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OSTROWSKI TYPE INEQUALITIES FROM A LINEAR FUNCTIONAL POINT OF VIEW

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ABSTRACT. Inequalities are obtained using P_0-simple functionals. Applications to Lipschitzian mappings are given.

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1. INTRODUCTION

Let I be a bounded interval of the real axis. We denote by $B(I)$ the set of all functions which are bounded on $[a, b]$.

Let A be a positive linear functional $A : B(I) \to \mathbb{R}$, such that $A(e_0) = 1$, where $e_i : I \to \mathbb{R}$, $e_i(x) = x^i$, $\forall\, x \in I,\ i \in \mathbb{N}$.

The following inequality is known in literature as the Grüss inequality for the functional A.

Theorem 1.1. *Let $f, g : I \to \mathbb{R}$ be two bounded functions such that $m_1 \le f(x) \le M_1$ and $m_2 \le g(x) \le M_2$ for all $x \in I,\ m_1, M_1, m_2$ and M_2 are constants. Then the inequality:*

$$|A(fg) - A(f)A(g)| \le \frac{1}{4}(M_1 - m_1)(M_2 - m_2) \tag{1.1}$$

holds.

In 1938 Ostrowski (cf. for example [7, p. 468]) proved the following result:

Theorem 1.2. *Let $f : I \to \mathbb{R}$ be continuous on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e.*

$$\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty.$$

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Then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.2}$$

for all $x\in[a,b]$. *The constant* $\frac{1}{4}$ *is best.*

In the recent paper [4] S.S. Dragomir and S. Wang proved the following version of Ostrowski's inequality.

Theorem 1.3. *Let* $f:I\to\mathbb{R}$ *be a differentiable mapping in the interior of* I *and* $a,b\in int(I)$ *with* $a<b$. *If* $f'\in L_1[a,b]$ *and* $\gamma\le f'(x)\le\Gamma$ *for all* $x\in[a,b]$ *then we have the following inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\le\frac{1}{4}(b-a)(\Gamma-\gamma) \tag{1.3}$$

for all $x\in[a,b]$.

The following inequality for mappings with bounded variation can be found in [1]:

Theorem 1.4. *Let* $f:I\to\mathbb{R}$ *be a mapping of bounded variation. Then for all* $x\in[a,b]$ *we have the inequality*

$$\left|\int_a^b f(t)dt-f(x)(b-a)\right|\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b f, \tag{1.4}$$

where $\bigvee_a^b f$ *denotes the total variation of* f.

The constant $\frac{1}{2}$ *is the best possible one.*

In [2] S.S. Dragomir gave the following result for Lipschitzian mappings:

Theorem 1.5. *Let* $f:[a,b]\to\mathbb{R}$ *be an* L*-Lipschitzian mapping on* $[a,b]$*, i.e.*

$$|f(x)-f(y)|\le L|x-y|,\ \text{for all } x,y\in[a,b].$$

Then we have the inequality

$$\left|\int_a^b f(t)dt-f(x)(b-a)\right|\le L(b-a)^2\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.5}$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ *is the best possible one.*

S.S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang in [3] proved the following theorem:

Theorem 1.6. *Let* $f,w:(a,b)\subseteq\mathbb{R}\to\mathbb{R}$ *be so that* $w(s)\ge 0$ *on* (a,b), w *is integrable on* (a,b) *and* $\int_a^b w(s)ds>0$, f *is of* r*-Hölder type, i.e.*

$$|f(x)-f(y)|\le H|x-y|^r,\ \text{for all } x,y\in(a,b) \tag{1.6}$$

where $H>0$ *and* $r\in(0,1]$ *are given. If* $w,f\in L_1(a,b)$, *then we have the inequality:*

$$\left|f(x)-\frac{1}{\int_a^b w(s)ds}\int_a^b w(s)f(s)ds\right|\le H\frac{1}{\int_a^b w(s)ds}\int_a^b|x-s|^r w(s)ds \tag{1.7}$$

for all $x\in(a,b)$.

The constant factor 1 in the right hand side cannot be replaced by a smaller one.

The aim of this paper is to improve the results from Theorems 1.1 – 1.6 using an unitary method.

2. AUXILIARY RESULTS

Let $X = (X, d)$ be a compact metric space and $C(X)$ the Banach lattice of real-valued continuous functions on the compact metric space $X = (X, d)$, endowed with the max norm $\|\cdot\|_X$.

