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SOME INEQUALITIES ASSOCIATED WITH A LINEAR OPERATOR DEFINED FOR A CLASS OF ANALYTIC FUNCTIONS

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Abstract

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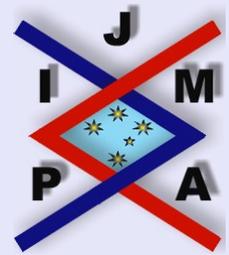


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Abstract

In this paper, we give a sufficient condition on a linear operator $L_p(a, c)g(z)$ which can guarantee that for α a complex number with $\text{Re}(\alpha) > 0$,

$$\text{Re} \left\{ (1 - \alpha) \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} + \alpha \frac{L_p(a + 1, c)f(z)}{L_p(a + 1, c)g(z)} \right\} > \rho, \quad \rho < 1,$$

in the unit disk E , implies

$$\text{Re} \left\{ \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} \right\} > \rho' > \rho, \quad z \in E.$$

Some interesting applications of this result are also given.

2000 Mathematics Subject Classification: 30C45.

Key words: Analytic functions, Differential subordination, Ruscheweyh derivatives, Linear operator.

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1. Introduction

Let $A(p, n)$ denote the class functions f normalized by

$$(1.1) \quad f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

which are analytic in the open unit disk $E = \{z : z \in \mathbb{C}, |z| < 1\}$.

In particular, we set $A(p, 1) = A_p$ and $A(1, 1) = A_1 = A$.

The Hadamard product $(f * g)(z)$ of two functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k \quad (p, n \in \mathbb{N}),$$

is defined, as usual, by

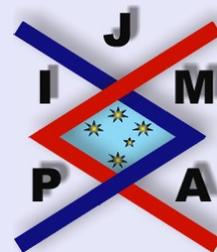
$$(f * g)(z) = z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k = (g * f)(z).$$

The Ruscheweyh derivative of $f(z)$ of order $\delta + p - 1$ is defined by

$$(1.2) \quad D^{\delta+p-1} f(z) = \frac{z^p}{(1-z)^{\delta+p}} * f(z) \quad (f \in A(p, n); \delta \in \mathbb{R} \setminus (-\infty, -p])$$

or, equivalently, by

$$(1.3) \quad D^{\delta+p-1} f(z) = z^p + \sum_{k=p+n}^{\infty} \binom{\delta+k-1}{k-p} a_k z^k,$$



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where $f(z) \in A(p, n)$ and $\delta \in \mathbb{R} \setminus (-\infty, -p]$. In particular, if $\delta = l \in \mathbb{N} \cup \{0\}$, we find from (1.2) or (1.3) that

$$D^{l+p-1}f(z) = \frac{z^p}{(l+p-1)!} \frac{d^{l+k-1}}{dz^{l+p-1}} \{z^{l-1}f(z)\}.$$

The author has proved the following result in [4].

Theorem A. *Let α be a complex number satisfying $\operatorname{Re}(\alpha) > 0$ and $\rho < 1$. Let $\delta > -p$, $f, g \in A_p$ and*

$$\operatorname{Re} \left\{ \alpha \frac{D^{\delta+p-1}g(z)}{D^{\delta+p}g(z)} \right\} > \gamma, \quad 0 \leq \gamma < \operatorname{Re}(\alpha), \quad z \in E.$$

Then

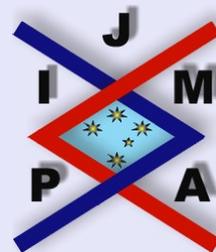
$$\operatorname{Re} \left\{ \frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)} \right\} > \frac{2\rho(\delta+p) + \gamma}{2(\delta+p) + \gamma}, \quad z \in E,$$

whenever

$$\operatorname{Re} \left\{ (1-\alpha) \frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)} + \alpha \frac{D^{\delta+p}f(z)}{D^{\delta+p}g(z)} \right\} > \rho, \quad z \in E.$$

The Pochhammer symbol $(\lambda)_k$ or the shifted factorial is given by $(\lambda)_0 = 1$ and $(\lambda)_k = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1)$, $k \in \mathbb{N}$. In terms of $(\lambda)_k$, we now define the function $\phi_p(a, c; z)$ by

$$\phi_p(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p}, \quad z \in E,$$



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where $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus z_0^-$; $z_0^- = \{0, -1, -2, \dots\}$.

