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SOME INEQUALITIES EXHIBITING CERTAIN PROPERTIES OF SOME SUBCLASSES OF MULTIVALENTLY ANALYTIC FUNCTIONS

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ABSTRACT. This paper introduces a new subclass and investigates the sufficiency conditions for a function to belong to this subclass. Certain types of inequalities are also studied exhibiting the well-known geometric properties of multivalently analytic functions in the unit disk. Several interesting consequences of the main results are also mentioned.

Key words and phrases: Open unit disk, analytic, multivalently analytic functions, multivalently starlike, multivalently convex, rational and complex inequalities, rational functions with complex variable and Jack's Lemma.

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1. INTRODUCTION AND DEFINITION

Let $\mathcal{T}(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathcal{N} = \{1, 2, 3, \dots\}), \tag{1.1}$$

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which are *analytic* and *multivalent* in the open disk $\mathcal{U} = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$. A function $f(z)$ belonging to $\mathcal{T}(p)$ is said to be *multivalently starlike order* α in $\mathcal{U}$ if it satisfies the inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathcal{U};\ 0 \leqslant \alpha < p;\ p \in \mathcal{N}), \tag{1.2}$$

and, a function $f(z) \in \mathcal{T}(p)$ is said to be *multivalently convex of order* α in $\mathcal{U}$ if it satisfies the inequality:

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{U};\ 0 \leqslant \alpha < p;\ p \in \mathcal{N}). \tag{1.3}$$

For the aforementioned definitions, one may refer to [1] (see also [11]). Further, a function $f(z) \in \mathcal{T}(p)$ is said to be in the subclass $\mathcal{TSK}_{\lambda}^{\delta}(p;\alpha)$ if it satisfies the inequality:

$$\Re\left\{\left(\frac{zf'(z) + \lambda z^2 f'(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right)^{\delta}\right\} > \alpha, \tag{1.4}$$

$$(z \in \mathcal{U};\ \delta \neq 0;\quad 0 \leqslant \lambda \leqslant 1;\quad 0 \leqslant \alpha < p;\quad p \in \mathcal{N}).$$

Here, and throughout this paper, the value of expressions like

$$\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right)^{\delta},$$

is considered to be its principal value. We mention below some of the subclasses of the functions $\mathcal{T}(p)$ from the families of functions $\mathcal{TSK}_{\lambda}^{\delta}(p;\alpha)$ (defined above). Indeed, we have

$$\mathcal{TS}^{\delta}(p;\alpha) \equiv \mathcal{TSK}_{0}^{\delta}(p;\alpha) \quad (\delta \neq 0,\ 0 \leqslant \alpha < p,\ p \in \mathcal{N}), \tag{1.5}$$

$$\mathcal{TK}^{\delta}(p;\alpha) \equiv \mathcal{TSK}_{1}^{\delta}(p;\alpha) \quad (\delta \neq 0,\ 0 \leqslant \alpha < p,\ p \in \mathcal{N}), \tag{1.6}$$

$$\mathcal{T}_{\lambda}(p;\alpha) \equiv \mathcal{TSK}_{\lambda}^{1}(p;\alpha) \quad (0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \alpha < p,\ p \in \mathcal{N}) \quad \text{(see [5])}. \tag{1.7}$$

The important subclasses in Geometric Function Theory such as multivalently starlike functions $\mathcal{S}_p(\alpha)$ of order α $(0 \leqslant \alpha < p;\ p \in \mathcal{N})$ in $\mathcal{U}$, multivalently convex functions $\mathcal{K}_p(\alpha)$ of order α $(0 \leqslant \alpha < p;\ p \in \mathcal{N})$ in $\mathcal{U}$, multivalently starlike functions $\mathcal{S}_p$ in $\mathcal{U}$, multivalently convex functions $\mathcal{K}_p$ in $\mathcal{U}$, starlike functions $\mathcal{S}(\alpha)$ of order α $(0 \leqslant \alpha < 1)$ in $\mathcal{U}$, convex functions $\mathcal{K}(\alpha)$ of order α $(0 \leqslant \alpha < 1)$ in $\mathcal{U}$, starlike functions $\mathcal{S}$ in $\mathcal{U}$ and convex functions $\mathcal{K}$ in $\mathcal{U}$, are seen to be easily identifiable with the aforementioned classes ([1], [5] and [11]).

By introducing a subclass $\mathcal{TSK}_{\lambda}^{\delta}(p;\alpha)$of functions $f(z) \in \mathcal{T}(p)$ satisfying the inequality (1.4), our motive in this paper is to obtain sufficient conditions for a function to belong to the above subclass. The other results investigated include certain inequalities for multivalent functions depicting the properties of starlikeness, close-to-convexity and convexity in the open unit disk. Several corollaries are deduced as worthwhile consequences of our main results.

