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BEST GENERALIZATION OF HARDY-HILBERT'S INEQUALITY WITH MULTI-PARAMETERS

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ABSTRACT. By introducing some parameters and the β function and improving the weight function, we obtain a generalization of Hilbert's integral inequality with the best constant factor. As its applications, we build its equivalent form and some particular results.

Key words and phrases: Hardy-Hilbert's inequality, Hölder's inequality, weight function, β function.

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1. INTRODUCTION

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, f, g are non-negative functions such that $0 < \int_0^\infty f^p(t)dt < \infty$ and $0 < \int_0^\infty g^q(t)dt < \infty$, then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \int_0^\infty f^p(t)dt \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(t)dt \right\}^{\frac{1}{q}}; \tag{1.1}$$

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y} dx \right]^p dy < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \int_0^\infty f^p(t)dt, \tag{1.2}$$

where the constant factors $\frac{\pi}{\sin(\pi/p)}$ and $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ are the best possible (see [1]). Inequality (1.1) is well known as Hardy-Hilbert's integral inequality, which is important in analysis and applications (see [2]). Inequality (1.1) is equivalent to (1.2).

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In 2002, Yang [3] gave some generalizations of (1.1) and (1.2) by introducing a parameter $\lambda > 0$ as:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda \sin(\frac{\pi}{p})} \left\{ \int_0^\infty t^{(p-1)(1-\lambda)} f^p(t) dt \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{(q-1)(1-\lambda)} g^q(t) dt \right\}^{\frac{1}{q}}; \tag{1.3}$$

$$\int_0^\infty y^{\lambda-1} \left[\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx \right]^p dy < \left[\frac{\pi}{\lambda \sin(\frac{\pi}{p})} \right]^p \int_0^\infty t^{(p-1)(1-\lambda)} f^p(t) dt, \tag{1.4}$$

where the constant factors $\frac{\pi}{\lambda \sin(\pi/p)}$ and $\left[\frac{\pi}{\lambda \sin(\pi/p)}\right]^p$ are the best possible. Inequality (1.3) is equivalent to (1.4).

When $\lambda = 1$, both (1.3) and (1.4) change to (1.1) and (1.2). Yang [4] gave another generalization of (1.1) by introducing a parameter λ and a β function.

In 2004, by introducing some parameters and estimating the weight function, Yang [5] gave some extensions of (1.1) and (1.2) with the best constant factors as:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda \sin(\frac{\pi}{r})} \left\{ \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x) dx \right\}^{\frac{1}{q}}; \tag{1.5}$$

$$\int_0^\infty y^{\frac{p\lambda}{s}-1} \left[\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx \right]^p dy < \left[\frac{\pi}{\lambda \sin(\frac{\pi}{r})} \right]^p \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx, \tag{1.6}$$

where the constant factors $\frac{\pi}{\lambda \sin(\pi/r)}$ and $\left[\frac{\pi}{\lambda \sin(\pi/r)}\right]^p$ are the best possible. Inequality (1.5) is equivalent to (1.6). Recently, [6, 7, 8, 9] considered some multiple extensions of (1.1).

Under the same conditions with (1.1), we still have (see [1, Th. 342]):

$$\int_0^\infty \int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)g(y)}{x-y} dxdy < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^2 \left\{ \int_0^\infty f^p(t) dt \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(t) dt \right\}^{\frac{1}{q}}; \tag{1.7}$$

$$\int_0^\infty \left[\frac{\ln\left(\frac{x}{y}\right) f(x)}{x-y} dx \right]^p dy < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^{2p} \int_0^\infty f^p(t) dt, \tag{1.8}$$

where the constant factors $\left[\frac{\pi}{\sin(\pi/p)}\right]^2$ and $\left[\frac{\pi}{\sin(\pi/p)}\right]^{2p}$ are the best possible. Inequality (1.7) is equivalent to (1.8). In recent years, by introducing a parameter λ, Kuang [10] gave an new extension of (1.7).

