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A UNIFIED TREATMENT OF CERTAIN SUBCLASSES OF PRESTARLIKE FUNCTIONS

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ABSTRACT. In this paper we introduce and study some properties of a unified class $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$ of prestarlike functions with negative coefficients in a unit disk U. These properties include growth and distortion, radii of convexity, radii of starlikeness and radii of close-to-convexity.

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1. INTRODUCTION

Let A denote the class of *normalized* analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n, \tag{1.1}$$

in the unit disk $U = \{z : |z| < 1\}$. Further let S denote the subclass of A consisting of analytic and univalent functions f in the unit disk U. A function f in S is said to be starlike of order α if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \tag{1.2}$$

for some α $(0 \le \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α. It is well-known that $S^*(\alpha) \subseteq S^*(0) \equiv S^*$.

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Let the function

$$S_\alpha(z)=\frac{z}{(1-z)^{2(1-\alpha)}},\quad (z\in U;\quad 0\le\alpha<1) \tag{1.3}$$

which is the extremal function for the class $S^*(\alpha)$. We also note that $S_\alpha(z)$ can be written in the form:

$$S_\alpha(z)=z+\sum_{n=2}^{\infty}|c_n(\alpha)|z^n, \tag{1.4}$$

where

$$c_n(\alpha)=\frac{\Pi_{j=2}^{n}(j-2\alpha)}{(n-1)!}\quad (n\in \mathbf{N}\,\{1\},\ \mathbf{N}:=\{1,2,3,\dots\}). \tag{1.5}$$

We note that $c_n(\alpha)$ is decreasing in α and satisfies

$$\lim_{n\to\infty}c_n(\alpha)=\begin{cases}\infty & \text{if } \alpha<\frac{1}{2}\ ,\\ 1 & \text{if } \alpha=\frac{1}{2}\ ,\\ 0 & \text{if } \alpha>\frac{1}{2}.\end{cases} \tag{1.6}$$

Also a function f in S is said to be convex of order α if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \tag{1.7}$$

for some α $(0\le\alpha<1)$. We denote by $K(\alpha)$ the class of all convex functions of order α. It is a fact that $f\in K(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$.

The well-known Hadamard product (or convolution) of two functions $f(z)$ given by (1.1) and $g(z)$ given by $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\quad (z\in U). \tag{1.8}$$

Let $\mathcal{R}[\mu,\alpha,\beta,\gamma,\lambda,A,B]$ denote the class of prestarlike functions satisfying the following condition

$$\left|\frac{\frac{zH'_\lambda(z)}{H_\lambda(z)}-1}{2\gamma(B-A)\left(\frac{zH'_\lambda(z)}{H_\lambda(z)}-\mu\right)-B\left(\frac{zH'_\lambda(z)}{H_\lambda(z)}-1\right)}\right|<\beta, \tag{1.9}$$

where $H_\lambda(z)=(1-\lambda)h(z)+\lambda zh'(z)$, $\lambda\ge0$, $h=f*S_\alpha$, $0<\beta\le1$, $0\le\mu<1$, and

$$\frac{B}{2(B-A)}<\gamma\le\begin{cases}\frac{B}{2(B-A)\mu}, & \mu\ne0,\\ 1, & \mu=0\end{cases}$$

for fixed $-1\le A\le B\le1$ and $0<B\le1$.

We also note that a function f is a so-called α-prestarlike $(0\le\alpha<1)$ function if, and only if, $h=f*S_\alpha\in S^*(\alpha)$ which was first introduced by Ruscheweyh [3], and was rigorously studied by Silverman and Silvia [4], Owa and Ahuja [5] and Uralegaddi and Sarangi [6]. Further, a function $f\in\mathcal{A}$ is in the class $\mathcal{C}[\mu,\alpha,\beta,\gamma,\lambda,A,B]$ if and only if, $zf'(z)\in\mathcal{R}[\mu,\alpha,\beta,\gamma,\lambda,A,B]$.