For a function $f \in C(X)$, the modulus of continuity (with respect to the metric d) is defined by:

$$\omega(f;t) = \omega_d(f;t) = \sup_{d(x,y)\le t} |f(x) - f(y)|, \quad t \ge 0.$$

The least concave majorant of this modulus with respect to the variable t is given by

$$\widetilde{\omega}(f;t) = \begin{cases} \sup\limits_{\substack{0\le x\le t\le y \\ x\ne y}} \frac{(t-x)\omega(f;y)+(y-t)\omega(f;x)}{y-x} & \text{for } 0 \le t \le d(X); \\ \\ \omega(f;d(X)) & \text{for } t > d(X), \end{cases}$$

where $d(X) < \infty$ is the diameter of the compact space X.

We denote by $Lip_M\alpha = Lip_M(\alpha; X)$ the set of all Lipschitzian functions of order $\alpha,\ \alpha \in [0,1]$ having the same Lipschitz constant M. That is $f \in Lip_M\alpha$ iff for all $x, y \in X$

$$|f(x) - f(y)| \le M d^\alpha(x,y).$$

We see that

$$Lip_M(\alpha; X) = \{g \in C(X) :\ \omega(g;t) \le Mt^\alpha\}.$$

Let $I = [a,b]$ be a compact interval of the real axis, S a subspace of $C(I)$, and A a linear functional defined on S. The following definition was given by T. Popoviciu in [8].

Definition 2.1. The linear functional A defined on the subspace S which contains all polynomials is P_n-simple $(n \ge -1)$ if

(i) $A(e_{n+1}) \ne 0$

(ii) for every $f \in S$ there are the distinct points $t_1, t_2, \ldots, t_{n+2}$ in $[a,b]$ such that

$$A(f) = A(e_{n+1})[t_1, t_2, \ldots, t_{n+2}; f],$$

where $[t_1, t_2, \ldots, t_{n+2}; f]$ is the divided difference of the function f on the points $t_1, t_2, \ldots, t_{n+2}$.

In [5] the following result is proved. The proof is reproduced here for completeness.

Theorem 2.1. *Let A be a bounded linear functional, $A : C(I) \to \mathbb{R}$. If A is P_0-simple then for all $f \in C(I)$ we have*

$$|A(f)| \le \frac{\|A\|}{2}\widetilde{\omega}\left(f; \frac{2|A(e_1)|}{\|A\|}\right). \tag{2.1}$$

Proof. For $g \in C^1(I)$ we have

$$\begin{aligned} |A(f)| &= |A(f-g) + A(g)| \le \|A\|\|f-g\| + |A(g)| \\ &\le \|A\|\|f-g\| + |A(e_1)|\|g'\|. \end{aligned}$$

From this inequality we obtain

$$|A(f)| \le \inf_{g\in C^1(I)} (\|A\|\|f-g\| + |A(e_1)|\|g'\|)$$

and using the following result (see [10])

$$\inf_{g\in C^1(I)} \left(\|f-g\| + \frac{t}{2}\|g'\|\right) = \frac{1}{2}\widetilde{\omega}(f;t), \quad t \ge 0$$

we obtain the relation (2.1). □

The following result was proved by I. Raşa [9].

Theorem 2.2. *Let k be a natural number such that $0 \leq k \leq n$ and $A : C^{(k)}[a,b] \to \mathbb{R}$ a bounded linear functional, $A \neq 0$, $A(e_i) = 0$ for $i = 0, 1, \ldots, n$ such that for every $f \in C^{(k)}[a,b]$ P_n-nonconcave $A(f) \geq 0$. Then A is P_n-simple.*

A function $f \in C^{(k)}[a,b]$ is called P_0-nonconcave if for any $n+2$ points $t_1, t_2, \ldots, t_{n+2} \in [a,b]$ the inequality

$$[t_1, t_2, \ldots, t_{n+2}; f] \geq 0$$

holds.

Another criterion for P_n-simple functionals was given by A. Lupaş in [6]. He proved that a bounded linear functional $A : C[a,b] \to \mathbb{R}$ for which $A(e_k) = 0$, $k = 0, 1, \ldots, n$ and $A(e_{n+1}) \neq 0$ is P_n-simple if and only if A is P_n-simple on $C^{(n+1)}[a,b]$.