In 2003, by introducing a parameter $\lambda > 0$ and the weight function, Yang [11] gave another generalisation of (1.7) and the extended equivalent form as:

$$\int_0^\infty \int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)g(y)}{x^\lambda - y^\lambda} dxdy < \left[\frac{\pi}{\lambda \sin(\frac{\pi}{p})}\right]^2 \left\{\int_0^\infty t^{(p-1)(1-\lambda)} f^p(t)dt\right\}^{\frac{1}{p}} \left\{\int_0^\infty t^{(q-1)(1-\lambda)} g^q(t)dt\right\}^{\frac{1}{q}}; \tag{1.9}$$

$$\int_0^\infty y^{\lambda-1} \left[\frac{\ln\left(\frac{x}{y}\right) f(x)}{x^\lambda - y^\lambda} dx\right]^p dy < \left[\frac{\pi}{\lambda \sin(\frac{\pi}{p})}\right]^{2p} \int_0^\infty t^{(p-1)(1-\lambda)} f^p(t)dt, \tag{1.10}$$

where the constant factors $\left[\frac{\pi}{\lambda \sin(\pi/p)}\right]^2$ and $\left[\frac{\pi}{\lambda \sin(\pi/p)}\right]^{2p}$ are the best possible. Inequality (1.9) is equivalent to (1.10).

In this paper, by using the β function and obtaining the expression of the weight function, we give a new extension of (1.7) with some parameters as (1.5). As applications, we also consider the equivalent form and some other particular results.

2. SOME LEMMAS

Lemma 2.1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{s} + \frac{1}{r} = 1$, $\lambda > 0$, *define the weight function* $\omega_\lambda(s, p, x)$ *as*

$$\omega_\lambda(s, p, x) = \int_0^\infty \frac{\ln\left(\frac{x}{y}\right)}{x^\lambda - y^\lambda} \cdot \frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{s}}} dy, \quad x \in (0, \infty). \tag{2.1}$$

Then we have

$$\omega_\lambda(s, p, x) = x^{p(1-\frac{\lambda}{r})-1} \left[\frac{\pi}{\lambda \sin(\frac{\pi}{r})}\right]^2. \tag{2.2}$$

Proof. For fixed x, setting $u = (\frac{y}{x})^\lambda$ in the integral (2.1) and by [1] (see [1, Th. 342 Remark]), we have

$$\begin{aligned} \omega_\lambda(s, p, x) &= \frac{1}{\lambda^2} \int_0^\infty \frac{\ln u}{x^\lambda(u-1)} \cdot \frac{x^{(p-1)(1-\frac{\lambda}{r})}}{(u^{\frac{1}{\lambda}} x)^{1-\frac{\lambda}{s}}} x u^{\frac{1}{\lambda}-1} du \\ &= \frac{1}{\lambda^2} x^{p(1-\frac{\lambda}{r})-1} \int_0^\infty \frac{\ln u}{u-1} \cdot u^{-\frac{1}{r}} du \\ &= \left[\frac{\pi}{\lambda \sin(\frac{\pi}{r})}\right]^2 x^{p(1-\frac{\lambda}{r})-1}. \end{aligned} \tag{2.3}$$

Hence, (2.2) is valid and the lemma is proved. □

Note. By (2.3), we still have

$$\begin{aligned} \omega_\lambda(r, q, y) &= \int_0^\infty \frac{\ln\left(\frac{x}{y}\right)}{x^\lambda - y^\lambda} \cdot \frac{x^{(q-1)(1-\frac{\lambda}{s})}}{y^{1-\frac{\lambda}{r}}} dx \\ &= y^{q(1-\frac{\lambda}{s})-1} \left[\frac{\pi}{\lambda \sin(\frac{\pi}{s})}\right]^2. \end{aligned} \tag{2.4}$$