2. Main Results

Before stating and proving our main results, we require the following assertion (popularly known as Jack's Lemma).

Lemma 2.1 ([7]). *Let the function $w(z)$ be non-constant and regular in the unit disc $\mathcal{U}$ such that $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at the point z_0, then*

$$z_0 w'(z_0) = c\, w(z_0) \qquad (c \geqslant 1). \tag{2.1}$$

We begin now to prove the following:

Theorem 2.2. *Let* $\delta \in \mathbb{R}\backslash\{0\}, 0 \leqslant \alpha < p, p \in \mathcal{N}$ *and* $f(z) \in \mathcal{T}(p)$*. If a function* $F(z)$ *defined by*

$$F(z) = (1-\lambda)f(z) + \lambda z f'(z) \qquad (0 \leqslant \lambda \leqslant 1), \tag{2.2}$$

satisfies the inequality:

$$\Re\left\{\frac{1+z\left(\frac{F''(z)}{F'(z)}-\frac{F'(z)}{F(z)}\right)}{1-p^{\delta}\left(\frac{zF'(z)}{F(z)}\right)^{-\delta}}\right\}\left\{\begin{array}{l} <\frac{1}{\delta} \text{ when } \delta>0 \\ \\ >\frac{1}{\delta} \text{ when } \delta<0 \end{array}\right\} \qquad (z\in\mathcal{U}), \tag{2.3}$$

then $f(z) \in \mathcal{TSK}_{\lambda}^{\delta}(p;\beta),$ *where* $\beta = p^{\delta} - (p-\alpha)^{\delta}$.

Proof. Let $f(z) \in \mathcal{T}(p)$and $F(z)$ be defined by (2.2) . From (1.1) and (2.2), we have

$$\begin{aligned} \frac{zF'(z)}{F(z)} &= \frac{zf'(z)+\lambda\, z^2 f''(z)}{(1-\lambda)f(z)+\lambda\, zf'(z)} \\ &= \frac{p+\sum_{k=p+1}^{\infty}\frac{k[1+\lambda(k-1)]}{1+\lambda(p-1)}a_k z^{k-p}}{1+\sum_{k=p+1}^{\infty}\frac{1+\lambda(k-1)}{1+\lambda(p-1)}a_k z^{k-p}}. \end{aligned} \tag{2.4}$$

$$(z\in\mathcal{U};\ 0\leqslant\lambda\leqslant 1;\ p\in\mathcal{N})$$

Now, define a function $w(z)$ by

$$\left(\frac{zF'(z)}{F(z)}\right)^{\delta} - p^{\delta} = (p-\alpha)^{\delta} w(z), \qquad (z\in\mathcal{U};\ \delta\neq 0;\ 0\leqslant\alpha<p;\ p\in\mathcal{N}), \tag{2.5}$$

then the function $w(z)$ is analytic in $\mathcal{U}$ and $w(0)=0$. Differentiation of (2.5) gives

$$1+\frac{zF''(z)}{F'(z)}-\frac{zF'(z)}{F(z)} = \left(\frac{(p-\alpha)^{\delta}}{p^{\delta}+(p-\alpha)^{\delta}w(z)}\right)\frac{z\,w'(z)}{\delta}. \tag{2.6}$$

Hence, (2.5) and (2.6) yields

$$\frac{1+z\left(\frac{F''(z)}{F'(z)}-\frac{F'(z)}{F(z)}\right)}{1-p^{\delta}\left(\frac{zF'(z)}{F(z)}\right)^{-\delta}} = \frac{zw'(z)}{\delta w(z)}. \tag{2.7}$$

We claim that $|w(z)| < 1$ in $\mathcal{U}$. For otherwise (by Jack's Lemma), there exists a point $z_0 \in \mathcal{U}$ such that

$$z_0 w'(z_0) = c\, w(z_0), \text{ where } |w(z_0)| = 1 \qquad (c \geqslant 1).$$

