



## A FUNCTIONAL INEQUALITY FOR THE SURVIVAL FUNCTION OF THE GAMMA DISTRIBUTION

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*Received 12 January, 2008; accepted 18 February, 2008*

*Communicated by A. Laforgia*

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**ABSTRACT.** In this note we give a completely different proof to a functional inequality established by Ismail and Laforgia for the survival function of the gamma distribution and we show that the inequality in the question is in fact the so-called new-is-better-than-used property, which arises in economic theory. Moreover, we extend this result to arbitrary reliability functions and we present a new simple proof for the Esseen-Mitrinović inequality.

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*Key words and phrases:* Error function; Incomplete gamma function; Density function; Survival function; Complete monotonicity; Functional inequality; New-is-better-than-used property; Log-concavity.

2000 *Mathematics Subject Classification.* 33B20, 26D05.

### 1. FUNCTIONAL INEQUALITIES INVOLVING THE INCOMPLETE GAMMA FUNCTION

Let

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt, \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

and

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

denote, as usual, the distribution function of the standard normal law, the error function and the complementary error function. Esseen [6, p. 291] in 1961 proved the following interesting inequality related to the distribution function  $\Phi$ : for all  $x, y \leq 0$  we have

$$(1.1) \quad \Phi(x+y) \leq 2\Phi(x)\Phi(y).$$

Another interesting inequality, which was published by Mitrinović [6, p. 291] in 1968 and proved by Weinacht, is: for all real numbers  $x, y \geq 0$  we have

$$(1.2) \quad \operatorname{erf}(x)\operatorname{erf}(y) \geq \operatorname{erf}(x) + \operatorname{erf}(y) - \operatorname{erf}(x+y),$$

with equality if and only if  $x$  or  $y$  is an end point of the closed interval  $[0, +\infty]$ . Recently, in 2003, Alzer [1, Theorem 1] extended and complemented the inequality (1.2), showing in particular that (1.2) is valid for all real numbers  $x$  and  $y$ . Moreover, Alzer pointed out that

inequalities (1.1) and (1.2) are not only similar, but even equivalent. Observe that since  $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$ , inequality (1.2) is equivalent to the inequality

$$(1.3) \quad \operatorname{erfc}(x+y) \leq \operatorname{erfc}(x)\operatorname{erfc}(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Now for all  $p > 0$  and  $x \in \mathbb{R}$  let

$$\Gamma(p, x) = \int_x^\infty t^{p-1}e^{-t} dt, \quad \gamma(p, x) = \int_0^x t^{p-1}e^{-t} dt$$

and

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t} dt$$

denote the upper incomplete gamma function, the lower incomplete gamma function and the gamma function, respectively. Recently, in 2006, motivated by the inequality (1.2), Ismail and Laforgia [4, Theorem 1.1], with their clever use of Rolle's theorem, proved that the function  $q : [0, \infty) \rightarrow (0, 1]$ , defined by  $q(x) := \Gamma(p, x)/\Gamma(p)$ , when  $p \geq 1$  satisfies the following inequality

$$(1.4) \quad q(x+y) \leq q(x)q(y) \quad \text{for all } x, y \geq 0.$$

Moreover, they showed that when  $p \in (0, 1]$ , the above inequality is reversed. In this section our aim is to show that inequality (1.4) can be deduced easily using some well-known facts from probability theory. Before we state our main results we need the following technical lemma.

**Lemma 1.1.** *Let us consider the continuously differentiable function  $\varphi : [0, \infty) \rightarrow (0, \infty)$ . If  $\varphi(0) \geq 1$  and  $\varphi$  is log-concave, then for all  $x, y \geq 0$  we have  $\varphi(x+y) \leq \varphi(x)\varphi(y)$ . Moreover, if  $\varphi(0) \leq 1$  and  $\varphi$  is log-convex, then the above inequality is reversed.*

