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ON THE ABSOLUTE RIESZ SUMMABILITY FACTORS

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ABSTRACT. The conditions of a theorem of H. Bor pertaining to the absolute Riesz summability factors are replaced by new type ones.

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1. INTRODUCTION

Following R.P. Boas, Jr. [2] we say that a sequence $\{a_n\}$ is δ-*quasi-monotonic* if $a_n \to 0,\ a_n > 0$ ultimately, and $\Delta\, a_n \geq -\delta_n$. Here $\{\delta_n\}$ is a positive sequence whose properties are selected appropriately in different contexts. Several authors have used this definition for different topics, namely it is nearly as useful as the classical monotonicity. We shall also recall a theorem (see Theorem A) utilizing this notion, but our plan is to eliminate the condition given by this notion from Theorem A, because by way of this we hope to generalize Theorem A and pull out the key conditions of this theorem.

To recall the mentioned theorem we need the definition of the almost increasing sequence. A positive sequence $\{a_n\}$ is said to be *almost increasing* if there exist a positive increasing sequence $\{b_n\}$ and two positive constants A and B such that $A\, b_n \leq a_n \leq B\, b_n$ (see [1]). It is easy to verify that a sequence $\{a_n\}$ is almost increasing if and only if it is *quasi increasing*, that is, if there exists a constant $K = K(\{a_n\}) \geq 1$ such that

$$K\, a_n \geq a_m\ (\geq 0) \tag{1.1}$$

holds for all $n \geq m.$ We can consider e.g.

$$b_n := \min_{k\geq n} a_k \quad \text{with} \quad A = 1 \quad \text{and} \quad B = K,$$

the converse clearly holds with $K := B/A.$

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We prefer to use the notion of quasi increasing sequences, namely the definition (1.1) is very simple.

The theorem to be generalized is due to H. Bor [4] and its subject is Riesz summability, therefore we recall the definition of the $|\overline{N}, p_n|_k$ summability.

Let $\sum_{n=1}^{\infty} a_n$ be a given series with partial sums s_n. Let $\{p_n\}$ be a sequence of positive numbers such that

$$P_n := \sum_{\nu=0}^{n} p_\nu \to \infty, \quad (P_{-1} = p_{-1} = 0).$$

The series $\sum_{n=1}^{\infty} a_n$ is said to be summable $|\overline{N}, p_n|_k,\ k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty,$$

where

$$t_n := \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu\, s_\nu.$$

Now we can recall a theorem of H. Bor [4] (see also [5]).

Theorem A. *Let $\{X_n\}$ be an almost increasing sequence such that $n|\Delta X_n| = O(X_n)$, and $\lambda_n \to 0$. Suppose that there exists a sequence of numbers $\{A_n\}$ such that it is δ-quasi-monotone with $\sum n\, \delta_n\, X_n < \infty,\ \sum A_n\, X_n$ is convergent and $|\Delta \lambda_n| \leq |A_n|$ for all n. If*

$$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n| < \infty, \tag{1.2}$$

$$X_m^* := \sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m) \tag{1.3}$$

and

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m), \tag{1.4}$$

then the series $\sum a_n\, \lambda_n$ is summable $|\overline{N}, p_n|_k,\ k \geq 1$.

2. RESULT

We prove the following theorem.

Theorem 2.1. *Let $\lambda_n \to 0$. Suppose that there exists a positive quasi increasing sequence $\{X_n\}$ such that*

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} n\, X_n^* |\Delta(|\Delta \lambda_n|)| < \infty \tag{2.2}$$

hold. If the conditions (1.2), (1.3) and (1.4) are satisfied then the series $\sum a_n\, \lambda_n$ is summable $|\overline{N}, p_n|_k,\ k \geq 1$.

Proposition 2.2. *Theorem 2.1. moderates the hypotheses of Theorem A under the added assumption $\Delta(|\Delta \lambda_n|) \geq 0$ for all n.*

I do believe that our conditions without the additional requirement are weaker than the hypotheses of Theorem A, but I cannot prove it now.

3. LEMMA

Later on we shall use the notation $L \ll R$ if there exists a positive constant K such that $L \leq KR$ holds.

To avoid the needless repetition we collect the important partial results proved in [4] into the following lemma.

In [4] the following inequality is verified implicitly.

