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ON APPLICATION OF DIFFERENTIAL SUBORDINATION FOR CERTAIN SUBCLASS OF MEROMORPHICALLY p-VALENT FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY LINEAR OPERATOR

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ABSTRACT. This paper is mainly concerned with the application of differential subordinations for the class of meromorphic multivalent functions with positive coefficients defined by a linear operator satisfying the following:

$$-\frac{z^{p+1}(L^n f(z))'}{p} \prec \frac{1+Az}{1+Bz} \; (n \in \mathbb{N}_0;\; z \in U).$$

In the present paper, we study the coefficient bounds, δ-neighborhoods and integral representations. We also obtain linear combinations, weighted and arithmetic means and convolution properties.

Key words and phrases: Meromorphic functions, Differential subordination, convolution (or Hadamard product), p-valent functions, Linear operator, δ-Neighborhood, Integral representation, Linear combination, Weighted mean and Arithmetic mean.

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1. INTRODUCTION

Let $L(p,m)$ be a class of all meromorphic functions $f(z)$ of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \text{ for any } m \geq p, \quad p \in \mathbb{N} = \{1,2,\dots\}, \quad a_k \geq 0, \tag{1.1}$$

which are p-valent in the punctured unit disk

$$U^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = U/\{0\}.$$

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Definition 1.1. Let f, g be analytic in U. Then g is said to be subordinate to f, written $g \prec f$, if there exists a Schwarz function $w(z)$, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $g(z) = f(w(z))$ $(z \in U)$. Hence $g(z) \prec f(z)$ $(z \in U)$, then $g(0) = f(0)$ and $g(U) \subset f(U)$. In particular, if the function $f(z)$ is univalent in U, we have the following (e.g. [6]; [7]):

$$g(z) \prec f(z)(z \in U) \text{ if and only if } g(0) = f(0) \quad \text{and} \quad g(U) \subset f(U).$$

Definition 1.2. For functions $f(z) \in L(p, m)$ given by (1.1) and $g(z) \in L(p, m)$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k, \quad (b_k \geq 0, p \in \mathbb{N}, m \geq p), \tag{1.2}$$

we define the convolution (or Hadamard product) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k, \quad (p \in \mathbb{N}, m \geq p, z \in U). \tag{1.3}$$

Definition 1.3 ([9])**.** Let $f(z)$ be a function in the class $L(p, m)$ given by (1.1). We define a linear operator L^n by

$$L^0 f(z) = f(z),$$
$$L^1 f(z) = z^{-p} + \sum_{k=m}^{\infty} (p + k + 1) a_k z^k = \frac{(z^{p+1} f(z))'}{z^p}$$

and in general

$$\begin{aligned} L^n f(z) &= L(L^{n-1} f(z)) \\ &= z^{-p} + \sum_{k=m}^{\infty} (p + k + 1)^n a_k z^k \\ &= \frac{(z^{p+1} L^{n-1} f(z))'}{z^p}, \quad (n \in \mathbb{N}). \end{aligned} \tag{1.4}$$

It is easily verified from (1.4) that

$$\begin{gathered} z(L^n f(z))' = L^{n+1} f(z) - (p + 1) L^n f(z), \\ (f \in L(p, m), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \end{gathered} \tag{1.5}$$

(1) Liu and Srivastava [4] introduced recently the linear operator when $m = 0$, investigating several inclusion relationships involving various subclasses of meromorphically p-valent functions, which they defined by means of the linear operator L^n (see [4]).
(2) Uralegaddi and Somanatha [10] introduced the linear operator L^n when $p = 1$ and $m = 0$.
(3) Aouf and Hossen [2] obtained several results involving the linear operator L^n when $m = 0$ and $p \in \mathbb{N}$.

We introduce a subclass of the function class $L(p, m)$ by making use of the principle of differential subordination as well as the linear operator L^n.

