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ON THE RATE OF STRONG SUMMABILITY BY MATRIX MEANS IN THE GENERALIZED HÖLDER METRIC

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ABSTRACT. In the paper we generalize (and improve) the results of T. Singh [5], with mediate function, to the strong summability. We also apply the generalization of L. Leindler type [3].

Key words and phrases: Strong approximation, Matrix means, Special sequences.

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1. INTRODUCTION

Let f be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

be its Fourier series. Denote by $S_n(x) = S_n(f,x)$ the n-th partial sum of (1.1) and by $\omega(f,\delta)$ the modulus of continuity of $f \in C_{2\pi}$.

The usual supremum norm will be denoted by $\|\cdot\|_C$.

Let $\omega(t)$ be a nondecreasing continuous function on the interval $[0,2\pi]$ having the properties

$$\omega(0) = 0, \quad \omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2).$$

Such a function will be called a modulus of continuity.

Denote by H^ω the class of functions

$$H^\omega := \{ f \in C_{2\pi};\ |f(x+h) - f(x)| \le C\omega(|h|) \},$$

where C is a positive constant. For $f \in H^\omega$, we define the norm $\|\cdot\|_\omega = \|\cdot\|_{H^\omega}$ by the formula

$$\|f\|_\omega := \|f\|_C + \|f\|_{C,\omega},$$

where

$$\|f\|_{C,\omega} = \sup_{h \ne 0} \frac{\|f(\cdot + h) - f(\cdot)\|_C}{\omega(|h|)},$$

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and $\|f\|_{C,0}=0$. If $\omega(t)=C_1|t|^{\alpha}$ $(0<\alpha\le 1)$, where C_1 is a positive constant, then

$$H^{\alpha}=\{f\in C_{2\pi};\ |f(x+h)-f(x)|\le C_1|h|^{\alpha},\ 0<\alpha\le 1\}$$

is a Banach space and the metric induced by the norm $\|\cdot\|_{\alpha}$ on H^{α} is said to be a Hölder metric.

Let $A:=(a_{nk})$ $(k,n=0,1,\dots)$ be a lower triangular infinite matrix of real numbers satisfying the following condition:

$$a_{nk}\ge 0\ \ (k,n=0,1,\dots),\quad a_{nk}=0,\quad k>n\quad\text{and}\quad \sum_{k=0}^{n}a_{nk}=1. \tag{1.2}$$

Let the $A-$transformation of $(S_n(f;x))$ be given by

$$t_n(f):=t_n(f;x):=\sum_{k=0}^{n}a_{nk}S_k(f;x)\qquad (n=0,1,\dots) \tag{1.3}$$

and the strong A_r-transformation of $(S_n(f;x))$ for $r>0$ by

$$T_n(f,r):=T_n(f,r;x):=\left\{\sum_{k=0}^{n}a_{nk}|S_k(f;x)-f(x)|^{r}\right\}^{\frac{1}{r}}\qquad (n=0,1,\dots).$$

Now we define two classes of sequences ([3]).

A sequence $c:=(c_n)$ of nonnegative numbers tending to zero is called the Rest Bounded Variation Sequence, or briefly $c\in RBVS$, if it has the property

$$\sum_{k=m}^{\infty}|c_n-c_{n+1}|\le K(c)\,c_m \tag{1.4}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $c:=(c_n)$ of nonnegative numbers will be called a Head Bounded Variation Sequence, or briefly $c\in HBVS$, if it has the property

$$\sum_{k=0}^{m-1}|c_n-c_{n+1}|\le K(c)\,c_m \tag{1.5}$$

for all natural numbers m, or only for all $m\le N$ if the sequence c has only finite nonzero terms and the last nonzero term is c_N.

Therefore we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0\le K(\alpha_n)\le K$$

holds for all n, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (1.4) or (1.5) for the sequence $\alpha_n:=(a_{nk})_{k=0}^{\infty}$. Now we can give the conditions to be used later on. We assume that for all n and $0\le m\le n$,

$$\sum_{k=m}^{\infty}|a_{nk}-a_{nk+1}|\le K a_{nm} \tag{1.6}$$

and

$$\sum_{k=0}^{m-1}|a_{nk}-a_{nk+1}|\le K a_{nm} \tag{1.7}$$

hold if $\alpha_n:=(a_{nk})_{k=0}^{\infty}$ belongs to $RBVS$ or $HBVS$, respectively.

Let $\omega(t)$ and $\omega^*(t)$ be two given moduli of continuity satisfying the following condition (for $0 \le p < q \le 1$):

$$\frac{(\omega(t))^{\frac{p}{q}}}{\omega^*(t)} = O(1) \qquad (t \to 0_+). \tag{1.8}$$

