and Surfaces in Illumination Geometry

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Dedicated to Prof. Dr. Gerhard Geise
on the occasion of his 70th birthday

Abstract. A point like light source in $\mathbb{R}^d$ induces a certain illumination intensity at hypersurface elements of $\mathbb{R}^d$. Manifolds of such elements with the same intensity of illumination are called isophotic. As shown in [9], a uniformly radiating light source causes isophotic strips along sinusoidal spirals. In the present paper this investigation is extended in two directions.
First all isophotic C^2-hypersurfaces are found, and also manifolds of hypersurface elements which are isophotic with respect to two and more central illuminations are discussed. It suggests itself to treat such illumination problems also in non-Euclidean spaces.
The second part of the paper deals with the generating curves of isophotic strips. They belong to the well-known families of Clairaut curves and sinusoidal spirals. Their known relations to each other and to other curve families (such as Ribaucour curves and roses) are extended by some perhaps new aspects.

Key Words: cardioid, cassinoid, central illumination, Clairaut curve, cycloid (of higher order), hyperbolic geometry, inversion, isophotic strips and surfaces, isotropic geometry, (isotropic) logarithmic spiral, projections (of spatial curves), pedal curve, rhodonea, Ribaucour curve, rose, sinusoidal spiral, Steiner's three-cusped hypocycloid

MSC 2000: 53A07, 53A04

1. Introduction

In [8], [9] and [10] G. Geise and H. Martini investigated families and strips of hypersurface elements in the Euclidean d-space $\mathbb{R}^d$, $d \geq 2$, which are *isophotic* (i.e., have equal intensity

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of illumination) with respect to a given central illumination. As basic curves of such families and strips they derived *Clairaut curves* and *sinusoidal spirals*, respectively. So it was also possible to explain known kinematical relations between these curve classes in terms of classical illumination geometry (see also [18] and [16] for related investigations). This motivates the question whether also relations of other curve classes to sinusoidal spirals can be explained in the spirit of illumination geometry, yielding our results in Section 2 and in Subsection 6.3 below.

Among other things, our investigations will also reflect the geometry of projections. Namely, we will interpret planar curves (which are important in illumination geometry) as images of suitable spatial curves under (orthogonal and parallel) projections, cf. Subsections 6.1 and 6.2.

Essentially, [9] and [10] investigated one-dimensional families of isophotic surface elements, whereas H.-P. Paukowitsch [18] derived isophotic surfaces in $\mathbb{R}^3$, without giving a complete list of them. Therefore (to our best knowledge) a complete description of isophotic $(d-1)$-surfaces in $\mathbb{R}^d$ is still missing (see our Section 3 below).

One might also ask for "k-fold isophotic" manifolds in $\mathbb{R}^d$, i.e., for manifolds being isophotic with respect to k light sources. Exemplarily, we will discuss in Section 4 twofold isophotic strips in $\mathbb{R}^3$.

It is natural to look for analogous problems in non-Euclidean spaces. In particular, those (Euclidean) functions of illumination intensity are interesting, which are constant in the sense of a certain non-Euclidean geometry. Such questions are shortly discussed in Section 5.

2. Geometric Central Illumination

Let C be a point like light source in Euclidean d-space $\mathbb{R}^d$, $d \geq 2$, having constant luminous intensity I over all directions. The classical photometric law (extended to d-space) regulates the connections between the following quantities:

- the *illumination intensity* E of a *hypersurface element* $(X, \mathbf{n})$, ($\mathbf{x}$ being the position vector of point X, $\mathbf{c}$ that of the light source C and $\mathbf{n}$ the normed normal vector of the oriented hypersurface element),
- the length $r := \|\mathbf{x} - \mathbf{c}\|$ of the *"main light ray"*, and
- its *angle* $\varphi = \sphericalangle [(\mathbf{x} - \mathbf{c}), \mathbf{n}]$ *of incidence.*

This law (cf. [9]) is given by

$$E(\mathbf{x}, \mathbf{n}) = I \cdot \|\mathbf{x} - \mathbf{c}\|^{1-d} \cdot \cos\varphi \,. \tag{1}$$

At the beginning we will only consider families of elements in a sufficiently small neighborhood U of an illuminated hypersurface element. Thus we may assume that, without loss of generality, $\cos\varphi > 0$ in U.

a) Taking X as the pole and φ as the polar angle of a polar coordinate system we get from (1) a *Clairaut curve* as position of all light sources (in a 2-plane passing through X) generating the same illumination intensity at a fixed element $(X_0, \mathbf{n}_0)$, where X_0 is the pole and $\mathbf{n}_0$ the direction vector of the symmetry axis of that curve.