*Proof.* First suppose that  $\varphi(0) \geq 1$  and  $\varphi$  is log-concave. Let the function  $\phi : [0, \infty) \rightarrow \mathbb{R}$  be defined by  $\phi(x) := \log \varphi(x) - x\varphi'(x)/\varphi(x)$ . Clearly we have  $\phi'(x) = -x(\varphi'(x)/\varphi(x))' \geq 0$  and consequently  $\phi$  is increasing. Thus  $\phi(x) \geq \phi(0) = \log \varphi(0) \geq 0$  for all  $x \geq 0$ . Hence it is easy to verify that the function  $x \mapsto [\log \varphi(x)]/x$  is decreasing on  $(0, \infty)$ , which implies that the function  $x \mapsto \log \varphi(x)$  is sub-additive on  $[0, \infty)$ . Therefore for all  $x, y \geq 0$  we have  $\varphi(x+y) \leq \varphi(x)\varphi(y)$ .

Now suppose that  $\varphi(0) \leq 1$  and  $\varphi$  is log-convex. Then  $\phi$  is decreasing and this implies that  $\phi(x) \leq \phi(0) = \log \varphi(0) \leq 0$  for all  $x \geq 0$ . Hence the function  $x \mapsto [\log \varphi(x)]/x$  is increasing on  $(0, \infty)$ , which implies that the function  $x \mapsto \log \varphi(x)$  is super-additive on  $[0, \infty)$ . This completes the proof.  $\square$

Let  $f$  be a probability density function whose support is the interval  $[a, b]$  and let  $F : [a, b] \rightarrow [0, 1]$ , defined by

$$F(x) = \int_a^x f(t) dt,$$

be the corresponding cumulative distribution function. The function  $\bar{F} : [a, b] \rightarrow [0, 1]$ , defined by

$$\bar{F}(x) = 1 - F(x) = \int_x^b f(t) dt,$$

is known as the corresponding reliability function or the survival function. From the theory of probabilities – see for example Bagnoli and Bergstrom [3, Theorem 1.2] – it is well-known that if the density function  $f$  is continuously differentiable and log-concave on  $(a, b)$ , then the survival function  $\bar{F}$  is also log-concave on  $(a, b)$ . Moreover, if  $f$  is continuously differentiable and log-convex on  $(a, b)$  and if  $f(b) = 0$ , then the reliability function  $\bar{F}$  is also log-convex on  $(a, b)$ .

We are now in a position to present an alternative proof of (1.4) and its reverse.

*Proof of (1.4).* Recall that the gamma distribution has support  $[a, b] = [0, \infty)$  and density function  $f(x) = x^{p-1}e^{-x}/\Gamma(p)$ . From definitions, the gamma distribution has the cumulative distribution function  $x \mapsto \gamma(p, x)/\Gamma(p)$  and consequently the function  $q$  defined above is actually the survival function of the gamma distribution, since  $\Gamma(p, x) + \gamma(p, x) = \Gamma(p)$ . Easy computations show that  $[\log f(x)]'' = (1-p)/x^2$ . First suppose that  $p \geq 1$ . Then the density function  $f$  is log-concave and consequently the function  $q$  is log-concave too. But  $q(0) = 1$ , thus from Lemma 1.1 we conclude that (1.4) holds. Now assume that  $p \in (0, 1]$ . Then the density function  $f$  is log-convex and satisfies  $f(b) = f(\infty) = 0$ . Hence the reliability function  $q$  is log-convex too. Application of Lemma 1.1 yields the reverse of (1.4).  $\square$

The above argument yields the following general result which we state without proof, since the proof of the next theorem goes along the lines introduced above in the proof of (1.4).

**Theorem 1.2.** *Let  $f$  be a continuously differentiable density function which has support  $[0, \infty)$ . If  $f$  is log-concave, then for all  $x, y \geq 0$  we have*

$$(1.5) \quad \bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y).$$

Moreover, if  $f$  is log-convex, then the above inequality is reversed.

We note that after we finished the first draft of this manuscript we discovered that the inequality  $\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y)$  is in fact not new. More precisely, the above inequality is known in economic theory as the new-is-better-than-used property, since if  $X$  is the time of death of a physical object, then the probability  $P(X \geq x) = \bar{F}(x)$  that a new unit will survive to age  $x$ , is greater than the probability

$$\frac{P(X \geq x+y)}{P(X \geq y)} = \frac{\bar{F}(x+y)}{\bar{F}(y)}$$

that a survived unit of age  $y$  will survive for an additional time  $x$ . For more details, the interested reader is referred to An's paper [2, Section 4.2], where among other things a slightly different proof of (1.5) is given.