In [4] R. Mohapatra and P. Chandra obtained some results on the degree of approximation for the means (1.3) in the Hölder metric. Recently, T. Singh in [5] established the following two theorems generalizing some results of P. Chandra [1] with a mediate function H such that:

$$\int_u^\pi \frac{\omega(f;t)}{t^2} dt = O(H(u)) \quad (u \to 0_+), \quad H(t) \ge 0 \tag{1.9}$$

and

$$\int_0^t H(u)\,du = O(tH(t)) \qquad (t \to O_+). \tag{1.10}$$

Theorem 1.1. *Let $A = (a_{nk})$ satisfy the condition (1.2) and $a_{nk} \le a_{nk+1}$ for $k = 0, 1, \ldots, n-1$, and $n = 0, 1, \ldots$. Then for $f \in H^\omega$, $0 \le p < q \le 1$,*

$$\|t_n(f) - f\|_{\omega^*} = O\left[\{\omega(|x-y|)\}^{\frac{p}{q}} \{\omega^*(|x-y|)\}^{-1} \times \left\{\left(H\left(\frac{\pi}{n}\right)\right)^{1-\frac{p}{q}} a_{nn} \left(n^{\frac{p}{q}} + a_{nn}^{-\frac{p}{q}}\right)\right\}\right] + O\left(a_{nn} H\left(\frac{\pi}{n}\right)\right), \tag{1.11}$$

if $\omega(f;t)$ satisfies (1.9) and (1.10), and

$$\|t_n(f) - f\|_{\omega^*} = O\left[\{\omega(|x-y|)\}^{\frac{p}{q}} \{\omega^*(|x-y|)\}^{-1}\right] \times \left\{\left(\omega\left(\frac{\pi}{n}\right)\right)^{1-\frac{p}{q}} + a_{nn} n^{\frac{p}{q}} \left(H\left(\frac{\pi}{n}\right)\right)^{1-\frac{p}{q}}\right\} + O\left\{\omega\left(\frac{\pi}{n}\right) + a_{nn} H\left(\frac{\pi}{n}\right)\right\}, \tag{1.12}$$

if $\omega(f;t)$ satisfies (1.9), where $\omega^*(t)$ is the given modulus of continuity.

Theorem 1.2. *Let $A = (a_{nk})$ satisfy the condition (1.2) and $a_{nk} \le a_{nk+1}$ for $k = 0, 1, \ldots, n-1$, and $n = 0, 1, \ldots$. Also, let $\omega(f;t)$ satisfy (1.9) and (1.10). Then for $f \in H^\omega$, $0 \le p < q \le 1$,*

$$\|t_n(f) - f\|_{\omega^*} = O\left[\{\omega(|x-y|)\}^{\frac{p}{q}} \{\omega^*(|x-y|)\}^{-1} \times \left\{(H(a_{n0}))^{1-\frac{p}{q}} a_{n0} \left(n^{\frac{p}{q}} + a_{n0}^{-\frac{p}{q}}\right)\right\}\right] + O(a_{n0} H(a_{n0})), \tag{1.13}$$

where $\omega^*(t)$ is the given modulus of continuity.

The next generalization of another result of P. Chandra [2] was obtained by L. Leindler in [3]. Namely, he proved the following two theorems

Theorem 1.3. *Let (1.2) and (1.9) hold. Then for $f \in C_{2\pi}$*

$$\|t_n(f) - f\|_C = O\left(\omega\left(\frac{\pi}{n}\right)\right) + O\left(a_{nn} H\left(\frac{\pi}{n}\right)\right). \tag{1.14}$$

If, in addition $\omega(f;t)$ satisfies the condition (1.10), then

$$\|t_n(f) - f\|_C = O(a_{nn} H(a_{nn})). \tag{1.15}$$

Theorem 1.4. *Let (1.2), (1.9) and (1.10) hold. Then for $f \in C_{2\pi}$*

$$\|t_n(f) - f\|_C = O(a_{n0} H(a_{n0})). \tag{1.16}$$

In the present paper we will generalize (and improve) the mentioned results of T. Singh [5] to strong summability with a mediate function H defined by the following conditions:

$$\int_{u}^{\pi}\frac{\omega^{r}(f;t)}{t^{2}}dt=O\left(H\left(r;u\right)\right)\qquad\left(u\rightarrow0_{+}\right),\qquad H\left(t\right)\geq0\text{ and }r>0, \tag{1.17}$$

and

$$\int_{0}^{t}H\left(u\right)du=O\left(tH\left(r;t\right)\right)\qquad\left(t\rightarrow O_{+}\right). \tag{1.18}$$

We also apply a generalization of Leindler's type [3].

Throughout the paper we shall use the following notation:

$$\phi_{x}\left(t\right)=f\left(x+t\right)+f\left(x-t\right)-2f\left(x\right).$$

By K_1, $K_2, \ldots$ we shall designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same at each occurrence.

2. Main Results

Our main results are the following.