Definition 1.4. Let A and B $(-1 \leq B < A \leq 1)$ be fixed parameters. We say that a function $f(z) \in L(p, m)$ is in the class $L(p, m, n, A, B)$, if it satisfies the following subordination condition:

$$\frac{z^{p+1}(L^n f(z))'}{p} \prec \frac{1 + Az}{1 + Bz} \quad (n \in \mathbb{N}_0;\ z \in U). \tag{1.6}$$

By the definition of differential subordination, (1.6) is equivalent to the following condition:

$$\left|\frac{z^{p+1}(L^n f(z))' + p}{Bz^{p+1}(L^n f(z))' + pA}\right| < 1, \quad (z \in U).$$

We can write

$$L\left(p, m, n, 1 - \frac{2\beta}{p}, -1\right) = L(p, m, n, \beta),$$

where $L(p, m, n, \beta)$ denotes the class of functions in $L(p, m)$ satisfying the following:

$$\mathrm{Re}\{-z^{p+1}(L^n f(z))'\} > \beta \quad (0 \le \beta < p;\ z \in U).$$

2. COEFFICIENT BOUNDS

Theorem 2.1. *Let the function $f(z)$ of the form (1.1), be in $L(p, m)$. Then the function $f(z)$ belongs to the class $L(p, m, n, A, B)$ if and only if*

$$\sum_{k=m}^{\infty} k(1 - B)(p + k + 1)^n a_k < (A - B)p, \tag{2.1}$$

where $-1 \le B < A \le 1$, $p \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m \ge p$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A - B)p}{k(1 - B)(p + k + 1)^n}\, z^m, \quad m \ge p.$$

Proof. Assume that the condition (2.1) is true. We must show that $f \in L(p, m, n, A, B)$, or equivalently prove that

$$\left|\frac{z^{p+1}(L^n f(z))' + p}{Bz^{p+1}(L^n f(z))' + Ap}\right| < 1. \tag{2.2}$$

We have

$$\left|\frac{z^{p+1}(L^n f(z))' + p}{Bz^{p+1}(L^n f(z))' + Ap}\right| = \left|\frac{z^{p+1}(-pz^{-(p+1)} + \sum_{k=m}^{\infty} k(p + k + 1)^n a_k z^{k-1}) + p}{Bz^{p+1}(-pz^{-(p+1)} + \sum_{k=m}^{\infty} k(p + k + 1)^n a_k z^{k-1}) + Ap}\right|$$

$$= \left|\frac{\sum_{k=m}^{\infty} k(p + k + 1)^n a_k z^{k+p}}{(A - B)p + B\sum_{k=m}^{\infty} k(p + k + 1)^n a_k z^{k+p}}\right|$$

$$\le \left\{\frac{\sum_{k=m}^{\infty} k(p + k + 1)^n a_k}{(A - B)p + B\sum_{k=m}^{\infty} k(k + p + 1)^n a_k}\right\} < 1.$$

The last inequality by (2.1) is true.

Conversely, suppose that $f(z) \in L(p, m, n, A, B)$. We must show that the condition (2.1) holds true. We have

$$\left|\frac{z^{p+1}(L^n f(z))' + p}{Bz^{p+1}(L^n f(z))' + Ap}\right| < 1,$$

hence we get

$$\left|\frac{\sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k+p}}{(A-B)p + B\sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k+p}}\right| < 1.$$

Since $\mathrm{Re}(z) < |z|$, so we have

$$\mathrm{Re}\left\{\frac{\sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k+p}}{(A-B)p + B\sum_{k=m}^{\infty} k(p+k+1)^n a_k z^{k+p}}\right\} < 1.$$