b) Vice versa, isophotic families of elements, whose fixed normal vector $\mathbf{n}$ and whose position vectors $\mathbf{x}$ lie in a 2-plane ε through C, belong to an arc $\varepsilon \cap U$ which continues to the same Clairaut curve m from a), namely to

$$m \;\ldots\; r^{d-1} = a_m \cdot \cos\varphi \tag{2}$$

with $a_m =$ const. and (r, φ) as polar coordinates with pole C in φ.
Thus, the physical fact (1) directly yields the curves geometrically determined by (2) ($\equiv$ (1)).

c) The generating curves of isophotic C^1-strips, whose position vectors $\mathbf{x}$ and direction vectors $\mathbf{n}$ are again from a fixed 2-plane ε through C, are essentially *cassinoids*, i.e., special (algebraic) *sinusoidal spirals s*

$$s \ \ldots \ \ r^p = a_s \cdot \cos p\varphi \tag{3}$$

with $a_s =$ const. and $p = d - 1$, cf. [9]. If an isophotic sinusoidal spiral slides through a fixed line element then, following a), its pole runs through a Clairaut curve m (see [10]).

The complete classification of Clairaut curves is given in the monograph [12] (see also [5]), and different applications of these curves (e.g., in connection with the Delian problem) are collected in [14], §V. 11, [7], §V. 7, and [22].

The large variety of (characterizing) properties and applications of sinusoidal spirals is described in [14], §V. 18, [21], §18, and [4], §§37–46 (see also the dictionary [20], pp. 333–336, and [6][1].

3. Further curves with applications in illumination geometry

Depending on the dimension $d \geq 2$ we get (planar) algebraic solution curves m and s which, for $p \in \mathbb{R}$ in (3), are embedded in families m_p, s_p of analytical curves satisfying the relation described in c). So one it motivated to interpret the space $\mathbb{R}^d$ as an optically homogeneous medium, for which the extended photometric law (not directly depending on the dimension d)

$$E(\mathbf{x}, \mathbf{n}) = I \cdot r^{-p} \cos\varphi\,, \quad p \in \mathbb{R} \tag{1'}$$

holds true.

In the practical situation, the illumination intensity of a surface element clearly depends on the distance to the light source and on the angle φ of incidence, but there are also other influences (possibly also depending on φ), such as roughness, porosity or absorption of the material. These phenomena could be described by a modified photometric law in which $\cos\varphi$ in (1′) is replaced by a suitable function $f(\varphi)$. For example, the case $f(\varphi) = \cos^q\varphi$ ($q \in \mathbb{R}$) should be investigated. The analogues of the Clairaut curves with respect to this illumination law are *generalized sinusoidal spirals*. Due to G. Loria [14], p. 403 and p. 658, the whole class of such curves is not classified, yet. For $q = 1$, the Clairaut curves are obtained as a subclass, and also the family of *roses* (cf. [14], p. 297), discussed below, belongs to this class of curves.

Within the considered light plane ε, the (physically motivated) question for isophotic strips yields a quadratic differential equation of first order, having an extremal circle k (concentric to C) as singular solution.

For $p > 0$, each point $X \neq 0$ from the interior of k belongs to two open arcs of sinusoidal spirals as solutions, these arcs being symmetric with respect to CX. Therefore a C^0-solution curve can be composed from such arcs like a Fresnel lens (Fig. 1a). A C^1-solution curve can

[1] Note that the Figures 2 and 5–11 in [6], presenting sinusoidal spirals, are not correct.

consist of two arcs of sinusoidal spirals which are connected by an arc of k (Fig. 1b). But C^2-solutions have to be open arcs of sinusoidal spirals bounded by C and k, i.e., they have to be analytic. Analogous relations hold for the solution curves exterior to k if $p < 0$.