Theorem 2.1. *Let (1.2), (1.7) and (1.8) hold. Suppose $\omega(f;t)$ satisfies (1.17) for $r\geq1$. Then for $f\in H^{\omega}$,*

$$\left\|T_{n}\left(f,r\right)\right\|_{\omega^{*}}=O\Bigg(\left\{1+\ln\left(2\left(n+1\right)a_{nn}\right)\right\}^{\frac{p}{q}}\times\left\{\left(\left(n+1\right)a_{nn}\right)^{r-1}a_{nn}H\left(r;\frac{\pi}{n}\right)\right\}^{\frac{1}{r}\left(1-\frac{p}{q}\right)}\Bigg). \tag{2.1}$$

If, in addition $\omega(f;t)$ satisfies the condition (1.18), then

$$\left\|T_{n}\left(f,r\right)\right\|_{\omega^{*}}=O\Bigg(\left\{1+\ln\left(2\left(n+1\right)a_{nn}\right)\right\}^{\frac{p}{q}}\times\left\{\left(\ln\left(2\left(n+1\right)a_{nn}\right)\right)^{r-1}a_{nn}H\left(r;a_{nn}\right)\right\}^{\frac{1}{r}\left(1-\frac{p}{q}\right)}\Bigg). \tag{2.2}$$

Theorem 2.2. *Under the assumptions of above theorem, if there exists a real number $s>1$ such that the inequality*

$$\left\{\sum_{i=2^{k-1}}^{2^{k}-1}\left(a_{ni}\right)^{s}\right\}^{\frac{1}{s}}\leq K_{1}\left(2^{k-1}\right)^{\frac{1}{s}-1}\sum_{i=2^{k-1}}^{2^{k}-1}a_{ni} \tag{2.3}$$

for any $k=1,2,\ldots,m$, where $2^{m}\leq n+1<2^{m+1}$ holds, then the following estimates

$$\left\|T_{n}\left(f,r\right)\right\|_{\omega^{*}}=O\left(\left\{a_{nn}H\left(r;\frac{\pi}{n}\right)\right\}^{\frac{1}{r}\left(1-\frac{p}{q}\right)}\right) \tag{2.4}$$

and

$$\left\|T_{n}\left(f,r\right)\right\|_{\omega^{*}}=O\left(\left\{a_{nn}H\left(r;a_{nn}\right)\right\}^{\frac{1}{r}\left(1-\frac{p}{q}\right)}\right) \tag{2.5}$$

are true.

Theorem 2.3. *Let (1.2), (1.6), (1.8) and (1.17) for $r \geq 1$ hold. Then for $f \in H^{\omega}$*

$$\|T_n(f,r)\|_{\omega^*} = O\left(\left\{a_{n0} H\left(r; \frac{\pi}{n}\right)\right\}^{\frac{1}{r}\left(1-\frac{p}{q}\right)}\right). \tag{2.6}$$

If, in addition, $\omega(f;t)$ satisfies (1.18), then

$$\|T_n(f,r)\|_{\omega^*} = O\left(\left\{a_{n0} H(r; a_{n0})\right\}^{\frac{1}{r}\left(1-\frac{p}{q}\right)}\right). \tag{2.7}$$

Remark 2.4. We can observe, that for the case $r = 1$ under the condition (1.8) the first part of Theorem 1.1 (1.11) and Theorem 1.2 are the corollaries of the first part of Theorem 2.1 (2.1) and the second part of Theorem 2.3 (2.7), respectively. We can also note that the mentioned estimates are better in order than the analogical estimates from the results of T. Singh, since $\ln(2(n+1)a_{nn})$ in Theorem 2.1 is better than $(n+1)a_{nn}$ in Theorem 1.1. Consequently, if na_{nn} is not bounded our estimate (2.7) in Theorem 2.3 is better than (1.13) from Theorem 1.2.

Remark 2.5. If in the assumptions of Theorem 2.1 or 2.3 we take $\omega(|t|) = O(|t|^q)$, $\omega^*(|t|) = O(|t|^p)$ with $p = 0$, then from (2.1), (2.2) and (2.7) we have the same estimates such as (1.14), (1.15) and (1.16), respectively, but for the strong approximation (with $r = 1$).

3. COROLLARIES

In this section we present some special cases of our results. From Theorems 2.1, 2.2 and 2.3, putting $\omega^*(|t|) = O\left(|t|^{\beta}\right)$, $\omega(|t|) = O(|t|^{\alpha})$,

$$H(r;t) = \begin{cases} t^{r\alpha-1} & \text{if } \alpha r < 1, \\ \ln\frac{\pi}{t} & \text{if } \alpha r = 1, \\ K_1 & \text{if } \alpha r > 1 \end{cases}$$

where $r > 0$ and $0 < \alpha \leq 1$, and replacing p by β and q by α, we can derive Corollaries 3.1, 3.2 and 3.3, respectively.

Corollary 3.1. *Under the conditions (1.2) and (1.7) we have for $f \in H^{\alpha}$, $0 \leq \beta < \alpha \leq 1$ and $r \geq 1$,*

$$\|T_n(f,r)\|_{\beta} = \begin{cases} O\left(\{\ln(2(n+1)a_{nn})\}^{1+\frac{1}{r}\left(1-\frac{\beta}{\alpha}\right)} \{a_{nn}\}^{\alpha-\beta}\right) & \text{if } \alpha r < 1, \\ O\left(\{\ln(2(n+1)a_{nn})\}^{1+\alpha-\beta} \left\{\ln\left(\frac{\pi}{a_{nn}}\right) a_{nn}\right\}^{\alpha-\beta}\right) & \text{if } \alpha r = 1, \\ O\left(\{\ln(2(n+1)a_{nn})\}^{1+\frac{1}{r}\left(1-\frac{\beta}{\alpha}\right)} \{a_{nn}\}^{\frac{\alpha-\beta}{\alpha r}}\right) & \text{if } \alpha r > 1. \end{cases}$$

