Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 3, Article 75, 2004

RATIONAL IDENTITIES AND INEQUALITIES

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Received 25 December, 2003; accepted 27 June, 2004
Communicated by L. Tóth

ABSTRACT. Recently, in [4] the author studied some rational identities and inequalities involving Fibonacci and Lucas numbers. In this paper we generalize these rational identities and inequalities to involve a wide class of sequences.

Key words and phrases: Rational Identities and Inequalities, Fibonacci numbers, Lucas numbers, Pell numbers.

2000 *Mathematics Subject Classification.* 05A19, 11B39.

1. INTRODUCTION

The Fibonacci and Lucas sequences are a source of many interesting identities and inequalities. For example, Benjamin and Quinn [1], and Vajda [5] gave combinatorial proofs for many such identities and inequalities. Recently, Díaz-Barrero [4] (see also [2, 3]) introduced some rational identities and inequalities involving Fibonacci and Lucas numbers. A sequence $(a_n)_{n\geq 0}$ is said to be *positive increasing* if $0 < a_n < a_{n+1}$ for all $n \geq 1$, and *complex increasing* if $0 < |a_n| \leq |a_{n+1}|$ for all $n \geq 1$. In this paper, we generalize the identities and inequalities which are given in [4] to obtain several rational identities and inequalities involving positive increasing sequences or complex sequences.

2. IDENTITIES

In this section we present several rational identities and inequalities by using results on contour integrals.

Theorem 2.1. *Let $(a_n)_{n\geq 0}$ be any complex increasing sequence such that $a_p \neq a_q$ for all $p \neq q$. For all positive integers r,*

$$\sum_{k=1}^{n}\left(\frac{1+a_{r+k}^{\ell}}{a_{r+k}}\prod_{j=1,\,j\neq k}^{n}(a_{r+k}-a_{r_j})^{-1}\right)=\frac{(-1)^{n+1}}{\prod_{j=1}^{n}a_{r+j}}$$

ISSN (electronic): 1443-5756

The author is grateful to Díaz-Barrero for his careful reading of the manuscript.

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holds, with $0 \le \ell \le n-1$.

Proof. Let us consider the integral

$$I = \frac{1}{2\pi i}\oint_\gamma \frac{1+z^\ell}{zA_n(z)}dz,$$

where $\gamma = \{z \in \mathbb{C} : |z| < |a_{r+1}|\}$ and $A_n(z) = \prod_{j=1}^n (z - a_{r+j})$. Evaluating the integral I in the exterior of the γ contour, we get $I_1 = \sum_{k=1}^n R_k$ where

$$R_k = \lim_{z\to a_{r+k}} \left(\frac{1+z^\ell}{z} \prod_{j=1,\, j\neq k}^n (z - a_{r_j})^{-1} \right) = \frac{1+a_{r+k}^\ell}{a_{r+k}} \prod_{j=1,\, j\neq k}^n (a_{r+k} - a_{r_j})^{-1}.$$

On the other hand, evaluating I in the interior of the γ contour, we obtain

$$I_2 = \lim_{z\to 0} \frac{1+z}{A_n(z)} = \frac{1}{A_n(0)} = \frac{(-1)^n}{\prod_{j=1}^n a_{r+j}}.$$

Using Cauchy's theorem on contour integrals we get that $I_1 + I_2 = 0$, as claimed. □

Theorem 2.1 for $a_n = F_n$ the n Fibonacci number ($F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \ge 0$) gives [4, Theorem 2.1], and for $a_n = L_n$ the n Lucas number ($L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for all $n \ge 0$) gives [4, Theorem 2.2]. As another example, Theorem 2.1 for $a_n = P_n$ the nth Pell number ($P_0 = 0$, $P_1 = 1$, and $P_{n+2} = P_{n+1} + P_n$ for all $n \ge 0$) we get that

$$\sum_{k=1}^n \left(\frac{1+P_{r+k}^\ell}{P_{r+k}} \prod_{j=1,\, j\neq k}^n (P_{r+k} - P_{r_j})^{-1} \right) = \frac{(-1)^{n+1}}{\prod_{j=1}^n P_{r+j}}$$

holds, with $0 \le \ell \le n-1$. In particular, we obtain

Corollary 2.2. *For all* $n \ge 2$,

$$\frac{(P_n^2+1)P_{n+1}P_{n+2}}{(P_{n+1}-P_n)(P_{n+2}-P_n)} + \frac{P_n(P_{n+1}^2+1)P_{n+2}}{(P_n-P_{n+1})(P_{n+2}-P_{n+1})} + \frac{P_nP_{n+1}(P_{n+2}^2+1)}{(P_n-P_{n+2})(P_{n+1}-P_{n+2})} = 1.$$