We choose the values of z on the real axis and letting $z \to 1^-$, then we obtain

$$\left\{\frac{\sum_{k=m}^{\infty} k(p+k+1)^n a_k}{(A-B)p + B\sum_{k=m}^{\infty} k(p+k+1)^n a_k}\right\} < 1,$$

then

$$\sum_{k=m}^{\infty} k(1-B)(p+k+1)^n a_k < (A-B)p$$

and the proof is complete. □

Corollary 2.2. *Let $f(z) \in L(p,m,n,A,B)$, then we have*

$$a_k \le \frac{(A-B)p}{k(1-B)(p+k+1)^n}, \; k \ge m.$$

Corollary 2.3. *Let $0 \le n_2 < n_1$, then $L(p,m,n_2,A,B) \subseteq L(p,m,n_1,A,B)$.*

3. NEIGHBOURHOODS AND PARTIAL SUMS

Definition 3.1. Let $-1 \le B < A \le 1$, $m \ge p$, $n \in \mathbb{N}_0$, $p \in \mathbb{N}$ and $\delta \ge 0$. We define the δ -neighbourhood of a function $f \in L(p,m)$ and denote $N_\delta(f)$ such that

$$N_\delta(f) = \left\{ g \in L(p,m) : g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k, \text{ and } \sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)p}|a_k - b_k| \le \delta \right\}. \tag{3.1}$$

Goodman [3], Ruscheweyh [8] and Altintas and Owa [1] have investigated neighbourhoods for analytic univalent functions, we consider this concept for the class $L(p,m,n,A,B)$.

Theorem 3.1. *Let the function $f(z)$ defined by (1.1) be in $L(p,m,n,A,B)$. For every complex number μ with $|\mu| < \delta, \delta \ge 0$, let $\frac{f(z)+\mu z^{-p}}{1+\mu} \in L(p,m,n,A,B)$, then $N_\delta(f) \subset L(p,m,n,A,B)$, $\delta \ge 0$.*

Proof. Since $f \in L(p,m,n,A,B)$, f satisfies (2.1) and we can write for $\gamma \in \mathbb{C}$, $|\gamma| = 1$, that

$$\left[\frac{z^{p+1}(L^n f(z))' + p}{Bz^{p+1}(L^n f(z))' + pA}\right] \ne \gamma. \tag{3.2}$$

Equivalently, we must have

$$\frac{(f*Q)(z)}{z^{-p}} \neq 0, \quad z \in U^*, \tag{3.3}$$

where

$$Q(z) = z^{-p} + \sum_{k=m}^{\infty} e_k z^k,$$

such that $e_k = \frac{\gamma k(1-B)(p+k+1)^n}{(A-B)p}$, satisfying $|e_k| \leq \frac{k(1-B)(p+k+1)^n}{(A-B)p}$ and $k \geq m$, $p \in \mathbb{N}$, $n \in \mathbb{N}_0$.

Since $\frac{f(z)+\mu z^{-p}}{1+\mu} \in L(p,m,n,A,B)$, by (3.3),

$$\frac{1}{z^{-p}}\left(\frac{f(z)+\mu z^{-p}}{1+\mu} * Q(z)\right) \neq 0,$$

and then

$$\frac{1}{z^{-p}}\left(\frac{(f*Q)(z)+\mu z^{-p}}{1+\mu}\right) \neq 0. \tag{3.4}$$

Now assume that $\left|\frac{(f*Q)(z)}{z^{-p}}\right| < \delta$. Then, by (3.4), we have

$$\left|\frac{1}{1+\mu}\frac{f*Q}{z^{-p}} + \frac{\mu}{1+\mu}\right| \geq \frac{|\mu|}{|1+\mu|} - \frac{1}{|1+\mu|}\left|\frac{(f*Q)(z)}{z^{-p}}\right| > \frac{|\mu|-\delta}{|1+\mu|} \geq 0.$$

This is a contradiction as $|\mu| < \delta$. Therefore $\left|\frac{(f*Q)(z)}{z^{-p}}\right| \geq \delta$.