Corollary 3.2. *Under the assumptions of Corollary 3.1 and (2.3) we have*

$$\|T_n(f,r)\|_{\beta} = \begin{cases} O\left(\{a_{nn}\}^{\alpha-\beta}\right) & \text{if } \alpha r < 1, \\ O\left(\left\{\ln\left(\frac{\pi}{a_{nn}}\right) a_{nn}\right\}^{\alpha-\beta}\right) & \text{if } \alpha r = 1, \\ O\left(\{a_{nn}\}^{\frac{\alpha-\beta}{\alpha r}}\right) & \text{if } \alpha r > 1. \end{cases}$$

Corollary 3.3. *Under the conditions (1.2) and (1.6) we have, for $f \in H^{\alpha}$, $0 \leq \beta < \alpha \leq 1$ and $r \geq 1$,*

$$\left\| T_n(f,r) \right\|_{\beta} = \begin{cases} O\left(\{a_{n0}\}^{\alpha-\beta} \right) & \text{if } \alpha r < 1, \\ O\left(\left\{ \ln\left(\frac{\pi}{a_{n0}} \right) a_{n0} \right\}^{\alpha-\beta} \right) & \text{if } \alpha r = 1, \\ O\left(\{a_{n0}\}^{\frac{\alpha-\beta}{\alpha r}} \right) & \text{if } \alpha r > 1. \end{cases}$$

4. LEMMAS

To prove our theorems we need the following lemmas.

Lemma 4.1. *If (1.17) and (1.18) hold with $r > 0$ then*

$$\int_0^s \frac{\omega^r(f;t)}{t} dt = O\left(sH(r;s) \right) \qquad (s \to 0_+). \tag{4.1}$$

Proof. Integrating by parts, by (1.17) and (1.18) we get

$$\begin{aligned} \int_0^s \frac{\omega^r(f;t)}{t} dt &= \left[-t \int_t^{\pi} \frac{\omega^r(f;u)}{u^2} du \right]_0^s + \int_0^s dt \int_t^{\pi} \frac{\omega^r(f;u)}{u^2} du \\ &= O\left(sH(r;s) \right) + O(1) \int_0^s H(r;t)\, dt \\ &= O\left(sH(r;s) \right). \end{aligned}$$

This completes the proof. □

Lemma 4.2 ([7]). *If (1.2), (1.7) hold, then for $f \in C_{2\pi}$ and $r > 0$,*

$$\left\| T_n(f,r) \right\|_C \leq O\left(\left\{ \sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k} E_k^r(f) + \left(E_{\left[\frac{n+1}{4}\right]}(f) \ln\left(2(n+1) a_{nn} \right) \right)^r \right\}^{\frac{1}{r}} \right). \tag{4.2}$$

If, in addition, (2.3) holds, then

$$\left\| T_n(f,r) \right\|_C \leq O\left(\left\{ \sum_{k=0}^{\left[\frac{n+1}{2}\right]} a_{n,2k} E_k^r(f) \right\}^{\frac{1}{r}} \right). \tag{4.3}$$

Lemma 4.3 ([7]). *If (1.2), (1.6) hold, then for $f \in C_{2\pi}$ and $r > 0$,*

$$\left\| T_n(f,r) \right\|_C \leq O\left(\left\{ \sum_{k=0}^{n} a_{nk} E_k^r(f) \right\}^{\frac{1}{r}} \right). \tag{4.4}$$

Lemma 4.4. *If (1.2), (1.7) hold and $\omega(f;t)$ satisfies (1.17) with $r > 0$ then*

$$\sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k} \omega^r\left(f; \frac{\pi}{k+1} \right) = O\left(a_{nn} H\left(r; \frac{\pi}{n} \right) \right). \tag{4.5}$$

If, in addition, $\omega(f;t)$ satisfies (1.18) then

$$\sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k} \omega^r\left(f; \frac{\pi}{k+1} \right) = O\left(a_{nn} H(r; a_{nn}) \right). \tag{4.6}$$

Proof. First we prove (4.5). If (1.7) holds then

$$a_{n\mu} - a_{nm} \leq |a_{n\mu} - a_{nm}| \leq \sum_{k=\mu}^{m-1} |a_{nk} - a_{nk+1}| \leq K a_{nm}$$

for any $m \geq \mu \geq 0$, whence we have

$$a_{n\mu} \leq (K+1)\, a_{nm}. \tag{4.7}$$

From this and using (1.17) we get

$$\begin{aligned}
\sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k}\omega^r\left(f;\frac{\pi}{k+1}\right) &\leq (K+1)\, a_{nn} \sum_{k=0}^{n} \omega^r\left(f;\frac{\pi}{k+1}\right) \\
&\leq K_1 a_{nn} \int_1^{n+1} \omega^r\left(f;\frac{\pi}{t}\right) dt \\
&= \pi K_1 a_{nn} \int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega^r(f;u)}{u^2} du \\
&= O\left(a_{nn} H\left(r;\frac{\pi}{n}\right)\right).
\end{aligned}$$