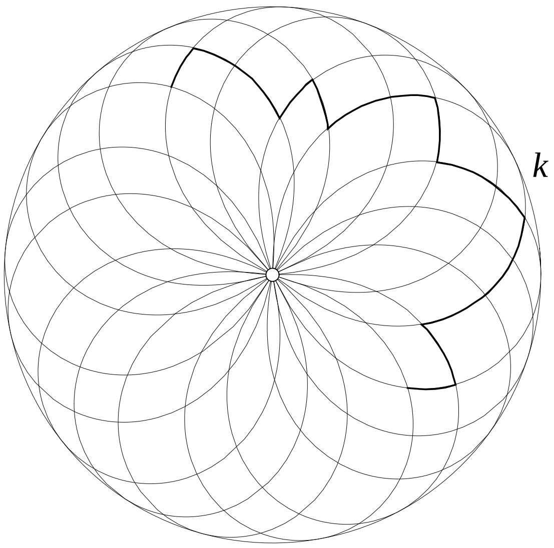

Figure 1a: Isophotic curve of class C^0

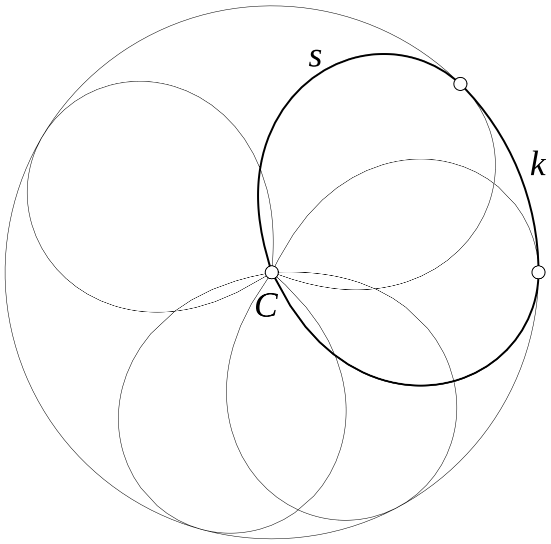

Figure 1b: Isophotic curve of class C^1

In [9] it is assumed that the normals n of the hypersurface elements of the considered strips belong to the 2-plane ε of the basic curve $\{X\}$. We will see below in Section 3 that this convenient assumption is not essential. To see this, we follow E. Bohne and R. Möller [2]. Their approach for $d = 3$ can be immediately generalized to arbitrary dimension $d \geq 2$.

Namely, let $X \in \mathbb{R}^d$ be a point different from the light source C. (In the following we assume that C is the pole and origin of the polar and the cartesian coordinate system, respectively.) Then the hyperplanes of all hypersurface elements $(X, \mathbf{n})$ with equal illumination intensity are supporting hyperplanes of a (imaginary or real) hypercone Λ_X of revolution, or they coincide with a unique hyperplane which is normal to CX. In the latter case, all these hyperplanes envelop a hypersphere S^{d-1} as integral hypersurface, which is concentric to C and therefore presents a singular solution of the integration problem referring to the elements $(X, \mathbf{n})$. In the following we assume the luminous intensity I to be normed such that the radius of S^{d-1} equals one. Then we get with $(1')$

$$\cos\varphi = \|4\mathbf{x}\|^p \,. \qquad (1'')$$

Thus, the isophotic hypersurface elements $(X, \mathbf{n})$ of a fixed point X (with $\|\mathbf{x}\| \neq 0, 1$) envelop the rotational hypercone Λ_X which is circumscribed to the hypersphere Φ_X of radius

$$\rho = \|\mathbf{x}\|^{p+1} \,. \qquad (4)$$

Hence this hypercone has a $(d-2)$-dimensional "base sphere" which is the intersection of Φ_X and the polar hyperplane of X with respect to Φ_X. So it is a $(d-2)$-dimensional manifold of linear generators. For $d = 3$, the whole family of all generators of all cones Λ_X, $(X \in \mathbb{R}^3 \setminus \{C\})$ is forming a *line complex* (cf. [13], p. 248). The general complex cones Λ_X of this line complex are rotational cones whose axis passes through X, in each case. Choosing coaxial cones Λ_X to be coaxial, they envelop a certain rotational surface. (It should be mentioned that surfaces of revolution of such a type and with Clairaut curves as meridians were found by W.

Wunderlich in [22], when he looked for spatial curves which are pseudo-geodesics on two different (conic) strips.) Taking the common axis as x-axis of a cartesian coordinate system $(C; x, y, z)$, the meridian m of this rotational surface has the coordinate representation (in the xy-plane)

$$m(p,\varphi)\ \ldots\ \ (x,y)=\left(\frac{1}{p}(\cos\varphi)^{\frac{1}{p}}\cdot(sin^2\varphi+p),\ \ \frac{1}{p}(\cos\varphi)^{\frac{1}{p}+1}\cdot\sin\varphi\right),\ \text{(see Fig. 2)}. \tag{5}$$

Thus the meridians $m(p,q)$ are polar reciprocal to Clairaut curves, cf. [22]. Among these curves $m(p,\varphi)$ (which are three-cusped in the complex extension), one can find the famous *three-cusped hypocycloid of Steiner*, namely for $p=\frac{1}{2}$ (cf. [20], pp. 344–345). Since the isophotic sinusoidal spiral with $p=\frac{1}{2}$ is a *cardioid* (see [10]), we get a possibly new remarkable relation between these two cycloids.