3. MAIN RESULTS AND APPLICATIONS

Theorem 3.1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{s} + \frac{1}{r} = 1$, $\lambda > 0$, $f, g \geq 0$ *such that* $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx < \infty$, *and* $0 < \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx < \infty$, *then we have*

$$\int_0^\infty \int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)g(y)}{x^\lambda - y^\lambda} dxdy < \left[\frac{\pi}{\lambda \sin(\frac{\pi}{r})}\right]^2 \left\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx\right\}^{\frac{1}{q}}, \tag{3.1}$$

where the constant factor $\left[\frac{\pi}{\lambda \sin(\pi/r)}\right]^2$ *is the best possible. In particular,*

(a) *for* $r = s = 2$, *we have*

$$\int_0^\infty \int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)g(y)}{x^\lambda - y^\lambda} dxdy < \left(\frac{\pi}{\lambda}\right)^2 \left\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1} g^q(x)dx\right\}^{\frac{1}{q}}, \tag{3.2}$$

(b) *for* $\lambda = 1$, *we have*

$$\int_0^\infty \int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)g(y)}{x - y} dxdy < \left[\frac{\pi}{\sin(\frac{\pi}{r})}\right]^2 \left\{\int_0^\infty x^{\frac{p}{s}-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{\frac{q}{r}-1} g^q(x)dx\right\}^{\frac{1}{q}}. \tag{3.3}$$

Proof. By Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)g(y)}{x^\lambda - y^\lambda} dxdy \\ &= \int_0^\infty \int_0^\infty \left\{\left[\frac{\ln\left(\frac{x}{y}\right)}{x^\lambda - y^\lambda}\right]^{\frac{1}{p}} \cdot \frac{x^{(1-\frac{\lambda}{r})/q}}{y^{(1-\frac{\lambda}{s})/p}} f(x)\right\} \left\{\left[\frac{\ln\left(\frac{x}{y}\right)}{x^\lambda - y^\lambda}\right]^{\frac{1}{q}} \cdot \frac{y^{(1-\frac{\lambda}{s})/p}}{x^{(1-\frac{\lambda}{r})/q}} g(y)\right\} dxdy \\ &\leq \left\{\int_0^\infty \left[\int_0^\infty \frac{\ln\left(\frac{x}{y}\right)}{x^\lambda - y^\lambda} \cdot \frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{(1-\frac{\lambda}{s})}} dy\right] f^p(x)dx\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\int_0^\infty \left[\int_0^\infty \frac{\ln\left(\frac{x}{y}\right)}{x^\lambda - y^\lambda} \cdot \frac{y^{(q-1)(1-\frac{\lambda}{s})}}{x^{(1-\frac{\lambda}{r})}} dx\right] g^q(y)dy\right\}^{\frac{1}{q}} \\ &= \left\{\int_0^\infty \omega_\lambda(s,p,x) f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \omega_\lambda(r,q,y) g^q(y)dy\right\}^{\frac{1}{q}}. \end{aligned} \tag{3.4}$$

If (3.4) takes the form of equality, then there exist constants A and B, such that they are not all zero and (see [12])

$$A\frac{\ln\left(\frac{x}{y}\right)}{x^{\lambda}-y^{\lambda}}\cdot\frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{s}}}f^{p}(x)=B\frac{\ln\left(\frac{x}{y}\right)}{x^{\lambda}-y^{\lambda}}\cdot\frac{y^{(q-1)(1-\frac{\lambda}{s})}}{x^{1-\frac{\lambda}{r}}}g^{q}(y),$$
$$\text{a.e. in } (0,\infty)\times(0,\infty).$$

We find that $Ax\cdot x^{p(1-\frac{\lambda}{r})-1}f^{p}(x)=By\cdot y^{q(1-\frac{\lambda}{s})-1}g^{q}(y)$, a.e. in $(0,\infty)\times(0,\infty)$. Hence there exists a constant C, such that

$$Ax\cdot x^{p(1-\frac{\lambda}{r})-1}f^{p}(x)=C=By\cdot y^{q(1-\frac{\lambda}{s})-1}g^{q}(y),\quad \text{a.e. in } (0,\infty).$$