Now we prove (4.6). Since

$$(K+1)(n+1)\, a_{nn} \geq \sum_{k=0}^{n} a_{nk} = 1,$$

we can see that

$$\begin{aligned}
\sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k}\omega^r\left(f;\frac{\pi}{k+1}\right) \leq &\sum_{k=0}^{\left[\frac{1}{4(K+1)a_{nn}}\right]-1} a_{n,4k}\omega^r\left(f;\frac{\pi}{k+1}\right) \\
&+ \sum_{k=\left[\frac{1}{4(K+1)a_{nn}}\right]-1}^{n} a_{n,4k}\omega^r\left(f;\frac{\pi}{k+1}\right) \\
= &\Sigma_1 + \Sigma_2.
\end{aligned} \tag{4.8}$$

Using again (4.7), (1.2) and the monotonicity of the modulus of continuity, we can estimate the quantities Σ_1 and Σ_2 as follows

$$\begin{aligned}
\Sigma_1 &\leq (K+1)\, a_{nn} \sum_{k=0}^{\left[\frac{1}{4(K+1)a_{nn}}\right]-1} \omega^r\left(f;\frac{\pi}{k+1}\right) \\
&\leq K_2 a_{nn} \int_1^{\frac{1}{4(K+1)a_{nn}}} \omega^r\left(f;\frac{\pi}{t}\right) dt \\
&= \pi K_2 a_{nn} \int_{4\pi(K+1)a_{nn}}^{\pi} \frac{\omega^r(f;u)}{u^2} du \\
&\leq \pi K_2 a_{nn} \int_{a_{nn}}^{\pi} \frac{\omega^r(f;u)}{u^2} du
\end{aligned} \tag{4.9}$$

and

$$\Sigma_2 \leq K_3 \omega^r \left(f; 4\pi \left(K+1\right) a_{nn}\right) \sum_{k=\left[\frac{1}{4(K+1)a_{nn}}\right]-1}^{n} a_{n,4k} \tag{4.10}$$
$$\leq K_3 \left(8\pi \left(K+1\right)\right)^r \omega^r \left(f; a_{nn}\right)$$
$$\leq K_3 \left(32\pi \left(K+1\right)\right)^r \omega^r \left(f; \frac{a_{nn}}{2}\right)$$
$$\leq 2K_3 \left(32\pi \left(K+1\right)\right)^r \int_{\frac{a_{nn}}{2}}^{a_{nn}} \frac{\omega^r \left(f;t\right)}{t} dt$$
$$\leq K_4 \int_0^{a_{nn}} \frac{\omega^r \left(f;t\right)}{t} dt.$$

If (1.17) and (1.18) hold then from (4.8) – (4.10) we obtain (4.6). This completes the proof. □

Lemma 4.5. *If (1.2), (1.7) hold and $\omega\left(f;t\right)$ satisfies (1.17) with $r \geq 1$ then*

$$\omega\left(f, \frac{\pi}{n+1}\right) \ln\left(2\left(n+1\right) a_{nn}\right) = O\left(\left\{\left(n+1\right) a_{nn}\right\}^{1-\frac{1}{r}} \left\{a_{nn} H\left(r; \frac{\pi}{n}\right)\right\}^{\frac{1}{r}}\right). \tag{4.11}$$

If, in addition, $\omega\left(f;t\right)$ satisfies (1.18) then

$$\omega\left(f, \frac{\pi}{n+1}\right) \ln\left(2\left(n+1\right) a_{nn}\right) = O\left(\left\{\ln\left(2\left(n+1\right) a_{nn}\right)\right\}^{1-\frac{1}{r}} \left\{a_{nn} H\left(r; a_{nn}\right)\right\}^{\frac{1}{r}}\right). \tag{4.12}$$

Proof. Let $r = 1$. Using the monotonicity of the modulus of continuity

$$\omega\left(f, \frac{\pi}{n+1}\right) \ln\left(2\left(n+1\right) a_{nn}\right) \leq 2a_{nn} \omega\left(f, \frac{\pi}{n+1}\right)\left(n+1\right)$$
$$\leq 4a_{nn} \omega\left(f, \frac{\pi}{n+1}\right) \int_1^{n+1} dt$$
$$\leq 4a_{nn} \int_1^{n+1} \omega\left(f, \frac{\pi}{t}\right) dt$$
$$= 4\pi a_{nn} \int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega\left(f,u\right)}{u^2} du$$

and by (1.17) we obtain that (4.11) holds. Now we prove (4.12). From (1.2) and (1.7) we get

$$\omega\left(f, \frac{\pi}{n+1}\right) \ln\left(2\left(n+1\right) a_{nn}\right) \leq K_1 \omega\left(f, \frac{\pi}{n+1}\right) \int_{\frac{\pi}{n+1}}^{\pi(K+1)a_{nn}} \frac{1}{t} dt,$$

$$K_1 \int_{\frac{\pi}{n+1}}^{\pi(K+1)a_{nn}} \frac{\omega\left(f,t\right)}{t} dt \leq 2K_1 \left(K+1\right) \pi \int_{\frac{1}{(K+1)(n+1)}}^{a_{nn}} \frac{\omega\left(f,u\right)}{u} du$$
$$\leq K_2 \int_0^{a_{nn}} \frac{\omega\left(f,u\right)}{u} du$$

and by Lemma 4.1 we obtain (4.12).