Saitoh [3] introduced a linear operator $L_P(a, c)$, which is defined by

$$(1.4) \quad L_p(a, c)f(z) = \phi_p(a, c, ; z) * f(z), \quad z \in E,$$

or, equivalently by

$$(1.5) \quad L_p(a, c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+p} z^{k+p}, \quad z \in E,$$

where $f(z) \in A_p$, $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus z_0^-$.

For $f(z) \in A(p, n)$ and $\delta \in \mathbb{R} \setminus (-\infty, -p]$, we obtain

$$(1.6) \quad L_p(\delta + p, 1)f(z) = D^{\delta+p-1}f(z),$$

which can easily be verified by comparing the definitions (1.3) and (1.5).

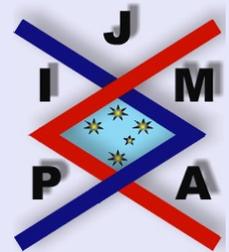
The main object of this paper is to present an extension of Theorem A to hold true for a linear operator $L_P(a, c)$ associated with the class $A(p, n)$.

The basic tool in proving our result is the following lemma.

Lemma 1.1 (cf. Miller and Mocanu [2, p. 35, Theorem 2.3 i(i)]). *Let Ω be a set in the complex plane C . Suppose that the function $\Psi : C^2 \times E \rightarrow C$ satisfies the condition $\Psi(ix_2, y_1; z) \notin \Omega$ for all $z \in E$ and for all real x_2 and y_1 such that*

$$(1.7) \quad y_1 \leq -\frac{1}{2}n(1 + x_2^2).$$

If $p(z) = 1 + c_n z^n + \dots$ is analytic in E and for $z \in E$, $\Psi(p(z), zp'(z); z) \subset \Omega$, then $\text{Re}(p(z)) > 0$ in E .



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2. Main Results

Theorem 2.1. Let α be a complex number satisfying $\operatorname{Re}(\alpha) > 0$ and $\rho < 1$. Let $a > 0, f, g \in A(p, n)$ and

$$(2.1) \quad \operatorname{Re} \left\{ \alpha \frac{L_p(a, c)g(z)}{L_p(a+1, c)g(z)} \right\} > \gamma, \quad 0 \leq \gamma < \operatorname{Re}(\alpha), \quad z \in E.$$

Then

$$\operatorname{Re} \left\{ \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} \right\} > \frac{2a\rho + n\gamma}{2a + n\gamma}, \quad z \in E,$$

whenever

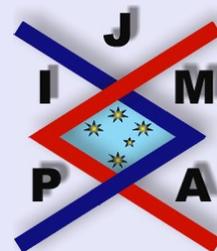
$$(2.2) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} + \alpha \frac{L_p(a+1, c)f(z)}{L_p(a+1, c)g(z)} \right\} > \rho, \quad z \in E.$$

Proof. Let $\tau = (2a\rho + n\gamma)/(2a + n\gamma)$ and define the function $p(z)$ by

$$(2.3) \quad p(z) = (1 - \tau)^{-1} \left\{ \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} - \tau \right\}.$$

Then, clearly, $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots$ and is analytic in E . We set $u(z) = \alpha L_p(a, c)g(z)/L_p(a+1, c)g(z)$ and observe from (2.1) that $\operatorname{Re}(u(z)) > \gamma, z \in E$. Making use of the familiar identity

$$z(L_p(a, c)f(z))' = aL_p(a+1, c)f(z) - (a-p)L_p(a, c)f(z),$$



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we find from (2.3) that

$$(2.4) \quad (1 - \alpha) \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} + \alpha \frac{L_p(a + 1, c)f(z)}{L_p(a + 1, c)g(z)} \\ = \tau + (1 - \tau) \left[p(z) + \frac{u(z)}{a} zp'(z) \right].$$

If we define $\Psi(x, y; z)$ by

$$(2.5) \quad \Psi(x, y; z) = \tau + (1 - \tau) \left(x + \frac{u(z)}{a} y \right),$$

then, we obtain from (2.2) and (2.4) that

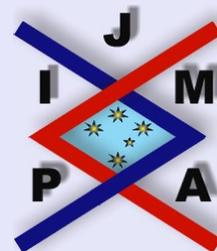
$$\{ \Psi(p(z), zp'(z); z) : |z| < 1 \} \subset \Omega = \{ w \in C : \operatorname{Re}(w) > \rho \}.$$