Let T denote the subclass of A consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,\quad (a_n\ge0). \tag{1.10}$$

Let us write

$$\mathcal{R}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B] = \mathcal{R}[\mu, \alpha, \beta, \gamma, \lambda, A, B] \cap T$$

and

$$\mathcal{C}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B] = \mathcal{C}[\mu, \alpha, \beta, \gamma, \lambda, A, B] \cap T$$

where T is the class of functions of the form (1.10) that are analytic and univalent in U. The idea of unifying the study of classes $\mathcal{R}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$ and $\mathcal{C}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$ thus, forming a new class $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$ is somewhat or rather motivated from the work of [1] and [2].

In this paper, we will study the unified presentation of prestarlike functions belonging to $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$ which include growth and distortion theorem, radii of convexity, radii of starlikeness and radii of close-to-convexity.

2. COEFFICIENT INEQUALITY

Our main tool in this paper is the following result, which can be easily proven, and the details are omitted.

Lemma 2.1. *Let the function f be defined by (1.10). Then $f \in \mathcal{R}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$ if and only if*

$$\sum_{n=2}^{\infty} \Lambda(n, \lambda) D[n, \beta, \gamma, A, B] |a_n| c_n(\alpha) \leq E[\beta, \gamma, \mu, A, B] \tag{2.1}$$

where

$$\Lambda(n, \lambda) = (1 + (n-1)\lambda),$$
$$D[n, \beta, \gamma, A, B] = n - 1 + 2\beta\gamma(n - \mu)(B - A) - B\beta(n-1),$$
$$E[\beta, \gamma, \mu, A, B] = 2\beta\gamma(1 - \mu)(B - A).$$

The result is sharp.

Next, by observing that

$$f \in \mathcal{C}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B] \Leftrightarrow zf'(z) \in \mathcal{R}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B], \tag{2.2}$$

we gain the following Lemma 2.2.

Lemma 2.2. *Let the function f be defined by (1.10). Then $f \in \mathcal{C}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$ if and only if*

$$\sum_{n=2}^{\infty} n\Lambda(n, \lambda) D[n, \beta, \gamma, A, B] |a_n| c_n(\alpha) \leq E[\beta, \gamma, \mu, A, B] \tag{2.3}$$

where

$$\Lambda(n, \lambda) = (1 + (n-1)\lambda),$$
$$D[n, \beta, \gamma, A, B] = n - 1 + 2\beta\gamma(n - \mu)(B - A) - B\beta(n-1),$$
$$E[\beta, \gamma, \mu, A, B] = 2\beta\gamma(1 - \mu)(B - A)$$

and $c_n(\alpha)$ given by (1.5).

In view of Lemma 2.1 and Lemma 2.2, we unified the classes $\mathcal{R}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,\lambda,A,B]$ and $\mathcal{C}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,\lambda,A,B]$ and so a new class $\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]$ is formed. Thus we say that a function f defined by (1.10) belongs to $\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]$ if and only if,

$$\sum_{n=2}^{\infty}(1-\eta+n\eta)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B]|a_n|c_n(\alpha)\le E[\beta,\gamma,\mu,A,B], \tag{2.4}$$

$$(0\le\alpha<1;\ 0<\beta\le 1;\ \eta\ge 0;\ \lambda\ge 0;\ -1\le A\le B\le 1 \text{ and } 0<B\le 1),$$

where $\Lambda(n,\lambda)$, $D[n,\beta,\gamma,A,B]$, $E[\beta,\gamma,\mu,A,B]$ and $c_n(\alpha)$ are given in (Lemma 2.1 and Lemma 2.2) and given by (1.5), respectively.

