EXAMPLES OF GIBBS STATES OF MECHANICAL SYSTEMS WITH SYMMETRIES

CHARLES-MICHEL MARLE

Communicated by Ivailo M. Mladenov

Abstract. In a previous paper, the notion of Gibbs state for the Hamiltonian action of a Lie group on a symplectic manifold was given, together with its applications in Statistical Mechanics, and the works in this field of the French mathematician and physicist Jean-Marie Souriau were presented. Using an adaptation of the cross product for pseudo-Euclidean three-dimensional vector spaces, we present several examples of such Gibbs states, together with the associated thermodynamic functions, for various two-dimensional symplectic manifolds, including the pseudo-spheres, the Poincaré disk and the Poincaré half-plane.

MSC: 53D05, 53D20, 53D17, 82B03, 82B30

Keywords: Gibbs states, Hamiltonian systems, Liouville measure, Hodge operator, Möbius transformations, moment maps, Poincaré disk, Poincaré half-plane, symplectic and Poisson manifolds, thermodynamic equilibrium

Contents

1 Introduction 56
2 Three-Dimensional Real Oriented Vector Spaces with a Scalar Product 56
   2.1 Admissible Bases and Symmetry Groups . . . . . . . . . . . . . . . . . . 56
   2.2 A Remarkable Lie Algebras Isomorphism . . . . . . . . . . . . . . . . . 57
   2.3 Expression of the Map $j$ in Terms of the Hodge Star Operator . . . 59
   2.4 Metric, Lie Algebra and Lie-Poisson Structures of $F$ ..................... 60
   2.5 Coadjoint Orbits of $G$ as Submanifolds of $F$ ......................... 61
3 Gibbs States on Some Two-Dimensional Symplectic Manifolds 62
   3.1 Gibbs States on Two-Dimensional Spheres . . . . . . . . . . . . . . . . . 63
   3.2 Gibbs States on Pseudo-Spheres and Other $SO(2, 1)$ Coadjoint Orbits . 64
   3.3 Gibbs States on the Poincaré Disk ................................. 66
   3.4 Gibbs States on the Poincaré Half-Plane ........................... 71
   3.5 No Gibbs State Can Exist on a Two-Dimensional Symplectic Vector Space 74
   3.6 Gibbs States on an Affine Euclidean and Symplectic Plane ............... 75
4 Final Comments 77
References 78