Therefore, (2.7) yields

$$\Re\left\{\left.\frac{1+z\left(\frac{F''(z)}{F'(z)}-\frac{F'(z)}{F(z)}\right)}{1-p^{\delta}\left(\frac{zF'(z)}{F(z)}\right)^{-\delta}}\right|_{z=z_0}\right\} = \frac{1}{\delta}\Re\left\{\frac{z_0 w'(z_0)}{w(z_0)}\right\} = \frac{c}{\delta}\left\{\begin{array}{l} \geqslant\frac{1}{\delta} \text{ when } \delta>0 \\ \\ \leqslant\frac{1}{\delta} \text{ when } \delta<0, \end{array}\right. \tag{2.8}$$

which contradicts our assumption (2.3) . Therefore, $|w(z)| < 1$ holds true for all $z \in \mathcal{U}$, and we conclude from (2.5) that

$$\left|\left(\frac{zF'(z)}{F(z)}\right)^{\delta} - p^{\delta}\right| = (p-\alpha)^{\delta}\,|w(z)| < (p-\alpha)^{\delta}, \tag{2.9}$$

which evidently implies that

$$\Re\left\{\left(\frac{zF'(z)}{F(z)}\right)^{\delta}\right\} > p^{\delta} - (p-\alpha)^{\delta}, \tag{2.10}$$

and hence $f(z) \in \mathcal{TSK}_{\lambda}^{\delta}(p;\alpha)$. □

Theorem 2.3. *Let* $\delta \in \mathbb{R}\backslash\{0\}$; $0 \leqslant \alpha < p$; $n, m, p \in \mathcal{N}$; $q = n - m$; $f(z) \in \mathcal{T}(n)$ *and* $g(z) \in \mathcal{T}(m)$. *If* $f(z)$ *satisfies the inequality:*

$$\Re\left(\frac{zf'(z)}{f(z)}\right)\begin{cases} < q+\alpha+\frac{1}{2\delta} & \textit{when} \quad \delta>0 \quad \textit{and}\ g(z)\in\mathcal{S}_m(\alpha) \\ > q+\alpha+\frac{1}{2\delta} & \textit{when} \quad \delta<0 \quad \textit{and}\ g(z)\notin\mathcal{S}_m(\alpha), \end{cases} \tag{2.11}$$

then

$$\Re\left\{\left(z^{-q}\frac{f(z)}{g(z)}\right)^{\delta}\right\} > 0, \tag{2.12}$$

where the value of $\left(z^{-q}\frac{f(z)}{g(z)}\right)^{\delta}$ *is taken to be its principle value.*

Proof. Let $f(z) \in \mathcal{T}(n)$ and $g(z) \in \mathcal{T}(m)$ with $n - m \in \mathcal{N}$. Since

$$\frac{f(z)}{g(z)} = z^q + c_1 z^{q+1} + c_2 z^{q+2} + \cdots \in \mathcal{T}(q) \qquad (q = n - m \in \mathcal{N}),$$

we define $w(z)$ by

$$\left(z^{-q}\frac{f(z)}{g(z)}\right)^{\delta} = 1 + w(z) \qquad (z \in \mathcal{U}; \delta \neq 0). \tag{2.13}$$

It is clear that the function $w(z)$ is an analytic function in $\mathcal{U}$ *and* $w(0) = 0$. Differentiating (2.13), we have

$$\frac{zf'(z)}{f(z)} = q + \frac{zw'(z)}{\delta\,(1 + w(z))} + \frac{zg'(z)}{g(z)}. \tag{2.14}$$

If we suppose that there exists a point $z_0 \in \mathcal{U}$ such that $z_0 w'(z_0) = c\,w(z_0)$ where $|w(z_0)| = 1$ $(c \geqslant 1)$, i.e. $w(z_0) = e^{i\theta}$ $(\theta \in [0, 2\pi) - \{\pi\})$, then

$$\begin{aligned} \Re\left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} &= q + \frac{1}{\delta}\Re\left\{\frac{z_0 w'(z_0)}{1 + w(z_0)} + \frac{\delta\, z_0 g'(z)}{g(z_0)}\right\} \\ &= q + \frac{1}{\delta}\Re\left\{\frac{ce^{i\theta}}{1 + e^{i\theta}}\right\} + \Re\left\{\frac{z_0 g'(z)}{g(z_0)}\right\}. \end{aligned} \tag{2.15}$$

From (2.15) it follows that

$$\Re\left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} \geqslant q + \alpha + \frac{1}{2\delta} \qquad (\delta > 0), \tag{2.16}$$

provided that

$$\Re\left\{\frac{z_0 g'(z_0)}{g(z_0)}\right\} > \alpha,$$

and

$$\Re\left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} \leqslant q + \alpha + \frac{1}{2\delta} \qquad (\delta < 0), \tag{2.17}$$

provided that

$$\Re\left\{\frac{z_0 g'(z_0)}{g(z_0)}\right\} \leqslant \alpha.$$

But the inequalities in (2.16) and (2.17) contradict the inequalities in (2.11). Hence $|w(z)| < 1$, for all $z \in \mathcal{U}$, and therefore (2.13) yields