Now for all $z \in E$ and for all real x_2 and y_1 constrained by the inequality (1.7), we find from (2.5) that

$$\operatorname{Re}\{ \Psi(ix_2, y_1; z) \} = \tau + \frac{(1 - \tau)}{a} y_1 \operatorname{Re}(u(z)) \\ \leq \tau - \frac{(1 - \tau)n\gamma}{2a} \equiv \rho.$$

Hence $\Psi(ix_2, y_1; z) \notin \Omega$. Thus by Lemma 1.1, $\operatorname{Re}(p(z)) > 0$ and hence $\operatorname{Re} \left\{ \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} \right\} > \tau$ in E . This proves our theorem. \square

Remark 1. Theorem A is a special case of Theorem 2.1 obtained by taking $a = \delta + p$ and $c = n = 1$, which reduces to Theorem 2.1 of [1], when $p = 1$.



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Corollary 2.2. Let α be a real number with $\alpha \geq 1$ and $\rho < 1$. Let $a > 0$, $f, g \in A(p, n)$ and

$$\operatorname{Re} \left\{ \frac{L_p(a, c)g(z)}{L_p(a+1, c)g(z)} \right\} > \gamma, \quad 0 \leq \gamma < 1, \quad z \in E.$$

Then

$$\operatorname{Re} \left\{ \frac{L_p(a+1, c)f(z)}{L_p(a+1, c)g(z)} \right\} > \frac{\alpha(2a\rho + n\gamma) - (1 - \rho)n\gamma}{\alpha(2a + n\gamma)}, \quad z \in E,$$

whenever

$$\operatorname{Re} \left\{ (1 - \alpha) \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} + \alpha \frac{L_p(a+1, c)f(z)}{L_p(a+1, c)g(z)} \right\} > \rho, \quad z \in E.$$

Proof. Proof follows from Theorem 2.1 (Since $\alpha \geq 1$). □

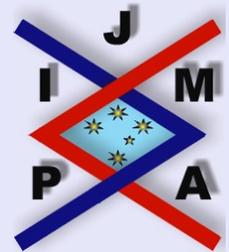
In its special case when $\alpha = 1$, Theorem 2.1 yields:

Corollary 2.3. Let $a > 0$, $f, g \in A(p, n)$ and $\operatorname{Re} \left\{ \frac{L_p(a, c)g(z)}{L_p(a+1, c)g(z)} \right\} > \gamma, 0 \leq \gamma < 1$, then for $\rho < 1$,

$$\operatorname{Re} \left\{ \frac{L_p(a+1, c)f(z)}{L_p(a+1, c)g(z)} \right\} > \rho, \quad z \in E,$$

implies

$$\operatorname{Re} \left\{ \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} \right\} > \frac{2a\rho + n\gamma}{2a + n\gamma}, \quad z \in E.$$



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If we set

$$v(z) = \frac{L_p(a+1, c)f(z)}{L_p(a+1, c)g(z)} - \left(\frac{1}{\alpha} - 1\right) \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)},$$

then for $a > 0, \alpha > 0$ and $\rho = 0$, Theorem 2.1 reduces to

$$\operatorname{Re}(v(z)) > 0, \quad z \in E$$

implies

$$(2.6) \quad \operatorname{Re} \left\{ \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} \right\} > \frac{n\alpha\gamma}{2a + n\alpha\gamma}, \quad z \in E,$$

whenever $\operatorname{Re}(L_p(a, c)g(z)/L_p(a+1, c)g(z)) > \gamma, 0 \leq \gamma < 1$. Let $\alpha \rightarrow \infty$.

Then (2.6) is equivalent to

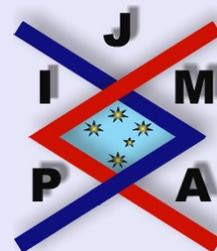
$$\operatorname{Re} \left\{ \frac{L_p(a+1, c)f(z)}{L_p(a+1, c)g(z)} - \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} \right\} > 0 \text{ in } E$$

implies

$$\operatorname{Re} \left\{ \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} \right\} > 1 \text{ in } E,$$

whenever $\operatorname{Re}(L_p(a, c)g(z)/L_p(a+1, c)g(z)) > \gamma, 0 \leq \gamma < 1$.

In the following theorem we shall extend the above result, the proof of which is similar to that of Theorem 2.1.