## 2. FUNCTIONAL INEQUALITIES INVOLVING THE SURVIVAL FUNCTIONS OF OTHER DISTRIBUTIONS

Let us consider the density function  $f_1 : [0, \infty) \rightarrow (0, \infty)$ , defined by

$$f_1(x) = \frac{e^{-x}u(x)}{\int_0^\infty e^{-t}u(t) dt},$$

where  $u : [0, \infty) \rightarrow (0, \infty)$  is a continuously differentiable function such that  $t \mapsto e^{-t}u(t)$  is integrable. Clearly we have that  $[\log f_1(x)]'' = [\log u(x)]''$ . Consider the survival function  $\bar{F}_1 : [0, \infty) \rightarrow (0, 1]$ , defined by

$$\bar{F}_1(x) = \int_x^\infty f_1(t) dt.$$

Then clearly  $\bar{F}_1(0) = 1$  and

$$\bar{F}_1(x) = \int_x^\infty e^{-t}u(t) dt \Big/ \int_0^\infty e^{-t}u(t) dt.$$

Thus, applying Theorem 1.2 we have the following generalization of (1.4). Note that it can be easily seen the first part of the next corollary is in fact equivalent to the first part of Theorem 1.3 due to Ismail and Laforgia in [4].

**Corollary 2.1.** *If  $u$  is log-concave, then for all  $x, y \geq 0$  we have  $\bar{F}_1(x+y) \leq \bar{F}_1(x)\bar{F}_1(y)$ . Moreover, if  $u$  is log-convex and  $e^{-x}u(x)$  tends to zero as  $x$  tends to infinity, then the above inequality is reversed.*

Now consider the following distributions: Weibull distribution, chi-squared distribution and chi distribution. These distributions have support  $[0, \infty)$  and density functions for  $p > 0$  as follows

$$f_2(x) = px^{p-1}e^{-x^p}, \quad f_3(x) = \frac{x^{(p-2)/2}e^{-x/2}}{2^{p/2}\Gamma(p/2)}$$

and

$$f_4(x) = \frac{x^{p-1}e^{-x^2/2}}{2^{(p-2)/2}\Gamma(p/2)}.$$

Recall that the Weibull distribution with  $p = 2$  – as well as the chi distribution with  $p = 2$  – is sometimes known as the Rayleigh distribution and the chi distribution with  $p = 3$  is sometimes called the Maxwell distribution. With some computations we get

$$[\log f_2(x)]'' = \frac{1-p}{x^2}(1+px^p), \quad [\log f_3(x)]'' = \frac{2-p}{2x^2}$$

and

$$[\log f_4(x)]'' = \frac{1-p}{x^2} - 1.$$

Thus the density function  $f_2$  of the Weibull distribution is log-concave if  $p \geq 1$  and is log-convex if  $p \in (0, 1]$ . Moreover, it is easy to verify that if  $p \in (0, 1]$ , then  $f_2(\infty) = 0$ . Analogously, the density function  $f_3$  of the chi-squared distribution is log-concave if  $p \geq 2$ , is log-convex if  $p \in (0, 2]$  and  $f_3(\infty) = 0$ . Finally, note that the density function  $f_4$  of the chi distribution is log-concave too when  $p \geq 1$ . For the log-concavity of the functions  $f_2, f_3, f_4$  and other known density functions, the interested reader is referred to Bagnoli's and Bergstrom's paper [3, Section 6]. Now, let us define the survival functions of these distributions  $\bar{F}_i : [0, \infty) \rightarrow (0, 1]$  by

$$\bar{F}_i(x) = \int_x^\infty f_i(t) dt,$$

where  $i = 2, 3, 4$ . Clearly we have  $\bar{F}_i(0) = 1$  for each  $i = 2, 3, 4$ . Thus, applying Theorem 1.2 we have the following result.

