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BOUNDED LINEAR OPERATORS IN PROBABILISTIC NORMED SPACE

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ABSTRACT. The notion of a probabilistic metric space was introduced by Menger in 1942. The notion of a probabilistic normed space was introduced in 1993. The aim of this paper is to give a necessary condition to get bounded linear operators in probabilistic normed space.

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1. INTRODUCTION

The purpose of this paper is to present a definition of bounded linear operators which is based on the new definition of a probabilistic normed space. This definition is sufficiently general to encompass the most important contraction function in probabilistic normed space. The concepts used are those of [1], [2] and [9].

A *distribution function* (briefly, a d.f.) is a function F from the extended real line $\bar{\mathbb{R}} = [-\infty, +\infty]$ into the unit interval $I = [0, 1]$ that is nondecreasing and satisfies $F(-\infty) = 0,\ F(+\infty) = 1$. We normalize all d.f.'s to be left-continuous on the unextended real line $\mathbb{R} = (-\infty, +\infty)$. For any $a \geq 0$, ε_a is the d.f. defined by

$$\varepsilon_a(x) = \begin{cases} 0, & \text{if } x \leq a \\ 1, & \text{if } x > a, \end{cases} \tag{1.1}$$

The set of all the d.f.s will be denoted by Δ and the subset of those d.f.s called positive d.f.s. such that $F(0) = 0$, by Δ^+.

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By setting $F \leq G$ whenever $F(x) \leq G(x)$ for all x in $\mathbb{R}$, the maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

A *triangle function* is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing and which has ε_0 as unit, that is, for all $F, G, H \in \Delta^+$, we have

$$\begin{aligned} \tau(\tau(F,G),H) &= \tau(F,\tau(G,H)), \\ \tau(F,G) &= \tau(G,F), \\ \tau(F,H) &\leq \tau(G,H), \quad \text{if} \quad F \leq G, \\ \tau(F,\varepsilon_0) &= F. \end{aligned}$$

Continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+.

Typical continuous triangle functions are convolution and the operations τ_T and τ_{T^*}, which are, respectively, given by

(1.2) $$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s), G(t)),$$

and

(1.3) $$\tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s), G(t)),$$

for all F, G in Δ^+ and all x in $\mathbb{R}$ [9, Sections 7.2 and 7.3], here T is a continuous t-norm, i.e. a continuous binary operation on $[0,1]$ that is associative, commutative , nondecreasing and has 1 as identity; T^* is a continuous t-conorm, namely a continuous binary operation on $[0,1]$ that is related to continuous t-norm through

(1.4) $$T^*(x,y) = 1 - T(1-x, 1-y).$$

It follows without difficulty from (1.1)–(1.4) that

$$\tau_T(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b} = \tau_{T^*}(\varepsilon_a, \tau_b),$$

for any continuous t-norm T, any continuous t-conorm T^* and any $a, b \geq 0$.

The most important t-norms are the functions W, $Prod$, and M which are defined, respectively, by

$$\begin{aligned} W(a,b) &= \max(a+b-1, 0), \\ prod(a,b) &= a \cdot b, \\ M(a,b) &= \min(a,b). \end{aligned}$$

Their corresponding t-norms are given, respectively, by

$$\begin{aligned} W^*(a,b) &= \min(a+b, 1), \\ prod^*(a,b) &= a + b - a \cdot b, \\ M^*(a,b) &= \max(a,b). \end{aligned}$$

Definition 1.1. A *probabilistic metric* (briefly PM) space is a triple (S, f, τ), where S is a nonempty set, τ is a triangle function, and f is a mapping from $S \times S$ into Δ^+ such that, if F_{pq} denoted the value of f at the pair (p,q), the following hold for all p, q, r in S:

(PM1) $F_{pq} = \varepsilon_0$ if and only if $p = q$.

(PM2) $F_{pq} = F_{qp}$.
(PM3) $F_{pr} \geq \tau\left(F_{pq}, F_{qr}\right)$.

Definition 1.2. A *probabilistic normed* space is a quadruple (V, ν, τ, τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions, and ν is a mapping from V into Δ^+ such that, for all p, q in V, the following conditions hold:

(PN1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$, θ being the null vector in V;
(PN2) $\nu_{-p} = \nu_p$;
(PN3) $\nu_{p+q} \geq \tau\left(\nu_p, \nu_q\right)$
(PN4) $\nu_p \leq \tau^*\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$ for all α in $[0, 1]$.