Assuming $r > 1$ we can use the Hölder inequality to estimate the following integrals

$$\int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega(f,u)}{u^2} du \leq \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega^r(f,u)}{u^2} du \right\}^{\frac{1}{r}} \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \frac{1}{u^2} du \right\}^{1-\frac{1}{r}} \leq \left(\frac{n+1}{\pi} \right)^{1-\frac{1}{r}} \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega^r(f,u)}{u^2} du \right\}^{\frac{1}{r}}$$

and

$$\int_{\frac{1}{(K+1)(n+1)}}^{a_{nn}} \frac{\omega(f,u)}{u} du \leq \left\{ \int_{\frac{1}{(K+1)(n+1)}}^{a_{nn}} \frac{\omega^r(f,u)}{u} du \right\}^{\frac{1}{r}} \left\{ \int_{\frac{1}{(K+1)(n+1)}}^{a_{nn}} \frac{1}{u} du \right\}^{1-\frac{1}{r}} \leq \{\ln(2(n+1)a_{nn})\}^{1-\frac{1}{r}} \left\{ \int_0^{a_{nn}} \frac{\omega^r(f,u)}{u} du \right\}^{\frac{1}{r}}.$$

From this, if (1.17) holds then

$$\omega\left(f, \frac{\pi}{n+1}\right) \ln(2(n+1)a_{nn}) \leq 4\pi a_{nn} \left(\frac{n+1}{\pi} \right)^{1-\frac{1}{r}} \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega^r(f,u)}{u^2} du \right\}^{\frac{1}{r}} = O\left(\{(n+1)a_{nn}\}^{1-\frac{1}{r}} \left\{ a_{nn} H\left(r; \frac{\pi}{n}\right) \right\}^{\frac{1}{r}} \right)$$

and if (1.17) and (1.18) hold then

$$\omega\left(f, \frac{\pi}{n+1}\right) \ln(2(n+1)a_{nn}) \leq 2K_1(K+1)\pi \{\ln(2(n+1)a_{nn})\}^{1-\frac{1}{r}} \left\{ \int_0^{a_{nn}} \frac{\omega^r(f,u)}{u} du \right\}^{\frac{1}{r}} = O\left(\{\ln(2(n+1)a_{nn})\}^{1-\frac{1}{r}} \left\{ a_{nn} H\left(r; \frac{\pi}{n}\right) \right\}^{\frac{1}{r}} \right).$$

This ends our proof. □

Lemma 4.6. *If (1.2), (1.6) hold and $\omega(f;t)$ satisfies (1.17) with $r > 0$ then*

$$\sum_{k=0}^{n} a_{nk} \omega^r \left(f; \frac{\pi}{k+1} \right) = O\left(a_{n0} H\left(r; \frac{\pi}{n}\right) \right). \tag{4.13}$$

If, in addition, $\omega(f;t)$ satisfies (1.18), then

$$\sum_{k=0}^{n} a_{nk} \omega^r \left(f; \frac{\pi}{k+1} \right) = O\left(a_{n0} H(r; a_{n0}) \right). \tag{4.14}$$

Proof. First we prove (4.13). If (1.6) holds then

$$a_{nn} - a_{nm} \leq |a_{nm} - a_{nn}| \leq \sum_{k=m}^{n-1} |a_{nk} - a_{nk+1}| \leq \sum_{k=m}^{\infty} |a_{nk} - a_{nk+1}| \leq K a_{nm}$$

for any $n \geq m \geq 0$, whence we have

$$a_{nn} \leq (K+1)\, a_{nm}. \tag{4.15}$$

From this and using (1.17) we get

$$\begin{aligned}\sum_{k=0}^{n} a_{nk}\omega^{r}\left(f;\frac{\pi}{k+1}\right) &\leq (K+1)\, a_{n0}\sum_{k=0}^{n}\omega^{r}\left(f;\frac{\pi}{k+1}\right)\\ &\leq K_{1}a_{n0}\int_{1}^{n+1}\omega^{r}\left(f;\frac{\pi}{t}\right)dt\\ &= \pi K_{1}a_{n0}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\omega^{r}\left(f;u\right)}{u^{2}}du\\ &= O\left(a_{n0}H\left(r;\frac{\pi}{n}\right)\right).\end{aligned}$$

Now, we prove (4.14). Since

$$(K+1)\,(n+1)\, a_{n0} \geq \sum_{k=0}^{n} a_{nk} = 1,$$

we can see that

$$\begin{aligned}\sum_{k=0}^{n} a_{nk}\omega^{r}\left(f;\frac{\pi}{k+1}\right) \leq &\sum_{k=0}^{\left[\frac{1}{(K+1)a_{n0}}\right]-1} a_{nk}\omega^{r}\left(f;\frac{\pi}{k+1}\right)\\ &+\sum_{k=\left[\frac{1}{(K+1)a_{n0}}\right]-1}^{n} a_{nk}\omega^{r}\left(f;\frac{\pi}{k+1}\right).\end{aligned}$$