Clearly, we obtain

$$\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]=(1-\eta)\mathcal{R}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,A,B]+\eta\mathcal{C}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,A,B],$$

so that

$$\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,0,A,B]=\mathcal{R}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,A,B],$$

and

$$\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,1,A,B]=\mathcal{C}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,A,B].$$

3. GROWTH AND DISTORTION THEOREM

A distortion property for function f in the class $\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]$ is given as follows:

Theorem 3.1. *Let the function f defined by (1.10) be in the class $\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]$, then*

$$r-\frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)}r^2 \le |f(z)|\le r+\frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)}r^2, \tag{3.1}$$

$$(\eta\ge 0;\quad 0\le\alpha<1;\quad 0<\beta\le 1;\quad z\in U)$$

and

$$1-\frac{E[\beta,\gamma,\mu,A,B]}{(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)}r \le |f'(z)|\le 1+\frac{E[\beta,\gamma,\mu,A,B]}{(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)}r, \tag{3.2}$$

$$(\eta\ge 0;\quad 0\le\alpha<1;\quad 0<\beta\le 1;\quad z\in U).$$

The bounds in (3.1) and (3.2) are attained for the function f given by

$$f(z)=z-\frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)}z^2.$$

Proof. Observing that $c_n(\alpha)$ defined by (1.5) is nondecreasing for $(0\le\alpha<1)$, we find from (2.4) that

$$\sum_{n=2}^{\infty}|a_n|\le\frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)}. \tag{3.3}$$

Using (1.10) and (3.3), we readily have $(z \in U)$

$$\begin{aligned}|f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n| c_n(\alpha) |z^n| \\ &\geq |z| - |z^2| \sum_{n=2}^{\infty} |a_n| c_n(\alpha), \\ &\geq r - \frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)} r^2, \qquad |z| = r < 1\end{aligned}$$

and

$$\begin{aligned}|f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n| c_n(\alpha) |z^n| \\ &\leq |z| + |z^2| \sum_{n=2}^{\infty} |a_n| c_n(\alpha), \\ &\leq r + \frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)} r^2, \qquad |z| = r < 1,\end{aligned}$$

which proves the assertion (3.1) of Theorem 3.1.

Also, from (1.10), we find for $z \in U$ that

$$\begin{aligned}|f'(z)| &\geq 1 - \sum_{n=2}^{\infty} n|a_n| c_n(\alpha) |z^{n-1}| \\ &\geq 1 - |z| \sum_{n=2}^{\infty} n|a_n| c_n(\alpha), \\ &\geq 1 - \frac{E[\beta,\gamma,\mu,A,B]}{(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)} r, \qquad |z| = r < 1\end{aligned}$$

and

$$\begin{aligned}|f'(z)| &\leq 1 + \sum_{n=2}^{\infty} n|a_n| c_n(\alpha) |z^{n-1}| \\ &\leq 1 + |z| \sum_{n=2}^{\infty} n|a_n| c_n(\alpha), \\ &\leq 1 + \frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)} r, \qquad |z| = r < 1,\end{aligned}$$

which proves the assertion (3.2) of Theorem 3.1. □

4. RADII CONVEXITY AND STARLIKENESS

The radii of convexity for class $\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]$ is given by the following theorem.

Theorem 4.1. *Let the function f be in the class $\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]$. Then the function f is convex of order ρ $(0 \leq \rho < 1)$ in the disk $|z| < r_1(\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B) = r_1$, where*

$$r_1 = \inf_n \left\{ \frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B](1-\eta+n\eta)}{n(n-\rho)E[\beta,\gamma,\mu,A,B]} \right\}^{\frac{1}{n-1}}. \tag{4.1}$$

Proof. It sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n |z|^{1-n}} \tag{4.2}$$

which implies that

$$(1-\rho) - \left|\frac{zf''(z)}{f'(z)}\right| \geq (1-\rho) - \frac{\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}$$

$$= \frac{(1-\rho) - \sum_{n=2}^{\infty} n(n-\rho)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n |z|^{n-1}}. \tag{4.3}$$

Hence from (4.1), if

$$|z|^{n-1} \leq \frac{(1-\rho)}{n(n-\rho)} \cdot \frac{2(1-\alpha)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B](1-\eta+n\eta)}{E[\beta,\gamma,\mu,A,B]}, \tag{4.4}$$

and according to (2.4)

$$1-\rho-\sum_{n=2}^{\infty} n(n-\rho)a_n|z|^{n-1} > 1-\rho-(1-\rho) = \rho. \tag{4.5}$$

Hence from (4.3), we obtain

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1-\rho$$

Therefore

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0,$$

which shows that f is convex in the disk $|z| < r_1(\mu,\alpha,\beta,\gamma,\lambda,\eta,\rho,A,B)$. □

By setting $\eta = 0$ and $\eta = 1$, we have the Corollary 4.2 and the Corollary 4.3, respectively.