Now we can prove the following result (see also [5]):

Theorem 2.3. *Let A be a bounded linear functional, $A : C(I) \to \mathbb{R}$. If $A(e_1) \neq 0$ and the inequality (2.1) holds for any $f \in C(I)$ then A is P_0-simple.*

Proof. We can assume that $A(e_1) > 0$. Combining the results of I. Raşa and A. Lupaş, it is sufficient, for the proof of the theorem, to show that

$$A(f) \geq 0 \tag{2.2}$$

for every nondecreasing differentiable function f defined on I.

For such a function we have

$$|A(f)| \leq A(e_1)\|f'\|.$$

Let B be the linear functional defined by

$$B(f) = \frac{A(F)}{A(e_1)},$$

where

$$F(t) = \int_0^t f(u)du, \quad f \in C[0,1].$$

The functional B is bounded and for any $f \in C(I)$ we have

$$|B(f)| \leq \|f\|$$

with $B(e_0) = 1$.

Let f be a continuous function such that $f \geq 0$, $f \neq 0$.

From the inequalities

$$0 \leq e_0 - \frac{f}{\|f\|} \leq 1$$

we obtain

$$1 - \frac{B(f)}{\|f\|} \leq \left| B\left(e_0 - \frac{f}{\|f\|}\right) \right| \leq 1.$$

These inequalities imply that

$$B(f) \geq 0. \tag{2.3}$$

Further, let f be a differentiable function on I such that $f' \geq 0$, then, from (2.3) we obtain

$$B(f') \geq 0.$$

Since $B(f') = A(f)$, the inequality (2.2) is thus proved. □

3. AN INTEGRAL INEQUALITY OF OSTROWSKI TYPE

The following inequality of Ostrowski type holds.

Theorem 3.1. *Let f be a continuous function on $[a,b]$ and $w:(a,b)\to\mathbb{R}_+$ an integrable function on (a,b) such that $\int_a^b \omega(s)ds = 1$. Then for any continuous function f the following inequality:*

$$\left|f(x)-\int_a^b w(s)f(s)ds\right| \le \left(\int_a^x w(t)dt\right)\widetilde{\omega}_{[a,x]}\left(f;\frac{\int_a^x w(t)(x-t)dt}{\int_a^x w(t)}\right) + \left(\int_x^b w(t)dt\right)\widetilde{\omega}_{[x,b]}\left(f;\frac{\int_x^b w(t)(t-x)dt}{\int_x^b w(t)dt}\right) \tag{3.1}$$

holds, where x is a fixed point in (a,b).

Proof. From Theorem 2.3 we get that the linear functionals

$$A_1 : C[a,x]\to\mathbb{R}, \quad A_2 : C[x,b]\to\mathbb{R}$$

defined by

$$A_1(f) = f(x)\int_a^x w(t)dt - \int_a^x f(t)w(t)dt$$

and

$$A_2(f) = f(x)\int_x^b w(t)dt - \int_x^b f(t)w(t)dt$$

are P_0-simple.

It is easy to see that:

$$\|A_1\| = 2\int_a^x w(t)dt \qquad \text{and} \qquad \|A_2\| = 2\int_x^b w(t)dt.$$

From the inequality:

$$\left|f(x)-\int_a^b w(s)f(s)ds\right| \le \left(\int_a^x w(t)dt\right)\widetilde{\omega}\left(f;\frac{|A_1(e_1)|}{\int_a^x w(t)dt}\right)+\left(\int_x^b w(t)dt\right)\widetilde{\omega}\left(f;\frac{A_2(e_1)}{\int_x^b w(t)dt}\right)$$

and from the results

$$|A_1(e_1)| = \int_a^x w(t)(x-t)dt \qquad \text{and} \qquad |A_2(e_1)| = \int_x^b w(t)(t-x)dt,$$

(3.1) follows. □

Corollary 3.2. *Let f be a continuous function on $[a,b]$, such that $f\in Lip_{M_1}(\alpha,[a,x])$ and $f\in Lip_{M_2}(\beta;[x,b])$. Then*

$$\left|f(x)-\int_a^b w(s)f(s)ds\right| \le M_1\left(\int_a^x w(t)dt\right)^{1-\alpha}\left[\int_a^x w(t)(x-t)dt\right]^{\alpha} + M_2\left(\int_x^b w(t)dt\right)^{1-\beta}\left[\int_x^b w(t)(t-x)dt\right]^{\beta}. \tag{3.2}$$

Proof. The proof follows from the inequality (3.1) and the fact that

$$\widetilde{\omega}_1(g;t)\le Mt^r$$

for any function $g,\ g\in Lip_M(\alpha,[c,d])$, where $\widetilde{\omega}_1$ is taken on the interval $[c,d]$. □

Corollary 3.2 is an improvement of the result of Theorem 1.6.