If, instead of (**PN1**), we only have $\nu_\theta = \varepsilon_\theta$, then we shall speak of a *Probabilistic Pseudo Normed Space*, briefly a PPN space. If the inequality (**PN4**) is replaced by the equality $V_p = \tau_M\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$, then the PN space is called a *Serstnev space*. The pair is said to be a Probabilistic *Seminormed* Space (briefly PSN space) if $\nu : V \to \Delta^+$ satisfies (**PN1**) and (**PN2**).

Definition 1.3. A PSN (V, ν) space is said to be *equilateral* if there is a d.f. $F \in \Delta^+$ different from ε_0 and from ε_∞, such that, for every $p \neq \theta$, $\nu_p = F$. Therefore, every equilateral PSN space (V, ν) is a PN space under $\tau = M$ and $\tau^* = M$ where is the triangle function defined for $G, H \in \Delta^+$ by

$$M\left(G, H\right)\left(x\right) = \min\left\{G\left(x\right), H\left(x\right)\right\} \qquad \left(x \in [0, \infty]\right).$$

An equilateral PN space will be denoted by (V, F, M).

Definition 1.4. Let $(V, \|\cdot\|)$ be a normed space and let $G \in \Delta^+$ be different from ε_0 and ε_∞; define $\nu : V \to \Delta^+$ by $\nu_\theta = \varepsilon_0$ and

$$\nu_p\left(t\right) = G\left(\frac{t}{\|p\|^\alpha}\right) \qquad \left(p \neq \theta,\ t > 0\right),$$

where $\alpha \geq 0$. Then the pair (V, ν) will be called the $\alpha-$*simple* space generated by $(V, \|\cdot\|)$ and by G.

The $\alpha-$simple space generated by $(V, \|\cdot\|)$ and by G is immediately seen to be a PSN space; it will be denoted by $(V, \|\cdot\|, G; \alpha)$.

Definition 1.5. There is a natural topology in PN space (V, ν, τ, τ^*), called the *strong topology*; it is defined by the neighborhoods,

$$N_p\left(t\right) = \left\{q \in V : \nu_{q-p}\left(t\right) > 1 - t\right\} = \left\{q \in d_L\left(\nu_{q-p}, \varepsilon_0\right) < t\right\},$$

where $t > 0$. Here d_L is the modified Levy metric ([9]).

2. BOUNDED LINEAR OPERATORS IN PROBABILISTIC NORMED SPACES

In 1999, B. Guillen, J. Lallena and C. Sempi [3] gave the following definition of bounded set in PN space.

Definition 2.1. Let A be a nonempty set in PN space (V, ν, τ, τ^*). Then

(a) A is *certainly bounded* if, and only if, $\varphi_A\left(x_0\right) = 1$ for some $x_0 \in (0, +\infty)$;
(b) A is *perhaps bounded* if, and only if, $\varphi_A\left(x_0\right) < 1$ for every $x_0 \in (0, +\infty)$ and $l^-\varphi_A\left(+\infty\right) = 1$;
(c) A is *perhaps unbounded* if, and only if, $l^-\varphi_A\left(+\infty\right) \in (0, 1)$;
(d) A is *certainly unbounded* if, and only if, $l^-\varphi_A\left(+\infty\right) = 0$; i.e., $\varphi_A\left(x\right) = 0$;

where $\varphi_A\left(x\right) = \inf\left\{\nu_p\left(x\right) : P \in A\right\}$ and $l^-\varphi_A\left(x\right) = \lim_{t \to x-} \varphi_A\left(t\right)$.

Moreover, A will be said to be D-bounded if either (a) or (b) holds.

Definition 2.2. Let (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. A linear map $T : V \to V'$ is said to be

(a) *Certainly bounded* if every certainly bounded set A of the space (V,ν,τ,τ^*) has, as image by T a certainly bounded set TA of the space (V',μ,σ,σ^*), i.e., if there exists $x_0 \in (0,+\infty)$ such that $\nu_p(x_0) = 1$ for all $p \in A$, then there exists $x_1 \in (0,+\infty)$ such that $\mu_{Tp}(x_1) = 1$ for all $p \in A$.