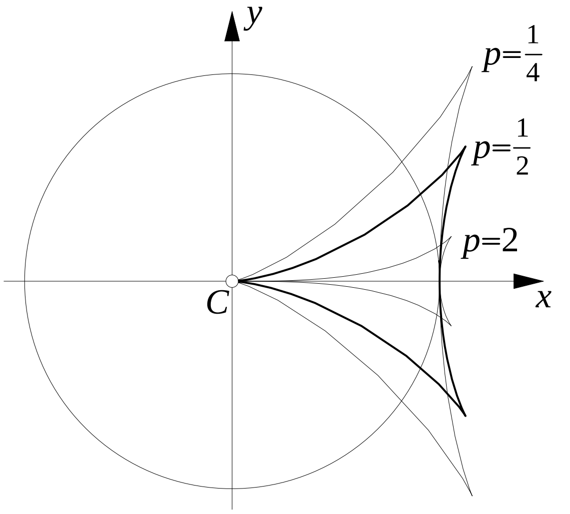

Figure 2: Meridians of rotational surfaces enveloped by the planes of isophotic elements with $X \in x$

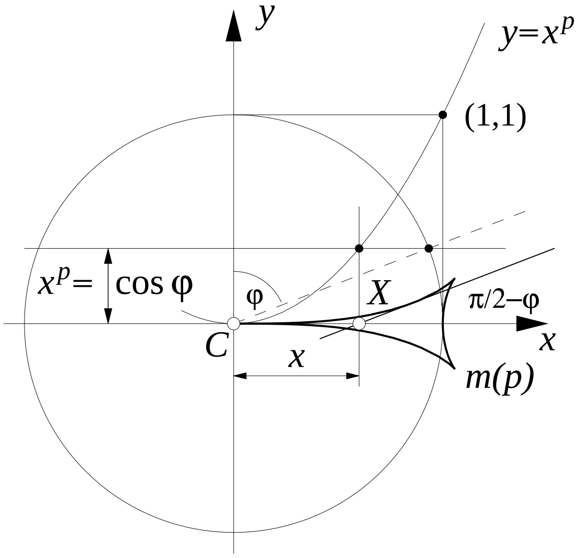

Figure 3: Construction of tangents to $m(p)$ with the help of the parabola (hyperbola) $y=x^p$

In Fig. 3 a simple construction of $m(p)$ by tangents and with the help of the unit parabola (or unit hyperbola) $y=x^p$ is shown.

4. Isophotic surfaces

Based on the above mentioned principles for the generation of isophotic families and strips of hypersurface elements we will study now the characteristic strips of the integration problem, which is determined by the element cones Λ_X for $X\in\mathbb{R}^3\setminus\{C\}$. Let $(X,\mathbf{g})$ be a starting line element, consisting of the cone apex X and the direction vector $\mathbf{g}$ of a generator.

If $\mathbf{n}$ denotes the normal vector of the hyperplane touching Λ_X along $\mathbf{g}$, then by $\dot{\mathbf{x}}=\lambda\mathbf{g}$, $\lambda\in\mathbb{R}$, the vectors $\{\mathbf{x},\mathbf{n},\dot{\mathbf{n}}\}$ are linearly dependent, and the integral curve is necessarily *planar*. This means that the sinusoidal spirals described by [9] are in fact the general characteristic curves of the integration problem.

For $d=3$, this configuration is discussed in [13], p. 510, namely in terms of the *geometry of differences*: Infinitesimally neighboring element cones of a characteristic strip of elements do not only have a common generator; along this common generator they even have a common tangential plane.

It is obvious that rotational hypersurfaces with sinusoidal spirals as meridians are isophotic hypersurfaces, i.e., solution hypersurfaces of the integration problem defined by the hypersurface elements $(X, \mathbf{n})$. Each general point X belongs to a $(d-2)$-dimensional manifold of such solution hypersurfaces. Due to a theorem of S. Lie (cf. [13], p. 535), we can derive now the general solution hypersurface (from the mentioned special – but complete – solutions), after giving an arbitrary function by the operations of differentiation and elimination alone. This is formulated in our

Theorem 1. *Let there be given a central illumination* $(1')$ *with light source* $C \in \mathbb{R}^d$. *Then an isophotic portion of a* C^2*-hypersurface in the neighborhood of a regular point is either a hypersphere concentric to* C, *or it is a rotational hypersurface whose meridian is an arc of a sinusoidal spiral* s_p *of index* p, *or it is the envelope of such a rotational surface which follows a* C^2*-bundle movement with center* C.