Corollary 4.2. *Let the function f be in the class $\mathcal{R}_{\mathcal{T}}(\mu,\alpha,\beta,\gamma,\lambda,\rho,A,B)$. Then the function f is convex of order ρ $(0 \leq \rho < 1)$ in the disk $|z| < r_2(\mu,\alpha,\beta,\gamma,\lambda,\rho,A,B) = r_2$, where*

$$r_2 = \inf_n \left\{\frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B]}{n(n-\rho)E[\beta,\gamma,\mu,A,B]}\right\}^{\frac{1}{n-1}}. \tag{4.6}$$

Corollary 4.3. *Let the function f be in the class $\mathcal{C}_{\mathcal{T}}(\mu,\alpha,\beta,\gamma,\lambda,\rho,A,B)$. Then the function f is convex of order ρ $(0 \leq \rho < 1)$ in the disk $|z| < r_3(\mu,\alpha,\beta,\gamma,\lambda,\rho,A,B) = r_3$, where*

$$r_3 = \inf_n \left\{\frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B]}{(n-\rho)E[\beta,\gamma,\mu,A,B]}\right\}^{\frac{1}{n-1}}. \tag{4.7}$$

Theorem 4.4. *Let the function f be in the class $\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]$. Then the function f is starlike of order ρ $(0 \leq \rho < 1)$ in the disk $|z| < r_4(\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B) = r_4$, where*

$$r_4 = \inf_n \left\{\frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B](1-\eta+n\eta)}{(n-\rho)E[\beta,\gamma,\mu,A,B]}\right\}^{\frac{1}{n-1}}. \tag{4.8}$$

Proof. It sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\rho$$

Using a similar method to Theorem 4.1 and making use of (2.4), we get (4.8). □

Letting $\eta = 0$ and $\eta = 1$, we have the Corollary 4.5 and the Corollary 4.6, respectively.

Corollary 4.5. *Let the function f be in the class $\mathcal{R}_{\mathcal{T}}(\mu, \alpha, \beta, \gamma, \lambda, \rho, A, B)$. Then the function f is starlike of order ρ $(0 \leq \rho < 1)$ in the disk $|z| < r_5(\mu, \alpha, \beta, \gamma, \lambda, \rho, A, B) = r_5$, where*

$$r_5 = \inf_n \left\{ \frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B]}{(n-\rho)E[\beta,\gamma,\mu,A,B]} \right\}^{\frac{1}{n-1}}. \tag{4.9}$$

Corollary 4.6. *Let the function f be in the class $\mathcal{C}_{\mathcal{T}}(\mu, \alpha, \beta, \gamma, \lambda, \rho, A, B)$. Then the function f is starlike of order ρ $(0 \leq \rho < 1)$ in the disk $|z| < r_6(\mu, \alpha, \beta, \gamma, \lambda, \rho, A, B) = r_6$, where*

$$r_6 = \inf_n \left\{ \frac{2n(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B]}{(n-\rho)E[\beta,\gamma,\mu,A,B]} \right\}^{\frac{1}{n-1}}. \tag{4.10}$$

Last, but not least we give the following result.

Theorem 4.7. *Let the function f be in the class $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$. Then the function f is close-to-convex of order ρ $(0 \leq \rho < 1)$ in the disk $|z| < r_7(\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B) = r_7$, where*

$$r_7 = \inf_n \left\{ \frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B](1-\eta+n\eta)}{nE[\beta,\gamma,\mu,A,B]} \right\}^{\frac{1}{n-1}}. \tag{4.11}$$

Proof. It sufficient to show that

$$|f'(z) - 1| < 1 - \rho.$$

Using a similar technique to Theorem 4.1 and making use of (2.4), we get (4.11). □

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