Letting

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \in N_\delta(f),$$

then

$$\begin{aligned}
\delta - \left|\frac{(g*Q)(z)}{z^{-p}}\right| &\leq \left|\frac{((f-g)*Q)(z)}{z^{-p}}\right| \\
&\leq \left|\sum_{k=m}^{\infty}(a_k-b_k)e_k z^k\right| \\
&\leq \sum_{k=m}^{\infty}|a_k-b_k||e_k||z|^k \\
&< |z|^m \sum_{k=m}^{\infty}\left[\frac{k(1-B)(p+k+1)^n}{(A-B)p}\right]|a_k-b_k| \\
&\leq \delta,
\end{aligned}$$

therefore $\frac{(g*Q)(z)}{z^{-p}} \neq 0$, and we get $g(z) \in L(p,m,n,A,B)$, so $N_\delta(f) \subset L(p,m,n,A,B)$. □

Theorem 3.2. *Let $f(z)$ be defined by (1.1) and the partial sums $S_1(z)$ and $S_q(z)$ be defined by $S_1(z) = z^{-p}$ and*

$$S_q(z) = z^{-p} + \sum_{k=m}^{m+q-2} a_k z^k, \qquad q > m,\ m \geq p,\ p \in \mathbb{N}.$$

Also suppose that $\sum_{k=m}^{\infty} C_k a_k \leq 1$, *where*

$$C_k = \frac{k(1-B)(p+k+1)^n}{(A-B)p}.$$

Then

$$\text{(i)} \qquad f \in L(p,m,n,A,B)$$

$$\text{(ii)} \qquad \operatorname{Re}\left\{\frac{f(z)}{S_q(z)}\right\} > 1 - \frac{1}{C_q}, \tag{3.5}$$

$$\operatorname{Re}\left\{\frac{S_q(z)}{f(z)}\right\} > \frac{C_q}{1+C_q}, \quad z \in U, q > m. \tag{3.6}$$

Proof.
(i) Since $\frac{z^{-p}+\mu z^{-p}}{1+\mu} = z^{-p} \in L(p,m,n,A,B)$, $|\mu| < 1$, then by Theorem 3.1, we have $N_1(z^{-p}) \subset L(p,m,n,A,B), p \in \mathbb{N}(N_1(z^{-p})$ denoting the 1-neighbourhood). Now since

$$\sum_{k=m}^{\infty} C_k a_k \leq 1,$$

then $f \in N_1(z^{-p})$ and $f \in L(p,m,n,A,B)$.
(ii) Since $\{C_k\}$ is an increasing sequence, we obtain

$$\sum_{k=m}^{m+q-2} a_k + C_q \sum_{k=q+m-1}^{\infty} a_k \leq \sum_{k=m}^{\infty} C_k a_k \leq 1. \tag{3.7}$$

Setting

$$G_1(z) = C_q\left(\frac{f(z)}{S_q(z)} - \left(1 - \frac{1}{C_q}\right)\right) = \frac{C_q \sum\limits_{k=q+m-1}^{\infty} a_k z^{k+p}}{1 + \sum\limits_{k=m}^{m+q-2} a_k z^{k+p}} + 1,$$

from (3.7) we get

$$\left|\frac{G_1(z)-1}{G_1(z)+1}\right| = \left|\frac{C_q \sum\limits_{k=q+m-1}^{\infty} a_k z^{k+p}}{2 + 2\sum\limits_{k=m}^{m+q-2} a_k z^{k+p} + C_q \sum\limits_{k=q+m-1}^{\infty} a_k z^{k+p}}\right|$$

$$\leq \frac{C_q \sum\limits_{k=q+m-1}^{\infty} a_k}{2 - 2\sum\limits_{k=m}^{m+q-2} a_k - C_q \sum\limits_{k=q+m-1}^{\infty} a_k} \leq 1.$$

This proves (3.5). Therefore, $\operatorname{Re}(G_1(z)) > 0$ and we obtain $\operatorname{Re}\left\{\frac{f(z)}{S_q(z)}\right\} > 1 - \frac{1}{C_q}$. Now, in the same manner, we can prove the assertion (3.6), by setting

$$G_2(z) = (1+C_q)\left(\frac{S_q(z)}{f(z)} - \frac{C_q}{1+C_q}\right).$$

This completes the proof. □

4. INTEGRAL REPRESENTATION

In the next theorem we obtain an integral representation for $L^n f(z)$.