Thus for $d = 3$ the general isophotic pieces of C^2-surfaces are *moulding surfaces*. They were not described by Paukowitsch [18]. One can generate them by an arc of a sinusoidal spiral s_p, whose plane is rolling without sliding on an arbitrarily given C^1-cone with apex C. (Obviously, for achieving an isophotic C^1-moulding surface, it is already sufficient to demand a differentiability class $r = 0$ of this base cone.)

5. Twofold isophotic strips in $\mathbb{R}^d$

The existence of isophotic (hyper-)surfaces suggests the question for manifolds of elements (i.e., for *element sets resp. strips*, cf. [13], p. 523) which are isophotic at the same time with respect to two or more light sources $C_1, \dots, C_k$ and illumination intensities $E_1, \dots, E_k$. Such a manifold is said to be *k-fold isophotic*. In general, element sets resp. strips of such a type are $(d-k+1)$-dimensional and $(d-k)$-dimensional, respectively.

In particular, one can expect that there exist certain strips in $\mathbb{R}^3$ which are isophotic with respect to two light sources C_1, C_2. The solution of this problem is obvious: One has to look for strips which are isophotic regarding to $C_2(E_2)$ and which belong to a surface Φ_1 that is "1-isophotic" with respect to $C_1(E_1)$. Then the tangential strip of Φ_1 along a 2-isophotic curve in Φ_1 must be a twofold isophotic strip.

If, in particular, $\Phi_1 \subset \mathbb{R}^3$ is a rotational surface and C_2 is a point at infinity (i.e., in this case the second illumination is a parallel illumination), then the 2-isophotic strips of Φ_1 can be found by graphic-constructive methods, cf. [17], p. 307. For central illumination from C_2, only computer-aided numerical approaches are known, see [1], p. 165.

It is geometrically obvious that in $\mathbb{R}^d$ a conic hyperplanar strip is twofold isophotic if and only if it belongs to a rotational hypercone whose axis a passes through the light sources C_1 and C_2. Then, for each point $C \in a$ different from C_1 and C_2, every such strip is isophotic with respect to a suitably chosen illumination intensity E.

We continue with considering in $\mathbb{R}^3$ the manifold of surface elements $(X, \mathbf{n})$ which are isophotic with respect to three central illuminations $C_i(E_i)$, $i = 1, 2, 3$. This manifold has (in general) a discrete set of elements in common with each 1-isophotic moulding surface Φ_1.

Therefore the set of required supporting points X is one-dimensional. The lines through X with direction $\mathbf{n}$ generate a ruled surface which can be determined by the following arguments: For each illumination $C_i(E_i)$ the normals n of the isophotic surface elements $(X, \mathbf{n})$ satisfy a line complex K_i with rotational cones $\Lambda_X^{(i)}$ as complex cones of general points X (see Fig. 3).

Hence the normal n of an element $(X, \mathbf{n})$ that is twofold isophotic with respect to $C_1(E_1)$ and $C_2(E_2)$ must belong to both line complexes K_1, K_2 and therefore is a common generator of the complex cones $\Lambda_X^{(1)}$ and $\Lambda_X^{(2)}$. Hence the element normals of twofold isophotic elements form the line congruence common to both the line complexes K_1, K_2. The element normals of threefold isophotic elements then belong to a certain ruled surface $\Psi = K_1 \cap K_2 \cap K_3$. The determination of these normals $\{n\} = \Psi$ and of the corresponding points $\{X\} \subset \Psi$ will be presented in a subsequent paper.

6. Non-Euclidean central illumination

The investigations of central illuminations in Euclidean d-space above also motivate two approaches in non-Euclidean spaces.

First, the Euclidean hypersphere S^{d-1} having a fixed light source C as its center (and presenting a singular solution regarding the determination of isophotic hypersurfaces) might be interpreted as the *absolute quadric* of a hyperbolic geometry. Then the Euclidean illumination law (1) can be rewritten in such a manner that r and φ are replaced by the hyperbolic quantities r_H and φ_H, respectively. Due to [15], p. 89, [3], p. 140, and [11], p. 177, we have

$$\begin{aligned} r_H &= \tfrac{1}{2} \ln \frac{1+r}{1-r}, \quad 0 \le r < 1\,, \\ \varphi_H &= \sphericalangle_H\, gh := 2 \arctan \sqrt{-\mathrm{CR}\,(G_1G_2, H_1H_2)}\,, \end{aligned} \tag{6}$$

where CR means the “cross-ratio” and G_i, H_i are the ideal points (in $\mathbb{C}$) of the lines g and h, respectively.