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Theorem 2.4. Let $a > 0$, $\rho < 1$, $f, g \in A(p, n)$ and $\operatorname{Re} \left\{ \frac{L_p(a, c)g(z)}{L_p(a+1, c)g(z)} \right\} > \gamma$,
 $0 \leq \gamma < 1$.
 If

$$\operatorname{Re} \left\{ \frac{L_p(a+1, c)f(z)}{L_p(a+1, c)g(z)} - \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} \right\} > -\frac{n\gamma(1-\rho)}{2a}, \quad z \in E,$$

then

$$\operatorname{Re} \left\{ \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} \right\} > \rho, \quad z \in E,$$

and

$$\operatorname{Re} \left\{ \frac{L_p(a+1, c)f(z)}{L_p(a+1, c)g(z)} \right\} > \frac{\rho(2a+n\gamma) - n\gamma}{2a}, \quad z \in E.$$

Using Theorem 2.1 and Theorem 2.4, we can generalize and improve several other interesting results available in the literature by taking $g(z) = z^p$. We illustrate a few in the following theorem.

Theorem 2.5. Let $a > 0$, $\rho < 1$ and $f(z) \in A(p, n)$. Then

(a) for α a complex number satisfying $\operatorname{Re}(\alpha) > 0$, we have

$$\operatorname{Re} \left\{ (1-\alpha) \frac{L_p(a, c)f(z)}{z^p} + \alpha \frac{L_p(a+1, c)f(z)}{z^p} \right\} > \rho, \quad z \in E,$$

implies

$$\operatorname{Re} \left\{ \frac{L_p(a, c)f(z)}{z^p} \right\} > \frac{2a\rho + n \operatorname{Re}(\alpha)}{2a + n \operatorname{Re}(\alpha)}, \quad z \in E.$$



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(b) for α real and $\alpha \geq 1$, we have

$$\operatorname{Re} \left\{ (1 - \alpha) \frac{L_p(a, c)f(z)}{z^p} + \alpha \frac{L_p(a + 1, c)f(z)}{z^p} \right\} > \rho, \quad \text{in } E$$

implies

$$\operatorname{Re} \left\{ \frac{L_p(a + 1, c)f(z)}{z^p} \right\} > \frac{(2a + n)\rho + n(\alpha - 1)}{2a + n\alpha} \quad \text{in } E$$

(c) for $z \in E$,

$$\operatorname{Re} \left\{ \frac{L_p(a + 1, c)f(z)}{z^p} - \frac{L_p(a, c)f(z)}{z^p} \right\} > -\frac{n(1 - \rho)}{2a}$$

implies

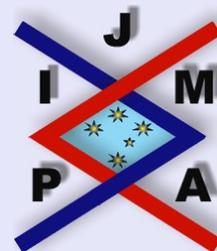
$$\operatorname{Re} \left\{ \frac{L_p(a + 1, c)f(z)}{z^p} \right\} > \frac{(2a + n)\rho - n}{2a}.$$

Remark 2. By taking $a = \delta + p, c = n = 1$ in Theorem 2.5 we obtain Theorem 1.6 of the author [4], which when $p = 1$ reduces to Theorem 2.3 of Bhoosnurmath and Swamy [1].

In a manner similar to Theorem 2.1, we can easily prove the following, which when $r = 1$ reduces to part (a) of Theorem 2.5.

Theorem 2.6. Let $a > 0, r > 0, \rho < 1$ and $f(z) \in A(p, n)$. Then for α a complex number with $\operatorname{Re}(\alpha) > 0$, we have

$$\operatorname{Re} \left\{ \left(\frac{L_p(a, c)f(z)}{z^p} \right)^r \right\} > \frac{2apr + n \operatorname{Re}(\alpha)}{2ar + n \operatorname{Re}(\alpha)}, \quad z \in E,$$



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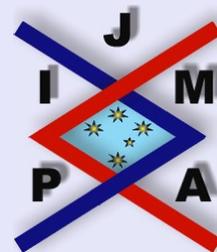
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whenever

$$\operatorname{Re} \left\{ (1 - \alpha) \left(\frac{L_p(a, c)f(z)}{z^p} \right)^r + \alpha \left(\frac{L_p(a + 1, c)f(z)}{z^p} \right) \left(\frac{L_p(a, c)f(z)}{z^p} \right)^{r-1} \right\} > \rho,$$

$z \in E$.



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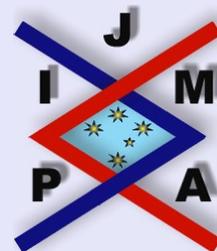
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