Theorem 4.1. *Let* $f \in L(p,m,n,A,B)$*, then*

$$L^n f(z) = \int_0^z \frac{p(A\psi(t)-1)}{t^{p+1}(1-B\psi(t))} dt,$$

where $|\psi(z)| < 1, z \in U^*$.

Proof. Let $f(z) \in L(p,m,n,A,B)$. Letting $-\frac{z^{p+1}(L^n f(z))'}{p} = y(z)$, we have

$$y(z) \prec \frac{1+Az}{1+Bz}$$

or we can write $\left|\frac{y(z)-1}{By(z)-A}\right| < 1$, so that consequently we have

$$\frac{y(z)-1}{By(z)-A} = \psi(z),\ |\psi(z)| < 1,\ z \in U.$$

We can write

$$\frac{-z^{p+1}(L^n f(z))'}{p} = \frac{1-A\psi(z)}{1-B\psi(z)},$$

which gives

$$(L^n f(z))' = \frac{p(A\psi(z)-1)}{z^{p+1}(1-B\psi(z))}.$$

Hence

$$L^n f(z) = \int_0^z \frac{p(A\psi(t)-1)}{t^{p+1}(1-B\psi(t))} dt,$$

and this gives the required result. □

5. LINEAR COMBINATION

In the theorem below, we prove a linear combination for the class $L(p,m,n,A,B)$.

Theorem 5.1. *Let*

$$f_i(z) = z^{-p} + \sum_{k=m}^{\infty} a_{k,i} z^k, \quad (a_{k,i} \geq 0, i = 1,2,\ldots,\ell, k \geq m, m \geq p)$$

belong to $L(p,m,n,A,B)$*, then*

$$F(z) = \sum_{i=1}^{\ell} c_i f_i(z) \in L(p,m,n,A,B),$$

where $\sum_{i=1}^{\ell} c_i = 1$.

Proof. By Theorem 2.1, we can write for every $i \in \{1,2,\ldots,\ell\}$

$$\sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)p} a_{k,i} < 1,$$

therefore

$$F(z) = \sum_{i=1}^{\ell} c_i \left(z^{-p} + \sum_{k=m}^{\infty} a_{k,i} z^k\right) = z^{-p} + \sum_{k=m}^{\infty} \left(\sum_{i=1}^{\ell} c_i a_{k,i}\right) z^k.$$

However,

$$\sum_{k=m}^{\infty}\frac{k(1-B)(p+k+1)^n}{(A-B)p}\left(\sum_{i=1}^{\ell}c_i a_{k,i}\right)=\sum_{i=1}^{\ell}\left[\sum_{k=m}^{\infty}\frac{k(1-B)(p+k+1)^n}{(A-B)p}a_{k,i}\right]c_i\leq 1,$$

then $F(z)\in L(p,m,n,A,B)$, so the proof is complete. □

6. WEIGHTED MEAN AND ARITHMETIC MEAN

Definition 6.1. Let $f(z)$ and $g(z)$ belong to $L(p,m)$, then the weighted mean $h_j(z)$ of $f(z)$ and $g(z)$ is given by

$$h_j(z)=\frac{1}{2}[(1-j)f(z)+(1+j)g(z)].$$

In the theorem below we will show the weighted mean for this class.