The second possibility is the following: From the beginning we might aim at a non-Euclidean space and leave the original (physically justified) illumination law (1) as it is.

As an example, we mention here only the isotropic case. For the planar situation H. Sachs [19], p. 101, gives corresponding results which (in view of our investigations above) can be extended to d-dimensional spaces, too. Then the isotropic analogues to the Euclidean sinusoidal spirals are the (general) parabolas and hyperbolas

$$y = x^p, \quad p \in \mathbb{R}\,. \tag{7}$$

Another related isotropic analogue, i.e., the curve analogous with the Euclidean logarithmic spiral, is the curve presented in Fig. 4, namely

$$y = ax \log(x). \qquad (8)$$

(The centrally symmetric continuation of this curve through its pole, given in [19], p. 96, does not make sense and gives confusion regarding the Euclidean analogue.)
Hence all the curves occurring as isotropic-isophotic ones are orbits of one-parameter groups of affinities, i.e., they are W-curves. It is remarkable that the construction of Euclidean line complex cones, presented in Fig. 3, is essentially based on these isotropic-isophotic W-curves.

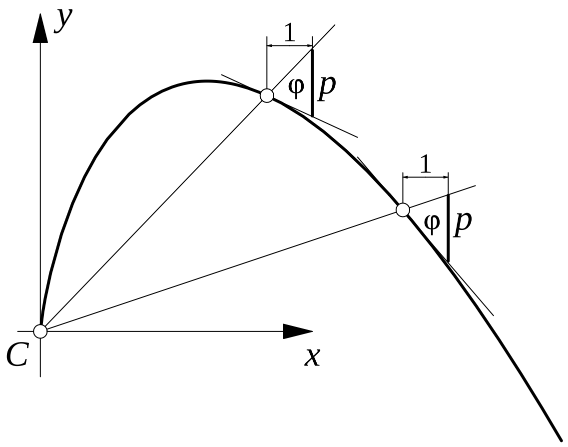

Figure 4: Isotropic spiral $y = ax \log(x)$

7. Relations of the considered curve classes to each other and to other types of curves

7.1. Clairaut and Ribaucour curves as images of spatial curves under parallel projections

There are various interesting relations between Clairaut curves, sinusoidal spirals and *Ribaucour curves*.[2] For example, we have the following relation: If a sinusoidal spiral slides through one of its (fixed) dihedrals $(X_0; p_1, p_2)$, then its pole runs through a Clairaut curve of the same index, with the dihedral apex X_0 as its own pole, and with the tangent $[p_1]$ of the spiral as tangent at its own pole, cf. [9]. On the other hand, if a sinusoidal spiral is rolling on one of its (fixed) tangents, e.g. on $[p_1]$, then its pole runs through a Ribaucour curve of the same index, see [21], p. 299.

Among the Clairaut curves we have the circle, and the ordinary cycloid is a special Ribaucour curve. These two special curves can be interpreted as images of an ordinary helix under orthogonal or oblique parallel projection onto the normal plane of the helical axis. The “intermediate cases”, namely prolate and curtate cycloids, can also be explained as images of the helix under oblique parallel projection. But there is still another interpretation: They are resulting curves of a “roll-and-slide procedure” of a circle accomplishing both movements on a tangent of it (with constant ratio of the rolling velocity and sliding velocity). The corresponding connection between Clairaut curves, sinusoidal spirals and Ribaucour curves can be explained as follows: Starting with a Clairaut curve m in the xy-plane, one has to erect an orthogonal cylinder over m. In this cylinder one has to construct a spatial curve c whose z-coordinate is proportional to the arc length of the sinusoidal spiral s which corresponds to m. A suitable oblique projection of c into the xy-plane yields planar curves which arise from s by a “roll-and-slide procedure” on a poltangent of m. A detailed investigation of this “roll-and-slide procedure” is also postponed to a subsequent paper.

[2]For the definition of Ribaucour curves and many of their interesting properties and relations to other curve classes, in particular to sinusoidal spirals, the interested reader is referred to [14], pp. 521–530, [21], pp. 309–311, and [4], pp. 54–66.