$$\left| \left(z^{-q} \frac{f(z)}{g(z)} \right)^{\delta} - 1 \right| = |w(z)| < 1, \tag{2.18}$$

which evidently implies (2.12), and this completes the proof of Theorem 2.3. □

Theorem 2.4. *Let* $\delta \in \mathbb{R} \setminus \{0\}; 0 \leqslant \alpha < p; n, m, p \in \mathcal{N}; q = n - m; f(z) \in \mathcal{T}(n)$, *and* $g(z) \in \mathcal{T}(m)$. *If* $f(z)$ *satisfies the inequality:*

$$\Re \left(1 + \frac{z f''(z)}{f'(z)} \right) \begin{cases} < q + \alpha + \frac{1}{2\delta} & \text{when } \delta > 0 \text{ and } g(z) \in \mathcal{K}_m(\alpha) \\ > q + \alpha + \frac{1}{2\delta} & \text{when } \delta < 0 \text{ and } g(z) \notin \mathcal{K}_m(\alpha), \end{cases} \tag{2.19}$$

then

$$\Re \left\{ \left(z^{-q} \frac{m f'(z)}{n g'(z)} \right)^{\delta} \right\} > 0, \tag{2.20}$$

where the value of $\left(z^{-q} \frac{m f'(z)}{n g'(z)} \right)^{\delta}$ *is taken its principle value.*

Proof. Let $f(z) \in \mathcal{T}(n)$ and $g(z) \in \mathcal{T}(m)$ with $n - m \in \mathcal{N}$. Since

$$\frac{m\, f'(z)}{n\, g'(z)} = z^q + k_1 z^{q+1} + k_2 z^{q+2} + \cdots \in \mathcal{T}(q) \qquad (q = n - m \in \mathcal{N}),$$

and if we define $w(z)$ by

$$\left(z^{-q} \frac{m\, f'(z)}{n\, g'(z)} \right)^{\delta} = 1 + w(z) \qquad (z \in \mathcal{U}), \tag{2.21}$$

then by appealing to the same technique as in the proof of Theorem 2.3, we arrive at the assertion (2.20) of Theorem 2.4 under the conditions stated with (2.19). □

3. SOME CONSEQUENCES OF MAIN RESULTS

Among the various interesting and important consequences of Theorems 2.2 – 2.4, we mention now some of the corollaries relating to the classes $\mathcal{T}_\lambda(p;\alpha)$, $\mathcal{T}_\lambda(\alpha)$, $\mathcal{S}_p(\alpha)$, $\mathcal{K}_p(\alpha)$, $\mathcal{S}_p$, $\mathcal{K}_p$, $\mathcal{S}(\alpha)$, $\mathcal{K}(\alpha)$, which are easily deducible form the main results. Inequalities concerning analytic and multivalent functions were also studied in [2] – [6], and in [8] – [10].

Firstly, if we take $\delta = 1$, then Theorem 2.2 by virtue of (1.7) gives the following:

Corollary 3.1. *Let a function* $F(z)$ *defined by (2.2) satisfy the condition:*

$$\Re \left\{ \frac{1 + z \left(\frac{F''(z)}{F'(z)} - \frac{F'(z)}{F(z)} \right)}{1 - p \left(\frac{F(z)}{zF'(z)} \right)} \right\} < 1, \tag{3.1}$$

$$(z \in \mathcal{U}; 0 \leqslant \alpha < p; p \in \mathcal{N}; f(z) \in \mathcal{T}(p))$$

then $f(z) \in \mathcal{T}_\lambda(p;\alpha)$.

Next, if we take $\delta - 1 = \lambda = 0$ in Theorem 2.2, so that $F(z) = f(z)$, then we get

Corollary 3.2. *If* $F(z) = f(z)$ *satisfies the condition in (3.1), then* $f(z) \in \mathcal{S}_p(\alpha)$, *i.e.* $f(z)$ *is* $p-$*valent starlike of order* $\alpha (0 \leqslant \alpha < p; p \in \mathcal{N})$ *in* $\mathcal{U}$.

If we take $\delta = \lambda = 1$ in Theorem 2.2, so that $F(z) = zf'(z)$, then we obtain the following:

Corollary 3.3. *If $f(z)$ satisfies the condition*

$$\Re\left\{\frac{1+z\left(\frac{(zf'(z))''}{(zf'(z))'}-\frac{(zf'(z))'}{zf'(z)}\right)}{1-p\left(\frac{zf'(z)}{(zf'(z))'}\right)}\right\}<1 \qquad (z\in\mathcal{U};0\leqslant\alpha<p;p\in\mathcal{N}), \tag{3.2}$$

then $f(z)\in\mathcal{K}_p(\alpha)$, that is $f(z)$ is $p-$valent convex of the order α $(0\leqslant\alpha<p;p\in\mathcal{N})$ in $\mathcal{U}$.