Theorem 6.1. *If $f(z)$ and $g(z)$ are in the class $L(p,m,n,A,B)$, then the weighted mean of $f(z)$ and $g(z)$ is also in $L(p,m,n,A,B)$.*

Proof. We have for $h_j(z)$ by Definition 6.1,

$$\begin{aligned}h_j(z)&=\frac{1}{2}\left[(1-j)\left(z^{-p}+\sum_{k=m}^{\infty}a_k z^k\right)+(1+j)\left(z^{-p}+\sum_{k=m}^{\infty}b_k z^k\right)\right]\\&=z^{-p}+\sum_{k=m}^{\infty}\frac{1}{2}((1-j)a_k+(1+j)b_k)z^k.\end{aligned}$$

Since $f(z)$ and $g(z)$ are in the class $L(p,m,n,A,B)$ so by Theorem 2.1 we must prove that

$$\begin{aligned}&\sum_{k=m}^{\infty}k(1-B)(p+k+1)^n\left[\frac{1}{2}(1-j)a_k+\frac{1}{2}(1+j)b_k\right]\\&=\frac{1}{2}(1-j)\sum_{k=m}^{\infty}k(1-B)(p+k+1)^n a_k+\frac{1}{2}(1+j)\sum_{k=m}^{\infty}k(1-B)(p+k+1)^n b_k\\&\leq\frac{1}{2}(1-j)(A-B)p+\frac{1}{2}(1+j)(A-B)p.\end{aligned}$$

The proof is complete. □

Theorem 6.2. *Let $f_1(z),f_2(z),\ldots,f_\ell(z)$ defined by*

$$f_i(z)=z^{-p}+\sum_{k=m}^{\infty}a_{k,i}z^k,\quad (a_{k,i}\geq 0, i=1,2,\ldots,\ell, k\geq m, m\geq p)\tag{6.1}$$

be in the class $L(p,m,n,A,B)$, then the arithmetic mean of $f_i(z)$ $(i=1,2,\ldots,\ell)$ defined by

$$h(z)=\frac{1}{\ell}\sum_{i=1}^{\ell}f_i(z)\tag{6.2}$$

is also in the class $L(p,m,n,A,B)$.

Proof. By (6.1), (6.2) we can write

$$h(z)=\frac{1}{\ell}\sum_{i=1}^{\ell}\left(z^{-p}+\sum_{k=m}^{\infty}a_{k,i}z^k\right)=z^{-p}+\sum_{k=m}^{\infty}\left(\frac{1}{\ell}\sum_{i=1}^{\ell}a_{k,i}\right)z^k.$$

Since $f_i(z) \in L(p,m,n,A,B)$ for every $i = 1, 2, \ldots, \ell$, so by using Theorem 2.1, we prove that

$$\sum_{k=m}^{\infty} k(1-B)(p+k+1)^n \left(\frac{1}{\ell} \sum_{i=1}^{\ell} a_{k,i} \right) = \frac{1}{\ell} \sum_{i=1}^{\ell} \left(\sum_{k=m}^{\infty} k(1-B)(p+k+1)^n a_{k,i} \right) \leq \frac{1}{\ell} \sum_{i=1}^{\ell} (A-B)p.$$

The proof is complete. □

7. CONVOLUTION PROPERTIES

Theorem 7.1. *If $f(z)$ and $g(z)$ belong to $L(p,m,n,A,B)$ such that*

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k, \qquad g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k, \tag{7.1}$$

then

$$T(z) = z^{-p} + \sum_{k=m}^{\infty} (a_k^2 + b_k^2) z^k$$

is in the class $L(p,m,n,A_1,B_1)$ such that $A_1 \geq (1-B_1)\mu^2 + B_1$, where

$$\mu = \frac{\sqrt{2}(A-B)}{\sqrt{m(m+2)^n}(1-B)}.$$

Proof. Since $f, g \in L(p,m,n,A,B)$, Theorem 2.1 yields

$$\sum_{k=m}^{\infty} \left(\left[\frac{k(1-B)(p+k+1)^n}{(A-B)p} \right] a_k \right)^2 \leq 1$$

and

$$\sum_{k=m}^{\infty} \left(\left[\frac{k(1-B)(p+k+1)^n}{(A-B)p} \right] b_k \right)^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{k=m}^{\infty} \frac{1}{2} \left[\frac{k(1-B)(p+k+1)^n}{(A-B)p} \right]^2 (a_k^2 + b_k^2) \leq 1. \tag{7.2}$$