7.2. Generating roses, sinusoidal spirals and Clairaut curves as images of spatial curves under normal projection

A further class of curves connected with our investigations is that of *roses*, cf. [14], pp. 297–306, [21], pp. 123–124, and [20], pp. 304–307. One usual way to introduce these curves is to consider them as special trochoids, see [14], p. 297, and [23], p. 153. In polar coordinates they are described by

$$r = a\cos p\varphi\,, \text{ with constant } a, p \in \mathbb{R}\setminus\{0\}\,, \tag{9}$$

and thus one can interpret them as normal projections of certain spherical curves c which themselves result from the superposition of two proportional rotations around orthogonal diameters of the respective sphere.

Namely, if φ and φ are the spherical parameters (roughly speaking, the geographic longitude and latitude), then the parametric representation of c in cartesian coordinates of the (unit) sphere $S^2 \subset \mathbb{R}^3$ is given by

$$\mathbf{x}(\varphi) = (\cos\varphi \cdot \cos\varphi(\varphi), \sin\varphi \cdot \cos\varphi(\varphi), \sin\varphi(\varphi))\,, \tag{10}$$

where $\varphi(\varphi) = p\varphi + \varphi_0$. Let $S^q \subset \mathbb{R}^{q+1}$ be the unit hypersphere, and $\varphi_1, \ldots, \varphi_q$ be the spherical Gaussian parameters of S^q. Then the parametric representation of S^q analogously to (10) reads

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_q \\ x_{q+1} \end{pmatrix} = \begin{pmatrix} \cos\varphi_1 \cdot \cos\varphi_2 \cdot \cos\varphi_3 \cdot \ldots \cdot \cos\varphi_q \\ \sin\varphi_1 \cdot \cos\varphi_2 \cdot \cos\varphi_3 \cdot \ldots \cdot \cos\varphi_q \\ \sin\varphi_2 \cdot \cos\varphi_3 \cdot \ldots \cdot \cos\varphi_q \\ \vdots \\ \sin\varphi_{q-1} \cdot \cos\varphi_q \\ \sin\varphi_q \end{pmatrix}. \tag{11}$$

In particular, for $\varphi_i(\varphi) := \varphi$, $i = 1, \ldots, q$, this yields in the x_1x_2-plane the Clairaut curve

$$m \;\ldots\; r = \cos^{q-1}\varphi$$

as top view of the spherical curve c determined by $\varphi_i = \varphi$.

For $\varphi_1(\varphi) := \varphi$ and $\varphi_i(\varphi) := p \cdot \varphi$ with $i = 2, \ldots, q$ we obtain for example the generalized sinusoidal spiral

$$s \;\ldots\; r = \cos^{q-1} p\varphi$$

as top view of the corresponding hyperspherical curve c. Thus, for suitably chosen values of $p \in \mathbb{N}$ one can generate certain algebraic sinusoidal spirals by projection.

The relations between the spherical curves c and *roulettes* of higher order (see, e.g., [23], p. 164) and between projections of the curves c into different coordinate planes should be taken into deeper consideration.

7.3. Geometric transformations of sinusoidal spirals and Clairaut curves

There exists a lot of mathematical literature about transformations which transfer curves of a given family into other representatives of the same family. In this sense we will discuss relations between Clairaut curves and also between sinusoidal spirals.

a) It is well-known that Clairaut curves m_1 and m_2, whose respective coefficients satisfy $p_1 = -p_2$, are inverse with respect to the unit circle having their common pole C as its center. One can interpret this unit circle as borderline case of a Clairaut curve with

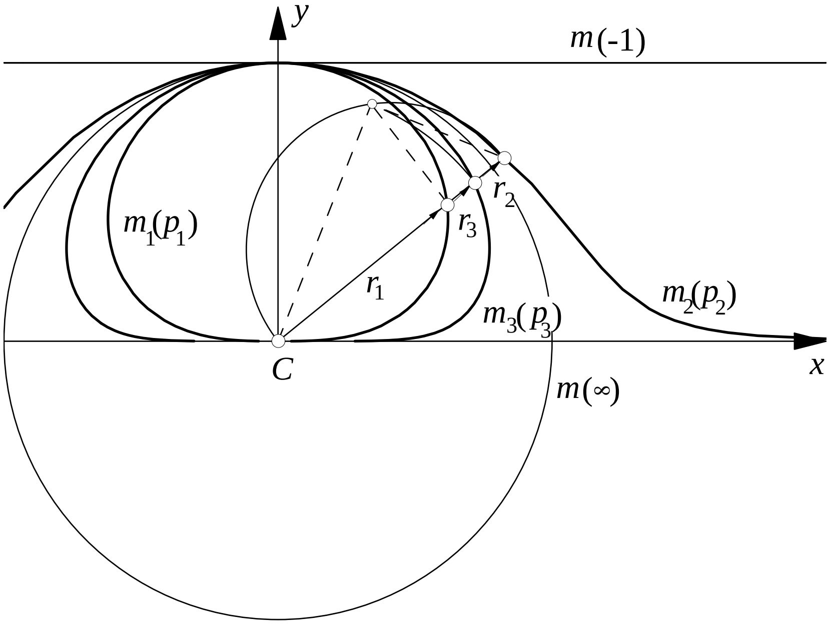

Figure 5: Inversion of a Clairaut curve $m_1(p_1)$ at a Clairaut curve $m_3(p_3)$