Remark 3.1. In the particular case when $w(t) = \frac{1}{b-a}$ the inequality (3.2) becomes:

$$\left|f(x) - \frac{\int_a^b f(s)ds}{b-a}\right| \leq \left[M_1 \frac{(x-a)^{\alpha+1}}{2^\alpha} + M_2 \frac{(b-x)^{\beta+1}}{2^\beta}\right] \frac{1}{b-a} \tag{3.3}$$
$$\leq \max(M_1, M_2) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a).$$

Inequality (3.3) improves the inequality (1.5).

Corollary 3.3. *Let $f : [a,b] \to \mathbb{R}$ be continuous on (a,b), whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b) and w a function as in Theorem 3.1. Then we have the following inequality:*

$$\left|f(x) - \int_a^b w(s)f(s)ds\right| \leq \left[\int_a^x w(t)(x-t)dt + \int_x^b w(t)(t-x)dt\right] \|f'\|_\infty. \tag{3.4}$$

Proof. The above inequality is a consequence of the inequality (3.1) and the fact that

$$\widetilde{\omega}(f;t) \leq \|f'\|_\infty t.$$

The inequality of Ostrowski follows from (3.4) if we consider

$$w(t) = \frac{1}{b-a}, \quad t \in [a,b].$$

□

Corollary 3.4. *Let $f : I \to \mathbb{R}$ be a mapping with bounded variation and w a function as in Theorem 3.1. Then for all $x \in [a,b]$ we have the inequalities*

$$\left|f(x) - \int_a^b w(s)f(s)ds\right| \leq \bigvee_a^x f \int_a^x w(t)dt + \bigvee_x^b f \int_x^b w(t)dt \tag{3.5}$$

$$\left|f(x) - \int_a^b w(s)f(s)ds\right| \leq \left(\frac{1}{2} + \frac{\left|\int_a^x w(t)dt - \int_x^b w(t)dt\right|}{2}\right) \bigvee_a^b f. \tag{3.6}$$

Proof. It is clear that

$$\widetilde{\omega}[a,x](f;t) \leq \bigvee_a^x f \qquad \text{and} \qquad \widetilde{\omega}[x,b](f,t) \leq \bigvee_x^b f \tag{3.7}$$

for every positive number t.

Thus, inequality (3.5) follows from (3.7).

For the proof of the inequality (3.6) we note that, if we suppose $\int_a^x w(t)dt \leq \frac{1}{2}$ then $\int_x^b w(t)dt \geq \frac{1}{2}$ and vice versa.

For definiteness we assume that

$$\int_a^x w(t)dt \leq \frac{1}{2} \qquad \text{and} \qquad \int_x^b w(t)dt \geq \frac{1}{2}.$$

We then have

$$\bigvee_a^x f \int_a^x w(t)dt + \bigvee_x^b f \int_x^b w(t)dt \leq \frac{1}{2}\bigvee_a^x f + \bigvee_x^b f \int_x^b w(t)dt$$
$$= \frac{1}{2}\bigvee_a^b f + \bigvee_x^b f \left(\int_x^b w(t)dt - \frac{1}{2}\right)$$

and so

$$\bigvee_a^x f \int_a^x w(t)dt + \bigvee_x^b f \int_x^b w(t)dt \le \left(\frac{1}{2} + \frac{\int_x^b w(t)dt - \int_a^x w(t)dt}{2}\right) \bigvee_a^b f. \tag{3.8}$$

From the inequalities (3.5) and (3.8), the inequality (3.6) follows. □

Remark 3.2. The inequality from Theorem 1.4 follows if we take in (3.6)

$$w(t) = \frac{1}{b-a}.$$

Theorem 3.5. *Let g be a continuous differentiable function on $[a,b]$ such that $g(a) = g(b) = 0$. Then the inequality*

$$\left|\frac{g(x)}{2} - \frac{1}{b-a}\int_a^b g(t)dt\right| \le \frac{(x-a)^2 + (b-x)^2}{8(b-a)} \widetilde{\omega}\left(g'; \frac{2}{3}\frac{(x-a)^3 + (y-b)^3}{(x-a)^2 + (y-b)^2}\right) \tag{3.9}$$

holds, where x is an arbitrary point in (a,b).