Without loss of generality, suppose $A\neq 0$, we may get $x^{p(1-\frac{\lambda}{r})-1}f^{p}(x)=C/(Ax)$, a.e. in $(0,\infty)$, which contradicts $0<\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^{p}(x)dx<\infty$. Hence (3.4) takes strict inequality as follows:

$$\int_0^\infty\int_0^\infty\frac{\ln\left(\frac{x}{y}\right)f(x)g(y)}{x^{\lambda}-y^{\lambda}}dxdy < \left\{\int_0^\infty \omega_\lambda(s,p,x)f^{p}(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \omega_\lambda(r,q,y)g^{q}(y)dy\right\}^{\frac{1}{q}}. \tag{3.5}$$

In view of (2.2) and (2.4), we have (3.1).

If the constant factor $\left[\frac{\pi}{\lambda\sin(\pi/r)}\right]^2$ in (3.1) is not the best possible, then there exists a positive constant K $\left(\text{with } K<\left[\frac{\pi}{\lambda\sin(\pi/r)}\right]^2\right)$ and an $a>0$. We have

$$\int_a^\infty\int_0^\infty\frac{\ln\left(\frac{x}{y}\right)f(x)g(y)}{x^{\lambda}-y^{\lambda}}dxdy < K\left\{\int_a^\infty x^{p(1-\frac{\lambda}{r})-1}f^{p}(x)dx\right\}^{\frac{1}{p}}\left\{\int_a^\infty x^{q(1-\frac{\lambda}{s})-1}g^{q}(x)dx\right\}^{\frac{1}{q}}. \tag{3.6}$$

For $\varepsilon>0$ small enough $\left(\varepsilon<\frac{p\lambda}{r}\right)$ and $0<b<a$, setting f_ε and g_ε as:

$$f_\varepsilon(x)=g_\varepsilon(x)=0,\quad x\in(0,b);$$
$$f_\varepsilon=x^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}},\quad g_\varepsilon=x^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}},\quad x\in[b,\infty),$$

then we find

$$\int_a^\infty\int_b^\infty\frac{\ln\left(\frac{x}{y}\right)f_\varepsilon(x)\cdot g_\varepsilon(y)}{x^{\lambda}-y^{\lambda}}dxdy=\int_a^\infty\int_b^\infty\frac{\ln\left(\frac{x}{y}\right)x^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}\cdot y^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}}}{x^{\lambda}-y^{\lambda}}dxdy. \tag{3.7}$$

In (3.7), for $b\to 0^+$, by (3.6), we have

$$\frac{1}{\lambda^2 a^\varepsilon}\int_0^\infty\frac{\ln u}{u-1}u^{-1+\frac{1}{s}-\frac{\varepsilon}{q\lambda}}du=\varepsilon\int_a^\infty\int_0^\infty\frac{\ln\left(\frac{x}{y}\right)f_\varepsilon(x)g_\varepsilon(y)}{x^{\lambda}-y^{\lambda}}dxdy\le\frac{K}{a^\varepsilon}.$$

For $\varepsilon^+\to 0$, by [1] (see [1, Th. 342 Remark]), it follows that $\left[\frac{\pi}{\lambda\sin(\pi/r)}\right]^2\le K$, which contradicts the fact that $K<\left[\frac{\pi}{\lambda\sin(\pi/r)}\right]^2$. Hence the constant factor $\left[\frac{\pi}{\lambda\sin(\pi/r)}\right]^2$ in (3.1) is the best possible. The theorem is proved. □