Theorem 2.3. *Let* $(a_n)_{n\ge 0}$ *be any complex increasing sequence such that* $a_p \neq a_q$ *for all* $p \neq q$. *For all* $n \ge 2$,

$$\sum_{k=1}^n \frac{1}{a_k^{n-2}} \prod_{j=1,\, j\neq k}^n \left(1 - \frac{a_j}{a_k}\right) = 0.$$

Proof. Let us consider the integral

$$I = \frac{1}{2\pi i}\oint_\gamma \frac{z}{A_n(z)}dz,$$

where $\gamma = \{z \in \mathbb{C} : |z| < |a_{n+1}|\}$ and $A_n(z) = \prod_{j=1}^n (z - a_{r+j})$. Evaluating the integral I in the exterior of the γ contour, we get $I_1 = 0$. Evaluating I in the interior of the γ contour, we obtain

$$I_2 = \sum_{k=1}^n \mathrm{Res}(z/A_n(z); z = a_k) = \sum_{k=1}^n \prod_{j=1,\, j\neq k}^n \frac{a_k}{a_k - a_j} = \sum_{k=1}^n \frac{1}{a_k^{n-2}} \prod_{j=1,\, j\neq k}^n \left(1 - \frac{a_j}{a_k}\right).$$

Using Cauchy's theorem on contour integrals we get that $I_1 + I_2 = 0$, as claimed. □

For example, Theorem 2.3 for $a_n = L_n$ the nth Lucas number gives [4, Theorem 2.5]. As another example, Theorem 2.3 for $a_n = P_n$ the nth Pell number obtains, for all $n \geq 2$,

$$\sum_{k=1}^{n} \frac{1}{P_k^{n-2}} \prod_{j=1,\, j\neq k}^{n} \left(1 - \frac{P_j}{P_k}\right) = 0.$$

3. INEQUALITIES

In this section we suggest some inequalities on positive increasing sequences.

Theorem 3.1. *Let* $(a_n)_{n\geq 0}$ *be any positive increasing sequence such that* $a_1 \geq 1$. *For all* $n \geq 1$,

$$a_n^{a_{n+1}} + a_{n+1}^{a_n} < a_n^{a_n} + a_{n+1}^{a_{n+1}}. \tag{3.1}$$

and

$$a_{n+1}^{a_{n+2}} - a_{n+1}^{a_n} < a_{n+2}^{a_{n+2}} - a_{n+2}^{a_n}. \tag{3.2}$$

Proof. To prove (3.1) we consider the integral

$$I = \int_{a_n}^{a_{n+1}} (a_{n+1}^x \log a_{n+1} - a_n^x \log a_n) dx.$$

Since a_n satisfies $1 \leq a_n < a_{n+1}$ for all $n \geq 1$, so for all x, $a_n \leq x \leq a_{n+1}$ we have that

$$a_n^x \log a_n < a_{n+1}^x \log a_n < a_{n+1}^x log a_{n+1},$$

hence $I > 0$. On the other hand, evaluating the integral I directly, we get that

$$I = (a_{n+1}^{a_{n+1}} - a_n^{a_{n+1}}) - (a_{n+1}^{a_n} - a_n^{w_n}),$$

hence

$$a_n^{a_{n+1}} + a_{n+1}^{a_n} < a_n^{a_n} + a_{n+1}^{a_{n+1}}$$

as claimed in (3.1). To prove (3.2) we consider the integral

$$J = \int_{a_n}^{a_{n+2}} (a_{n+2}^x \log a_{n+2} - a_{n+1}^x \log a_{n+1}) dx.$$

Since a_n satisfies $1 \leq a_{n+1} < a_{n+2}$ for all $n \geq 0$, so for all x, $a_{n+1} \leq x \leq a_{n+2}$ we have that

$$a_{n+1}^x \log a_{n+1} < a_{n+2}^x \log a_{n+2},$$

hence $J > 0$. On the other hand, evaluating the integral J directly, we get that

$$I = (a_{n+2}^{a_{n+2}} - a_{n+2}^{a_n}) - (a_{n+1}^{a_{n+2}} - a_{n+1}^{a_n}),$$

hence

$$a_{n+1}^{a_{n+2}} - a_{n+1}^{a_n} < a_{n+2}^{a_{n+2}} - a_{n+2}^{a_n}$$

as claimed in (3.2). □

For example, Theorem 3.1 for $a_n = L_n$ the nth Lucas number gives [4, Theorem 3.1]. As another example, Theorem 3.1 for $a_n = P_n$ the nth Pell number obtains, for all $n \geq 1$,

$$P_n^{P_{n+1}} + P_{n+1}^{P_n} < P_n^{P_n} + P_{n+1}^{P_{n+1}},$$

where P_n is the nth Pell number.