Using again (1.2), (1.6) and the monotonicity of the modulus of continuity, we get

$$\begin{aligned}\sum_{k=0}^{n} a_{nk}\omega^{r}\left(f;\frac{\pi}{k+1}\right) &\leq (K+1)\, a_{n0}\sum_{k=0}^{\left[\frac{1}{(K+1)a_{n0}}\right]-1}\omega^{r}\left(f;\frac{\pi}{k+1}\right)\\ &\quad + K_{1}\omega^{r}\left(f;\pi\,(K+1)\, a_{no}\right)\sum_{k=\left[\frac{1}{(K+1)a_{n0}}\right]-1}^{n} a_{nk}\\ &\leq K_{2}a_{n0}\int_{1}^{\frac{1}{(K+1)a_{n0}}}\omega^{r}\left(f;\frac{\pi}{t}\right)dt + K_{1}\omega^{r}\left(f;\pi\,(K+1)\, a_{no}\right)\\ &\leq K_{3}\left(a_{n0}\int_{a_{n0}}^{\pi}\frac{\omega^{r}\left(f;u\right)}{u^{2}}du + \omega^{r}\left(f;a_{n0}\right)\right).\end{aligned} \tag{4.16}$$

According to

$$\omega^{r}\left(f;a_{n0}\right) \leq 4^{r}\omega^{r}\left(f;\frac{a_{n0}}{2}\right) \leq 2\cdot 4^{r}\int_{\frac{a_{n0}}{2}}^{a_{n0}}\frac{\omega^{r}\left(f;t\right)}{t}dt \leq 2\cdot 4^{r}\int_{0}^{a_{n0}}\frac{\omega^{r}\left(f;t\right)}{t}dt,$$

(1.17), (1.18) and (4.16) lead us to (4.14). □

5. PROOFS OF THE THEOREMS

In this section we shall prove Theorems 2.1, 2.2 and 2.3.

5.1. **Proof of Theorem 2.1.** Setting

$$R_n(x+h,x) = T_n(f,r;x+h) - T_n(f,r;x)$$

and

$$g_h(x) = f(x+h) - f(x)$$

and using the Minkowski inequality for $r \geq 1$, we get

$$\begin{aligned}
&|R_n(x+h,x)| \\
&= \left| \left\{ \sum_{k=0}^{n} a_{nk} |S_k(f;x+h) - f(x+h)|^r \right\}^{\frac{1}{r}} - \left\{ \sum_{k=0}^{n} a_{nk} |S_k(f;x) - f(x)|^r \right\}^{\frac{1}{r}} \right| \\
&\leq \left\{ \sum_{k=0}^{n} a_{nk} |S_k(g_h;x) - g_h(x)|^r \right\}^{\frac{1}{r}}.
\end{aligned}$$

By (4.2) we have

$$\begin{aligned}
&|R_n(x+h,x)| \\
&\leq K_1 \left\{ \sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k} E_k^r(g_h) + \left(E_{\left[\frac{n+1}{4}\right]}(g_h) \ln(2(n+1)a_{nn}) \right)^r \right\}^{\frac{1}{r}} \\
&\leq K_2 \left\{ \sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k} \omega^r \left(g_h, \frac{\pi}{k+1} \right) + \left(\omega \left(g_h, \frac{\pi}{n+1} \right) \ln(2(n+1)a_{nn}) \right)^r \right\}^{\frac{1}{r}}.
\end{aligned}$$

Since

$$|g_h(x+l) - g_h(x)| \leq |f(x+l+h) - f(x+h)| + |f(x+l) - f(x)|$$

and

$$|g_h(x+l) - g_h(x)| \leq |f(x+l+h) - f(x+l)| + |f(x+h) - f(x)| \leq 2\omega(|h|),$$

therefore, for $0 \leq k \leq n$,

$$\omega\left(g_h, \frac{\pi}{k+1}\right) \leq 2\omega\left(f, \frac{\pi}{k+1}\right) \tag{5.1}$$

and $f \in H^\omega$

$$\omega\left(g_h, \frac{\pi}{k+1}\right) \leq 2\omega(|h|). \tag{5.2}$$

From (5.2) and (1.2)

$$\begin{aligned}
|R_n(x+h,x)| &\leq 2K_2\omega(|h|) \left\{ \sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k} + (\ln(2(n+1)a_{nn}))^r \right\}^{\frac{1}{r}} \\
&\leq 2K_2\omega(|h|)(1 + \ln(2(n+1)a_{nn})).
\end{aligned} \tag{5.3}$$

On the other hand, by (5.1),

$$|R_n(x+h,x)| \le 2K_2\left\{\sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k}\omega^r\left(f,\frac{\pi}{k+1}\right)+\left(\omega\left(f,\frac{\pi}{n+1}\right)\ln\left(2(n+1)a_{nn}\right)\right)^r\right\}^{\frac{1}{r}}. \tag{5.4}$$