Lemma 3.1. *Let T_n denote the n-th $(\overline{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. If $\lambda_n \to 0$, (1.2) and (1.4) hold, then*

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k \ll |\lambda_m| X_m + \sum_{n=1}^{m} |\Delta \lambda_n| X_n + \sum_{n=1}^{m} |t_n|^k |\Delta \lambda_n|, \tag{3.1}$$

where the notations of Theorem A are used.

4. PROOFS

Proof of Theorem 2.1. In view of Lemma 3.1 it suffices to verify that the three terms on the right of (3.1) are uniformly bounded. Since $\lambda_n \to 0$, thus by (1.1) and (2.1)

$$|\lambda_m| X_m \ll X_m \sum_{n=m}^{\infty} |\Delta \lambda_n| \ll \sum_{n=m}^{\infty} X_n |\Delta \lambda_n| < \infty.$$

The second term is clearly bounded by (2.1).

To estimate the third term we use the Abel transformation as follows:

$$\begin{aligned} \sum_{n=1}^{m} |t_n|^k |\Delta \lambda_n| &= \sum_{n=1}^{m} n |\Delta \lambda_n| \frac{1}{n} |t_n|^k \\ &\ll \sum_{n=1}^{m-1} |\Delta(n |\Delta \lambda_n|)| \sum_{i=1}^{n} \frac{1}{i} |t_i|^k + m |\Delta \lambda_m| \sum_{n=1}^{m} \frac{1}{n} |t_n|^k \\ &\ll \sum_{n=1}^{m-1} n |\Delta(|\Delta \lambda_n|)| X_n^* + \sum_{n=1}^{m-1} |\Delta \lambda_{n+1}| X_{n+1}^* + m |\Delta \lambda_m| X_m^*. \end{aligned}$$

Here the first term is bounded by (2.2), the second one by (1.3) and (2.1), and next we show that the third term is also bounded by (2.2). Namely

$$m X_m^* |\Delta \lambda_m| \ll m X_m^* \sum_{n=m}^{\infty} |\Delta(|\Delta \lambda_n|)| \ll \sum_{n=m}^{\infty} n X_n^* |\Delta(|\Delta \lambda_n|)| < \infty.$$

Herewith we have verified that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty,$$

and this completes the proof of Theorem 2.1. □

5. PROOF OF PROPOSITION 2.2

We have to verify that under the hypotheses of Theorem A the assumptions of Theorem 2.1 are always satisfied assuming $\Delta(|\Delta\,\lambda_n|) \geq 0$.

In [4] it is proved that the conditions of Theorem A forever imply that

$$\sum_{n=1}^{\infty} X_n|A_n| < \infty \quad \text{see Lemma 3.1,} \tag{5.1}$$

and

$$\sum_{n=1}^{\infty} |t_n|^k|A_n| < \infty \quad \text{(see p. 5.)}. \tag{5.2}$$

Since $|\Delta\,\lambda_n| \leq |A_n|$ is assumed, thus (5.1) implies (2.1). Next we show that (2.2) follows from (5.2). Namely

$$\sum_{n=1}^{m} |t_n|^k|A_n| \geq \sum_{n=1}^{m} |t_n|^k|\Delta\,\lambda_n|. \tag{5.3}$$

Applying again the Abel transformation we get

$$\begin{aligned} \sum_{n=1}^{m} n|\Delta\,\lambda_n|\frac{1}{n}|t_n|^k &\geq \sum_{n=1}^{m-1} \Delta(n|\Delta\,\lambda_n|)\sum_{i=1}^{n}\frac{1}{i}|t_i|^k \\ &\geq \sum_{n=1}^{m-1} n\,\Delta(|\Delta\,\lambda_n|)X_n^* - \sum_{n=1}^{m-1}|\Delta\,\lambda_{n+1}|X_{n+1}^*. \end{aligned} \tag{5.4}$$

Since $\Delta(|\Delta\,\lambda_n|) \geq 0$ and by (5.3) the sum on the left of (5.4), and the sums

$$\sum_{n=1}^{m-1} |\Delta\,\lambda_{n+1}|X_{n+1}^*,$$

by (1.3) and (5.1), are uniformly bounded, thus (2.2) clearly follows from (5.4).

The proof is complete.

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