Theorem 3.2. *Let* $(a_n)_{n\geq 0}$ *be any positive increasing sequence such that* $a_1 \geq 1$. *For all* $n, m \geq 1$,

$$a_{n+m}^{a_n} \prod_{j=0}^{m-1} a_{n+j}^{a_{n+j+1}} < \prod_{j=0}^{m} a_{n+j}^{a_{n+j}}.$$

Proof. Let us prove this theorem by induction on m. Since $1 \le a_n < a_{n+1}$ for all $n \ge 1$ then $a_n^{a_{n+1}-a_n} < a_{n+1}^{a_{n+1}-a_n}$, equivalently, $a_n^{a_{n+1}} a_{n+1}^{a_n} < a_n^{a_n} a_{n+1}^{a_{n+1}}$, so the theorem holds for $m = 1$. Now, assume for all $n \ge 1$

$$a_{n+m-1}^{a_n} \prod_{j=0}^{m-2} a_{n+j}^{a_{n+j+1}} < \prod_{j=0}^{m-1} a_{n+j}^{a_{n+j}}.$$

On the other hand, similarly as in the case $m = 1$, for all $n \ge 1$,

$$a_{n+m-1}^{a_{n+m}-a_n} < a_{n+m}^{a_{n+m}-a_n}.$$

Hence,

$$a_{n+m-1}^{a_{n+m}-a_n} a_{n+m-1}^{a_n} \prod_{j=0}^{m-2} a_{n+j}^{a_{n+j+1}} < a_{n+m}^{a_{n+m}-a_n} \prod_{j=0}^{m-1} a_{n+j}^{a_{n+j}},$$

equivalently,

$$a_{n+m}^{a_n} \prod_{j=0}^{m-1} a_{n+j}^{a_{n+j+1}} < \prod_{j=0}^{m} a_{n+j}^{a_{n+j}},$$

as claimed. □

Theorem 3.2 for $a_n = L_n$ the nth Lucas number and $m = 3$ gives [4, Theorem 3.3].

Theorem 3.3. *Let* $(a_n)_{n\ge 0}$ *and* $(b_n)_{n\ge 0}$ *be any two sequences such that* $0 < a_n < b_n$ *for all* $n \ge 1$*. Then for all* $n \ge 1$*,*

$$\sum_{i=1}^{n} (b_j + a_j) \ge \frac{2n^{n+1}}{(n+1)^n} \prod_{i=1}^{n} \frac{b_j^{1+1/n} - a_j^{1+1/n}}{b_j - a_j}.$$

Proof. Using the AM-GM inequality, namely

$$\frac{1}{n} \sum_{i=1}^{n} x_i \ge \prod_{i=1}^{n} x_i^{1/n},$$

where $x_i > 0$ for all $i = 1, 2, \dots, n$, we get that

$$\int_{b_1}^{a_1} \cdots \int_{b_n}^{a_n} \frac{1}{n} \sum_{i=1}^{n} x_i dx_1 \cdots dx_n \ge \int_{b_1}^{a_1} \cdots \int_{b_n}^{a_n} \prod_{i=1}^{n} x_i^{1/n} dx_1 \cdots dx_n,$$

equivalently,

$$\frac{1}{2n} \sum_{i=1}^{n} (b_i^2 - a_i^2) \prod_{j=1,\, j\ne i}^{n} (b_j - a_j) \ge \prod_{i=1}^{n} \left(\frac{n}{n+1} (b_i^{1+1/n} - a_i^{1+1/n}) \right),$$

hence, on simplifying the above inequality we get the desired result. □

Theorem 3.3 for $a_n = L_n^{-1}$ where L_n is the nth Lucas number and $b_n = F_n^{-1}$ where F_n is the nth Fibonacci number gives [4, Theorem 3.4].

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