index $p = \infty$. This gives the motivation to extend the notion of inversion such that Clairaut curves themselves are taken as basic curves of inversion (Fig. 5).

Two Clairaut curves $m_1(p_1), m_2(p_2)$ of the family described by $(1'')$ (where the exponent $p \in \mathbb{R}$ is the parameter of this family) are said to be inverse to each other with respect to a third Clairaut curve $m_3(p_3)$ if the exponents satisfy the relation

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}.$$

This means that the radius r_3 is the geometric mean of the radii r_1 and r_2, as in the case of usual inversion. In Fig. 5 the generalized inversion at a Clairaut curve $m_3(p_3)$ is displayed.

b) The pedal curve of a sinusoidal spiral s (cf. (3)) of index p with respect to its pole C is a sinusoidal spiral s_1 of index $q = \dfrac{p}{p+1}$, see [14], p. 675, and [21], p. 137. Fig. 6 presents a simple construction of the series of pedal curves of a sinusoidal spiral.
Let P_1 be the foot of the tangent t of s at a point P, and $P^* \in t$ be the reflection of P at P_1, cf. Fig. 6. Then a simple calculation shows the following (up to now seemingly unknown) connection between s and its second pedal curve s_2 (as well as all the subsequent pedal curves).

Theorem 2. *If one reflects a point P of a sinusoidal spiral s at the polar normal g_1 to the tangent t of P, then the tangent t_1 of the first pedal curve s_1 at $P_1 = t \cdot g_1$ is normal to the reflex g^* of the polar ray $g = OP$, and it will intersect g^* at a point P_2 of the second pedal curve s_2 of s, where*

$$\overline{CP} \cdot \overline{CP}_2 = \left(\overline{CP}_1\right)^2 . \tag{12}$$

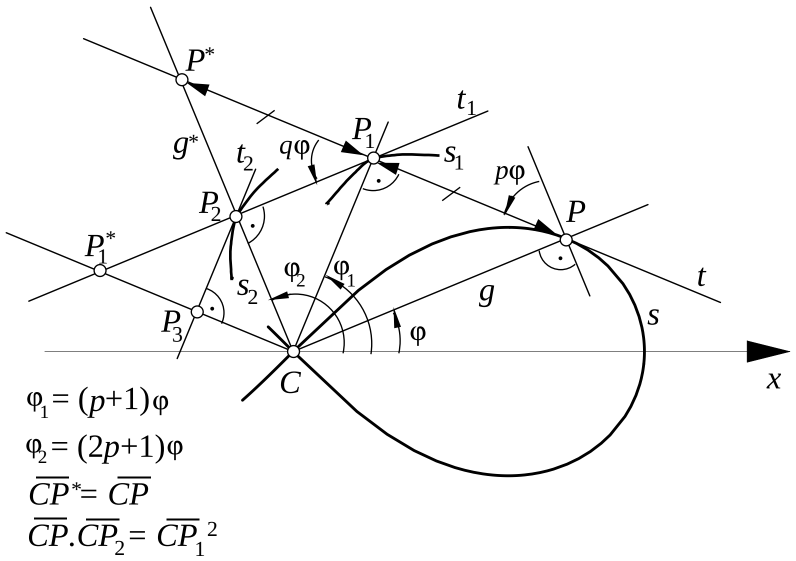

Figure 6: Successive construction of the family of pedal spirals to a given sinusoidal spiral s.

It is obvious that this construction can easily be used to derive sinusoidal spirals (cf. once more Fig. 6), and this construction is an extension of a well known one applied to logarithmic spirals which are borderline cases of sinusoidal spirals.

Finally we remark that the point P^* (depending on $P \in s$) runs through the *generalized sinusoidal spiral* s^* given by

$$r^p = a \cdot \cos \frac{p}{2p+1} \varphi \,, \tag{13}$$

and that the sequence of feet $\ldots, P_{-2}, P_{-1}, P, P_1, P_2, \ldots$ corresponding with a fixed starting point P belongs to a *logarithmic spiral.*

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