(b) *Bounded* if it maps every D-bounded set of V into a D-bounded set of V', i.e., if, and only if, it satisfies the implication,

$$\lim_{x\to+\infty} \varphi_A(x) = 1 \Rightarrow \lim_{x\to+\infty} \varphi_{TA}(x) = 1,$$

for every nonempty subset A of V.

(c) *Strongly **B**-bounded* if there exists a constant $k > 0$ such that, for every $p \in V$ and for every $x > 0$, $\mu_{Tp}(x) \geq \nu_p\left(\frac{x}{k}\right)$, or equivalently if there exists a constant $h > 0$ such that, for every $p \in V$ and for every $x > 0$,

$$\mu_{Tp}(hx) \geq \nu_p(x).$$

(d) *Strongly **C**-bounded* if there exists a constant $h \in (0,1)$ such that, for every $p \in V$ and for every $x > 0$,

$$\nu_p(x) > 1 - x \Rightarrow \mu_{Tp}(hx) > 1 - hx.$$

Remark 2.1. The identity map I between PN space (V,ν,τ,τ^*) into itself is strongly **C**-bounded. Also, all linear contraction mappings, according to the definition of [7, Section 1], are strongly **C**-bounded, i.e for every $p \in V$ and for every $x > 0$ if the condition $\nu_p(x) > 1 - x$ is satisfied then

$$\nu_{Ip}(hx) = \nu_p(hx) > 1 - hx.$$

But we note that when $k = 1$ then the identity map I between PN space (V,ν,τ,τ^*) into itself is a strongly **B**-bounded operator. Also, all linear contraction mappings, according to the definition of [9, Section 12.6], are strongly **B**-bounded.

In [3] B. Guillen, J. Lallena and C. Sempi present the following, every strongly **B**-bounded operator is also certainly bounded and every strongly **B**-bounded operator is also bounded. But the converses need not to be true.

Now we are going to prove that in the Definition 2.2, the notions of strongly **C**-bounded operator, certainly bounded, bounded and strongly **B**-bounded do not imply each other.

In the following example we will introduce a strongly **C**-bounded operator, which is not strongly **B**-bounded, not bounded nor certainly bounded.

Example 2.1. Let V be a vector space and let $\nu_\theta = \mu_\theta = \varepsilon_0$, while, if $p,q \neq \theta$ then, for every $p,q \in V$ and $x \in \mathbb{R}$, if

$$\nu_p(x) = \begin{cases} 0, & x \leq 1 \\ 1, & x > 1 \end{cases} \qquad \mu_p(x) = \begin{cases} \frac{1}{3}, & x \leq 1 \\ \frac{9}{10}, & 1 < x < \infty \\ 1, & x = \infty \end{cases}$$

and if

$$\tau(\nu_p(x), \nu_q(y)) = \tau^*(\nu_p(x), \nu_q(y)) = \min(\nu_p(x), \nu_q(x)),$$
$$\sigma(\mu_p(x), \mu_q(y)) = \sigma^*(\mu_p(x), \mu_q(y)) = \min(\mu_p(x), \mu_q(x)),$$

then (V,ν,τ,τ^*) and (V',μ,σ,σ^*) are equilateral PN spaces by Definition 1.3. Now let $I : (V,\nu,\tau,\tau^*) \to (V,\mu,\tau,\tau^*)$ be the identity operator, then I is strongly **C**-bounded but I is not strongly **B**-bounded, bounded and certainly bounded, it is clear that I is not certainly bounded and is not bounded. I is not strongly **B**-bounded, because for every $k > 0$ and for $x = \max\left\{2, \frac{1}{k}\right\}$,

$$\mu_{Ip}(kx) = \frac{9}{10} < 1 = \nu_p(x).$$

But I is strongly **C**-bounded, because for every $p > 0$ and for every $x > 0$, this condition $v_p(x) > 1 - x$ is satisfied only if $x > 1$ now if $h = \frac{7}{10}x$ then

$$\mu_{Ip}(hx) = \mu_{Ip}\left(\frac{7}{10x}x\right) = \mu_p\left(\frac{7}{10}\right) = \frac{1}{3} > \frac{3}{10} = 1 - \frac{7}{10} = 1 - \left(\frac{7}{10x}\right)x.$$

Remark 2.2. We have noted in the above example that there is an operator, which is strongly **C**-bounded, but it is not strongly **B**-bounded. Moreover we are going to give an operator, which is strongly **B**-bounded, but it is not strongly **C**-bounded.