**Corollary 2.2.** *If  $p \geq 1$  then for all  $x, y \geq 0$  we have the inequality  $\bar{F}_i(x+y) \leq \bar{F}_i(x)\bar{F}_i(y)$ , where  $i = 2, 4$ . When  $p \in (0, 1]$  and  $i = 2$  the above inequality is reversed. If  $p \geq 2$ , then for all  $x, y \geq 0$  the inequality  $\bar{F}_3(x+y) \leq \bar{F}_3(x)\bar{F}_3(y)$  holds. Moreover, when  $p \in (0, 2]$  the above inequality is reversed.*

### 3. CONCLUDING REMARKS

In this section we list some remarks related to the results of the previous sections.

**1.** First note that the density function  $x \mapsto e^{-x^2}/\sqrt{2\pi}$  of the normal distribution is clearly log-concave on  $\mathbb{R}$ . Thus we have that the tail function  $\bar{\Phi} : \mathbb{R} \rightarrow (0, 1)$ , defined by  $\bar{\Phi}(x) = 1 - \Phi(x)$ , is log-concave too on  $\mathbb{R}$ . Since  $2\Phi(x\sqrt{2}) = 1 + \operatorname{erf}(x)$  and  $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$ , we have that  $\operatorname{erfc}(x) = 2\bar{\Phi}(x\sqrt{2})$ , which implies that the complementary error function is log-concave as well on  $\mathbb{R}$ . Since  $\operatorname{erfc}(0) = 1$ , the application of Lemma 1.1 yields a new proof of inequality (1.2).

2. Recall that due to Petrović [7], [6, p. 22], we know that if  $\phi$  is a convex function on the domain which contains  $0, x_1, x_2, \dots, x_n \geq 0$ , then

$$\phi(x_1) + \phi(x_2) + \dots + \phi(x_n) \leq \phi(x_1 + \dots + x_n) + (n - 1)\phi(0).$$

If  $n = 2$  and  $\phi(0) = 0$ , then the last inequality shows that  $\phi$  is a super-additive function. Thus if  $\varphi$  is defined as in Lemma 1.1,  $\varphi(0) = 1$  and  $\varphi$  is log-convex, then from Petrović's result easily follows that  $x \mapsto \log \varphi(x)$  is super-additive.

3. A function  $f$  with domain  $(0, \infty)$  is said to be completely monotonic if it possesses derivatives  $f^{(n)}$  for all  $n = 1, 2, 3, \dots$  and if  $(-1)^n f^{(n)}(x) \geq 0$  for all  $x > 0$ . Due to Kimberling [5] we know that if the continuous function  $h : [0, \infty) \rightarrow (0, 1]$  is completely monotonic on  $(0, \infty)$ , then we get that  $x \mapsto \log h(x)$  is super-additive, i.e., for all  $x, y \geq 0$  we have  $h(x)h(y) \leq h(x + y)$ .

We note that the reverse of (1.4) is actually an immediate consequence of Kimberling's result. To prove this, first let us consider  $p = 1$ . Then  $q(x) = e^{-x}$  and clearly we have equality in (1.4). Now suppose that  $p \in (0, 1)$ . Then from the Leibniz rule for derivatives we have

$$\begin{aligned} (-1)^n q^{(n)}(x) \Gamma(p) &= (-1)^n \frac{\partial^n \Gamma(p, x)}{\partial x^n} \\ &= (-1)^n \frac{\partial^{n-1} [x^{p-1} (-e^{-x})]}{\partial x^{n-1}} \\ &= e^{-x} \sum_{k=0}^{n-1} C_{n-1}^k \prod_{m=1}^k (m - p) x^{p-k-1} \geq 0 \end{aligned}$$

for all  $x \geq 0$  and  $p \in (0, 1)$ . Thus the function  $q$  is completely monotonic. Now since  $q$  maps  $[0, \infty)$  into  $(0, 1]$ , from Kimberling's result the reverse of (1.4) holds. Moreover, using the above argument related to Corollary 2.1 we have the following result:

**Corollary 3.1.** *If the function  $u$  is completely monotonic, then  $\bar{F}_1$  satisfies the inequality*

$$\bar{F}_1(x) \bar{F}_1(y) \leq \bar{F}_1(x + y)$$

for all  $x, y \geq 0$ .

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