Theorem 3.2. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{s} + \frac{1}{r} = 1$, $\lambda > 0$, $f \geq 0$ *such that* $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx < \infty$, *then we have*

$$\int_0^\infty y^{\frac{p\lambda}{s}-1}\left[\int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)}{x^\lambda - y^\lambda}dx\right]^p dy < \left[\frac{\pi}{\lambda\sin(\frac{\pi}{r})}\right]^{2p} \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx, \tag{3.8}$$

where the constant $\left[\frac{\pi}{\lambda\sin(\pi/r)}\right]^{2p}$ *is the best possible. Inequality (3.8) is equivalent to (3.1). In particular,*

(a) *for* $r = s = 2$, *we have*

$$\int_0^\infty y^{\frac{p\lambda}{2}-1}\left[\int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)}{x^\lambda - y^\lambda}dx\right]^p dy < \left(\frac{\pi}{\lambda}\right)^{2p} \int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx, \tag{3.9}$$

(b) *for* $\lambda = 1$, *we have*

$$\int_0^\infty y^{\frac{p}{s}-1}\left[\int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)}{x - y}dx\right]^p dy < \left[\frac{\pi}{\sin(\frac{\pi}{r})}\right]^{2p} \int_0^\infty x^{\frac{p}{s}-1} f^p(x)dx. \tag{3.10}$$

Proof. Setting a real function $g(y)$ as

$$g(y) = y^{\frac{p\lambda}{s}-1}\left[\int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)}{x^\lambda - y^\lambda}dx\right]^{p-1}, \quad y \in (0,\infty),$$

then by (3.1), we find

$$\begin{aligned}
&\left[\int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy\right]^p \\
&= \left\{\int_0^\infty y^{\frac{p\lambda}{s}-1}\left[\int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)}{x^\lambda - y^\lambda}dx\right]^p dy\right\}^p \\
&= \left[\int_0^\infty \int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)g(y)}{x^\lambda - y^\lambda}dxdy\right]^p \\
&\leq \left[\frac{\pi}{\lambda\sin(\frac{\pi}{r})}\right]^{2p} \left\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx\right\}\left\{\int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx\right\}^{p-1}.
\end{aligned} \tag{3.11}$$

Hence we obtain

$$0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy \leq \left[\frac{\pi}{\lambda\sin(\frac{\pi}{r})}\right]^{2p} \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx < \infty. \tag{3.12}$$

By (3.1), both (3.11) and (3.12) take the form of strict inequality, and we have (3.8).

On the other hand, suppose that (3.8) is valid. By Hölder's inequality, we find

$$\begin{aligned}(3.13)\qquad &\int_0^\infty \int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)g(y)}{x^\lambda - y^\lambda} dxdy \\ &= \int_0^\infty \left[y^{\frac{\lambda}{s}-\frac{1}{p}} \int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)}{x^\lambda - y^\lambda} dx \right] \left[y^{-\frac{\lambda}{s}+\frac{1}{p}} g(y) \right] dy \\ &\le \left\{ \int_0^\infty y^{\frac{p\lambda}{s}-1} \left[\int_0^\infty \frac{\ln\left(\frac{x}{y}\right) f(x)}{x^\lambda - y^\lambda} dx \right]^p dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y) dy \right\}^{\frac{1}{q}}.\end{aligned}$$

Then by (3.8), we have (3.1). Hence (3.1) and (3.8) are equivalent.

If the constant $\left[\frac{\pi}{\lambda \sin(\pi/r)}\right]^{2p}$ in (3.8) is not the best possible, by using (3.13), we may get a contradiction that the constant factor in (3.1) is not the best possible. Thus we complete the proof of the theorem. □

Remark 3.3.

(a) For $r = q, s = p$, Inequality (3.1) reduces to (1.9) and (3.8) reduces to (1.10).
(b) Inequality (3.1) is an extension of (1.7) with parameters (λ, r, s).
(c) It is interesting that inequalities (1.9) and (3.2) are different, although they have the same parameters and possess a best constant factor.

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