However, $T(z) \in L(p,m,n,A_1,B_1)$ if and only if

$$\sum_{k=m}^{\infty} \left[\frac{k(1-B_1)(p+k+1)^n}{(A_1-B_1)p} \right] (a_k^2 + b_k^2) \leq 1, \tag{7.3}$$

where $-1 \leq B_1 < A_1 \leq 1$, but (7.2) implies (7.3) if

$$\frac{k(1-B_1)(p+k+1)^n}{(A_1-B_1)p} < \frac{1}{2} \left[\frac{k(1-B)(p+k+1)^n}{(A-B)p} \right]^2.$$

Hence, if

$$\frac{1-B_1}{A_1-B_1} < \frac{k(p+k+1)^n}{2p} \alpha^2, \quad \text{where } \alpha = \frac{1-B}{A-B}.$$

In other words,

$$\frac{1-B_1}{A_1-B_1} < \frac{k(k+2)^n}{2} \alpha^2.$$

This is equivalent to

$$\frac{A_1 - B_1}{1 - B_1} > \frac{2}{k(k+2)^n \alpha^2}.$$

So we can write

$$\frac{A_1 - B_1}{1 - B_1} > \frac{2(A-B)^2}{m(m+2)^n(1-B)^2} = \mu^2. \tag{7.4}$$

Hence we get $A_1 \geq (1 - B_1)\mu^2 + B_1$. □

Theorem 7.2. *Let $f(z)$ and $g(z)$ of the form (7.1) belong to $L(p, m, n, A, B)$. Then the convolution (or Hadamard product) of two functions f and g belong to the class, that is, $(f * g)(z) \in L(p, m, n, A_1, B_1)$, where $A_1 \geq (1 - B_1)v + B_1$ and*

$$v = \frac{(A-B)^2}{m(1-B)^2(m+2)^n}.$$

Proof. Since $f, g \in L(p, m, n, A, B)$, by using the Cauchy-Schwarz inequality and Theorem 2.1, we obtain

$$\sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)p} \sqrt{a_k b_k} \tag{7.5}$$

$$\leq \left(\sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)p} a_k \right)^{\frac{1}{2}} \left(\sum_{k=m}^{\infty} \frac{k(1-B)(p+k+1)^n}{(A-B)p} b_k \right)^{\frac{1}{2}} \leq 1 \,.$$

We must find the values of A_1, B_1 so that

$$\sum_{k=m}^{\infty} \frac{k(1-B_1)(p+k+1)^n}{(A_1-B_1)p} a_k b_k < 1. \tag{7.6}$$

Therefore, by (7.5), (7.6) holds true if

$$\sqrt{a_k b_k} \leq \frac{(1-B)(A_1-B_1)}{(1-B_1)(A-B)}, \quad k \geq m,\ m \geq p,\ a_k \neq 0,\ b_k \neq 0. \tag{7.7}$$

By (7.5), we have $\sqrt{a_k b_k} < \frac{(A-B)p}{k(1-B)(p+k+1)^n}$, therefore (7.7) holds true if

$$\frac{k(1-B_1)(p+k+1)^n}{(A_1-B_1)p} \leq \left[\frac{k(1-B)(p+k+1)^n}{(A-B)p} \right]^2,$$

which is equivalent to

$$\frac{(1-B_1)}{(A_1-B_1)} < \frac{k(1-B)^2(p+k+1)^n}{(A-B)^2 p}.$$

Alternatively, we can write

$$\frac{(1-B_1)}{(A_1-B_1)} < \frac{k(1-B)^2(k+2)^n}{(A-B)^2},$$

to obtain

$$\frac{A_1 - B_1}{1 - B_1} > \frac{(A-B)^2}{m(1-B)^2(m+2)^n} = v.$$

Hence we get $A_1 > v(1 - B_1) + B_1$. □

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