Definition 2.3. Let (V, ν, τ, τ^*) be PN space then we defined

$$B(p) = \inf\left\{h \in \mathbb{R} : \nu_p\left(h^+\right) > 1 - h\right\}.$$

Lemma 2.3. *Let $T : (V, \nu, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ be a strongly **B**-bounded linear operator, for every p in V and let μ_{Tp} be strictly increasing on $[0, 1]$, then $B(T_p) < B(p), \forall p \in V$.*

Proof. Let $\eta \in \left(0, \frac{1-\gamma}{\gamma}B(p)\right)$, where $\gamma \in (0, 1)$. Then $B(p) > \gamma[B(p) + \eta]$ and so

$$\mu_{Tp}(B(p)) > \mu_{Tp}(\gamma[B(p) + \eta]),$$

and where μ_{Tp} is strictly increasing on $[0, 1]$, then

$$\mu_{Tp}(\gamma[B(p) + \eta]) \geq \nu_p(B(p) + \eta) \geq \nu_p\left(B(p)^+\right) > 1 - B(p),$$

we conclude that

$$B(T_p) = \inf\left\{B(p) : \mu_{Tp}\left(B(p)^+\right) > 1 - B(p)\right\},$$

so $B(T_p) < B(p), \ \forall p \in V$. □

Theorem 2.4. *Let $T : (V, \nu, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ be a strongly **B**-bounded linear operator, and let μ_{Tp} be strictly increasing on $[0, 1]$, then T is a strongly **C**-bounded linear operator.*

Proof. Let T be a strictly **B**-bounded operator. Since, by Lemma 2.3, $B(T_p) < B(p), \forall p \in V$ there exist $\gamma_p \in (0, 1)$ such that $B(T_p) < \gamma_p B(p)$.

It means that

$$\begin{aligned}\inf\left\{h \in \mathbb{R} : \mu_{Tp}\left(h^+\right) > 1 - h\right\} &\leq \gamma \inf\left\{h \in \mathbb{R} : \nu_p\left(h^+\right) > 1 - h\right\} \\ &= \inf\left\{\gamma h \in \mathbb{R} : \nu_p\left(h^+\right) > 1 - h\right\} \\ &= \inf\left\{h \in \mathbb{R} : \nu_p\left(\frac{h^+}{\gamma}\right) > 1 - \frac{h}{\gamma}\right\}.\end{aligned}$$

We conclude that $\nu_p\left(\frac{h}{\gamma}\right) > 1 - \left(\frac{h}{\gamma}\right) \Longrightarrow \mu_{Tp}(h) > 1 - h$. Now if $x = \frac{h}{\gamma}$ then $\nu_p(x) > 1 - x \Longrightarrow \mu_{Tp}(xh) > 1 - xh$, so T is a strongly **C**-bounded operator. □

Remark 2.5. From Theorem 2.4 we have noted that under some additional condition every a strongly **B**-bounded operator is a strongly **C**-bounded operator. But in general, it is not true.

Example 2.2. Let $V = V' = \mathbb{R}$ and $v_0 = \mu_0 = \varepsilon_0$, while, if $p \neq 0$, then, for $x > 0$, let $v_p(x) = G\left(\frac{x}{|p|}\right)$, $\mu_p(x) = U\left(\frac{x}{|p|}\right)$, where

$$G(x) = \begin{cases} \frac{1}{2}, & 0 < x \leq 2, \\ 1, & 2 < x \leq +\infty, \end{cases} \qquad U(x) = \begin{cases} \frac{1}{2}, & 0 < x \leq \frac{3}{2}, \\ 1, & \frac{3}{2} < x \leq +\infty \end{cases}.$$

Consider now the identity map $I : (\mathbb{R}, |\cdot|, G, \mu) \to (\mathbb{R}, |\cdot|, G, \mu)$. Now

(a) I is a strongly **B**-bounded operator, such that for every $p \in \mathbb{R}$ and every $x > 0$ then

$$\mu_{Ip}\left(\frac{3}{4}x\right) = \mu_p\left(\frac{3}{4}x\right) = U\left(\frac{3x}{4|p|}\right) = \begin{cases} \frac{1}{2}, & 0 < x \leq 2|p|, \\ 1, & 2|p| < x \leq +\infty, \end{cases} = G\left(\frac{x}{|p|}\right) = v_p(x).$$