Proof. The following functional A defined on $C[a,b]$ by

$$A(f) = \frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right) f(t)dt$$

is a linear bounded functional having its norm equal to $\frac{b-a}{4}$. For every increasing function f we have:

$$A(f) \ge 0.$$

Using Theorem 2.3, we deduce that the functional A is P_0-simple with

$$A(e_1) = \frac{(b-a)^2}{12}.$$

From Theorem 2.1, we obtain the following inequality:

$$\left|\frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right) f(t)dt\right| \le \frac{b-a}{8} \widetilde{\omega}\left(f; \frac{2}{3}(b-a)\right). \tag{3.10}$$

Inequality (3.10) holds for every continuous function f.

Let us suppose that f is differentiable on $[a,b]$. From the inequality (3.10) (written for f') we obtain the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{8} \widetilde{\omega}\left(f'; \frac{2}{3}(b-a)\right). \tag{3.11}$$

Now, we can prove the inequality (3.9). We have the following identity:

$$\begin{aligned} -\frac{g(x)}{2} + \frac{1}{b-a}\int_a^b g(t)dt = \frac{x-a}{b-a}&\left(\frac{1}{x-a}\int_a^x g(t)dt - \frac{g(a)+g(x)}{2}\right) \\ &+ \frac{b-x}{b-a}\left(\frac{1}{b-x}\int_a^b g(t)dt - \frac{g(b)+g(x)}{2}\right). \end{aligned} \tag{3.12}$$

Using the relations (3.11) and (3.12) we obtain

$$\begin{aligned} &\left|\frac{g(x)}{2} - \frac{1}{b-a}\int_a^b g(t)dt\right| \\ &\quad \le \frac{(x-a)^2}{8(b-a)} \widetilde{\omega}\left(g'; \frac{2}{3}(x-a)\right) + \frac{(b-x)^2}{8(b-a)} \widetilde{\omega}\left(g'; \frac{2}{3}(b-x)\right). \end{aligned} \tag{3.13}$$

As the function $\widetilde{\omega}(g';\cdot)$, is concave, then from (3.13) and using Jensen's inequality, we obtain the inequality (3.9). □

Corollary 3.6. *Let g be a continuous differentiable function on $[a,b]$ such that $g(a)=g(b)=0$, then the following inequality*

$$\left|\frac{g(x)}{2}-\frac{1}{b-a}\int_a^b g(t)dt\right| \leq \left[\frac{1}{8}+\frac{\left(x-\frac{a+b}{2}\right)^2}{2(b-a)}\right](b-a)\|g'\|_\infty \tag{3.14}$$

is valid for all $x\in[a,b]$.

Proof. It is well known that

$$\widetilde{\omega}(g';t)\leq 2\|g'\|_\infty, \tag{3.15}$$

for every positive number t.

The inequality (3.15) then readily follows from the inequality (3.14). □

Remark 3.3. The result from the Theorem 1.3 can be written in terms of $\widetilde{\omega}$ using the inequality (3.13) for the function

$$g(x)=f(x)-\frac{x-a}{b-a}f(b)-\frac{b-x}{b-a}f(a).$$

In [5] the following result was proved:

Let A be a linear positive functional $A: C[0,1]\to\mathbb{R}$, $A(e_0)=1$ and $\varphi,\ \varphi:[0,1]\to\mathbb{R}$ a continuous increasing function such that $A(e_1\varphi)-A(e_1)A(\varphi)>0$. Then the following Grüss type inequality

$$|A(\varphi f)-A(\varphi)A(f)|\leq\frac{A(|\varphi-A(\varphi)|)}{2}\widetilde{\omega}\left(f;\frac{2(A(e_1\varphi)-A(e_1)A(\varphi))}{A(|\varphi-A(\varphi)|)}\right) \tag{3.16}$$

holds.

We are interested in the following open problem:

Open problem. Let A be a linear positive functional defined on $C[0,1]$ and f,g be two continuous functions. Do positive numbers $\delta_1=\delta_1(f)<1$ and $\delta_2=\delta_2(f)<1$ exist such that

$$|A(fg)-A(f)A(g)|\leq\frac{1}{4}\widetilde{\omega}(f;\delta_1)\widetilde{\omega}(f,\delta_2)?$$

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