Using (5.3) and (5.4) we get

$$\begin{aligned}
&\sup_{h\neq 0}\frac{\|T_n(f,r;\cdot+h)-T_n(f,r;\cdot)\|_C}{\omega(|h|)} \\
&= \sup_{h\neq 0}\frac{\left(\|R_n(\cdot+h,\cdot)\|_C\right)^{\frac{p}{q}}}{\omega(|h|)}\left(\|R_n(\cdot+h,\cdot)\|_C\right)^{1-\frac{p}{q}} \\
&\le K_3\left(1+\ln\left(2(n+1)a_{nn}\right)\right)^{\frac{p}{q}} \\
&\quad\times\left\{\sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k}\omega^r\left(f,\frac{\pi}{k+1}\right)+\left(\omega\left(f,\frac{\pi}{n+1}\right)\ln\left(2(n+1)a_{nn}\right)\right)^r\right\}^{\frac{1}{r}\left(1-\frac{p}{q}\right)}.
\end{aligned} \tag{5.5}$$

Similarly, by (4.2) we have

$$\begin{aligned}
&\|T_n(f,r)\|_C \\
&\le K_4\left\{\sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k}\omega^r\left(f,\frac{\pi}{k+1}\right)+\left(\omega\left(f,\frac{\pi}{n+1}\right)\ln\left(2(n+1)a_{nn}\right)\right)^r\right\}^{\frac{1}{r}} \\
&\le K_4\left\{\sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k}\omega^r\left(f,\frac{\pi}{k+1}\right)+\left(\omega\left(f,\frac{\pi}{n+1}\right)\ln\left(2(n+1)a_{nn}\right)\right)^r\right\}^{\frac{1}{r}\frac{p}{q}} \\
&\quad\times\left\{\sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k}\omega^r\left(f,\frac{\pi}{k+1}\right)+\left(\omega\left(f,\frac{\pi}{n+1}\right)\ln\left(2(n+1)a_{nn}\right)\right)^r\right\}^{\frac{1}{r}\left(1-\frac{p}{q}\right)} \\
&\le K_5\left(1+\ln\left(2(n+1)a_{nn}\right)\right)^{\frac{p}{q}} \\
&\quad\times\left\{\sum_{k=0}^{\left[\frac{n+1}{4}\right]} a_{n,4k}\omega^r\left(f,\frac{\pi}{k+1}\right)+\left(\omega\left(f,\frac{\pi}{n+1}\right)\ln\left(2(n+1)a_{nn}\right)\right)^r\right\}^{\frac{1}{r}\left(1-\frac{p}{q}\right)}.
\end{aligned} \tag{5.6}$$

Collecting our partial results (5.5), (5.6) and using Lemma 4.4 and Lemma 4.5 we obtain that (2.1) and (2.2) hold. This completes our proof. □

5.2. **Proof of Theorem 2.2.** Using (4.3) and the same method as in the proof of Lemma 4.4 we can show that

$$\sum_{k=0}^{\left[\frac{n+1}{2}\right]} a_{n,2k}\omega^r\left(f,\frac{\pi}{k+1}\right)=O\left(a_{nn}H\left(r;\frac{\pi}{n}\right)\right) \tag{5.7}$$

holds, if $\omega(t)$ satisfies (1.17) and (1.18), and

(5.8)
$$\sum_{k=0}^{\left[\frac{n+1}{2}\right]} a_{n,2k}\omega^r\left(f,\frac{\pi}{k+1}\right) = O\left(a_{nn}H\left(r;a_{nn}\right)\right)$$

if $\omega\left(t\right)$ satisfies (1.17).

The proof of Theorem 2.2 is analogously to the proof of Theorem 2.1. The only difference being that instead of (4.2), (4.5) and (4.6) we use (4.3), (5.7) and (5.8) respectively. This completes the proof. □

5.3. **Proof of Theorem 2.3.** Using the same notations as in the proof of Theorem 2.1, from (4.4) and (5.2) we get

(5.9)
$$\begin{aligned}\left|R_n\left(x+h,x\right)\right| &\le K_1\left\{\sum_{k=0}^{n} a_{nk}E_k^r\left(g_h\right)\right\}^{\frac{1}{r}}\\ &\le K_2\left\{\sum_{k=0}^{n} a_{nk}\omega^r\left(g_h,\frac{\pi}{k+1}\right)\right\}^{\frac{1}{r}}\\ &\le 2K_2\omega\left(\left|h\right|\right).\end{aligned}$$

On the other hand, by (4.4) and (5.1), we have

(5.10)
$$\begin{aligned}\left|R_n\left(x+h,x\right)\right| &\le K_2\left\{\sum_{k=0}^{n} a_{nk}\omega^r\left(g_h,\frac{\pi}{k+1}\right)\right\}^{\frac{1}{r}}\\ &\le 2K_2\left\{\sum_{k=0}^{n} a_{nk}\omega^r\left(f,\frac{\pi}{k+1}\right)\right\}^{\frac{1}{r}}.\end{aligned}$$

Similarly, we can show that

(5.11)
$$\left\|T_n\left(f,r\right)\right\|_C \le K_3\left\{\sum_{k=0}^{n} a_{nk}\omega^r\left(f,\frac{\pi}{k+1}\right)\right\}^{\frac{1}{r}}.$$

Finally, using the same method as in the proof of Theorem 2.1 and Lemma 4.6, (2.6) and (2.7) follow from (5.9) – (5.11). □

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