(b) I is not a strongly **C**-bounded operator, such that for every $h \in (0,1)$, let $x = \frac{3}{8h}$, $p = \frac{1}{4}$. If $x > 2|p|$ then the condition $v_p(x) > 1 - x$ will be satisfied, but we note that

$$\mu_{Ip}(hx) = \mu_p(hx) = U\left(\frac{hx}{|p|}\right) = U\left(\frac{3}{2}\right) = \frac{1}{2} < \frac{5}{8} = 1 - h\left(\frac{3}{8h}\right) = 1 - hx.$$

Now we introduce the relation between the strongly **B**-bounded and strongly **C**-bounded operators with boundedness in normed space.

Theorem 2.6. *Let G be strictly increasing on $[0,1]$, then $T : (V, \|\cdot\|, G, \alpha) \to (V', \|\cdot\|, G, \alpha)$ is a strongly **B**-bounded operator if, and only if, T is a bounded linear operator in normed space.*

Proof. Let $k > 0$ and $x > 0$. Then for every $p \in V$

$$G\left(\frac{kx}{\|T_p\|^{\alpha}}\right) = \mu_{Tp}(kx) \geq v_p(x) = G\left(\frac{x}{\|p\|^{\alpha}}\right),$$

if and only if

$$\|T_p\| \leq k^{\frac{1}{\alpha}} \|p\|.$$

□

Theorem 2.7. *Let $T : (V, \|\cdot\|, G, \alpha) \to (V', \|\cdot\|, G, \alpha)$ be strongly **C**-bounded, and let G be strictly increasing on $[0,1]$ then T is a bounded linear operator in normed space.*

Proof. If v_p is strictly increasing for every $p \in V$, then the quasi-inverse v_p^{Λ} is continuous and $B(p)$ is the unique solution of the equation $x = v_p^{\Lambda}(1 - x)$ i.e.

$$B(p) = v_p^{\Lambda}(x)(1 - B(p)). \tag{2.1}$$

If $v_p(x) = G\left(\frac{x}{\|p\|^{\alpha}}\right)$, then $v_p^{\Lambda}(x) = \|p\|^{\alpha} G^{\Lambda}(x)$ and from (2.1) it follows that

$$B(p) = \|p\|^{\alpha} G^{\Lambda}(1 - B(p)). \tag{2.2}$$

Suppose that T is strongly **C**-bounded, i.e. that

$$B(T_p) \leq kB(p), \quad \forall p \in V, \tag{2.3}$$

where $k \in (0,1)$.

Then (2.2) and (2.3) imply

$$\|T_p\|^{\alpha} \leq \frac{B(T_p)}{G^{\Lambda}(1 - B(T_p))} \leq \frac{kB(p)}{G^{\Lambda}(1 - kB(p))} \leq \frac{kB(p)}{G^{\Lambda}(1 - B(p))} = k\|p\|^{\alpha}.$$

Which means that T is a bounded in normed space. □

The converse of the above theorem is not true, see Example 2.2.

We recall the following theorems from [3].

Theorem 2.8. *Let (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. A linear map $T : V \to V'$ is either continuous at every point of V or at no point of V.*

Corollary 2.9. *If $T : (V, \nu, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ is linear, then T is continuous if, and only if, it is continuous at θ.*

Theorem 2.10. *Every strongly **B**-bounded linear operator T is continuous with respect to the strong topologies in (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$, respectively.*

In the following theorem we show that every strongly **C**-bounded linear operator T is continuous.

Theorem 2.11. *Every strongly **C**-bounded linear operator T is continuous.*

Proof. Due to Corollary 3.1 [3], it suffices to verify that T is continuous at θ. Let $N_{\theta'}(t)$, with $t > 0$, be an arbitrary neighbourhood of θ'. If T is strongly **C**-bounded linear operator then there exist $h \in (0, 1)$ such that for every $t > 0$ and $p \in N_\theta(s)$ we note that

$$\mu_{Tp}(t) \geq \nu_p(ht) \geq 1 - ht > 1 - t,$$

so $T_p \in N_{\theta'}(t)$; in other words, T is continuous. □

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