For $p=1$ in Corollaries 3.1 – 3.3 give the following:

Corollary 3.4. *Let a function $F(z)$ defined by (2.2) satisfy the condition*

$$\Re\left\{\frac{1+z\left(\frac{F''(z)}{F'(z)}-\frac{F'(z)}{F(z)}\right)}{1-\frac{F(z)}{zF'(z)}}\right\}<1, \tag{3.3}$$

$$(z\in\mathcal{U};\ 0\leqslant\alpha<1;\ \ f(z)\in\mathcal{T})$$

then $f(z)\in\mathcal{T}_\lambda(\alpha)$.

Corollary 3.5. *If $F(z)=f(z)$ satisfies the condition (3.3), then $f(z)\in\mathcal{S}(\alpha)$, i.e. $f(z)$ is starlike of order α $(0\leqslant\alpha<1)$ in $\mathcal{U}$.*

Corollary 3.6. *If $f(z)$ satisfies the condition*

$$\Re\left\{\frac{1+z\left(\frac{(zf'(z))''}{(zf'(z))'}-\frac{(zf'(z))'}{zf'(z)}\right)}{1-\frac{zf'(z)}{(zf'(z))'}}\right\}<1 \qquad (z\in\mathcal{U};\ 0\leqslant\alpha<1), \tag{3.4}$$

then $f(z)\in\mathcal{K}(\alpha)$, i.e., $f(z)$ is convex of order α $(0\leqslant\alpha<1)$ in $\mathcal{U}$.

Let us take $\delta=1$ in Theorems 2.3 and 2.4, then we get the following:

Corollary 3.7. *Let $z\in\mathcal{U}$; $0\leqslant\alpha<p$; $n,m,p\in\mathcal{N}$; $f(z)\in\mathcal{T}(n)$ and a function $g(z)\in\mathcal{T}(m)$ belong to the class $\mathcal{S}_m(\alpha)$ with $q=n-m\in\mathcal{N}$. If $f(z)$ satisfies the inequality:*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}<q+\alpha+\frac{1}{2}, \tag{3.5}$$

then

$$\Re\left\{z^{-q}\frac{f(z)}{g(z)}\right\}>0. \tag{3.6}$$

Corollary 3.8. *Let $z\in\mathcal{U}$; $0\leqslant\alpha<p$; $n,m,p\in\mathcal{N}$; $f(z)\in\mathcal{T}(n)$ and a function $g(z)$ in $\mathcal{T}(m)$ belong to the class $\mathcal{K}_m(\alpha)$ with $q=n-m\in\mathcal{N}$. If $f(z)$ satisfies the inequality:*

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}<q+\alpha+\frac{1}{2}, \tag{3.7}$$

then

$$\Re\left\{z^{-q}\frac{m\ f'(z)}{n\ g'(z)}\right\}>0. \tag{3.8}$$

Lastly, setting $\delta=-1$ in Theorems 2.3 and 2.4, we obtain the following:

Corollary 3.9. *Let $z\in\mathcal{U}$; $0\leqslant\alpha<p$; $n,m,p\in\mathcal{N}$; $f(z)\in\mathcal{T}(n)$ and suppose a function $g(z)\in\mathcal{T}(m)$ does not belong to the class $\mathcal{S}_m(\alpha)$ with $q=n-m\in\mathcal{N}$. If $f(z)$ satisfies the inequality:*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>q+\alpha-\frac{1}{2}, \tag{3.9}$$

then

$$\Re\left\{z^q\frac{g(z)}{f(z)}\right\} > 0. \tag{3.10}$$

Corollary 3.10. *Let* $z \in \mathcal{U}$; $0 \leqslant \alpha < p$; $n, m, p \in \mathcal{N}$; $f(z) \in \mathcal{T}(n)$ *and suppose a function* $g(z)$ *in* $\mathcal{T}(m)$ *does not belong to the class* $\mathcal{K}_m(\alpha)$ *with* $q = n - m \in \mathcal{N}$. *If* $f(z)$ *satisfies the inequality:*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > q + \alpha - \frac{1}{2}, \tag{3.11}$$

then

$$\Re\left\{z^q\frac{n\,g'(z)}{m\,f'(z)}\right\} > 0. \tag{3.12}$$

REFERENCES

[1] P.L. DUREN, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer- Verlag, New York Berlin, Heidelberg, and Tokyo, 1983.

[2] H. IRMAK AND S. OWA, Certain inequalities involving analytic and univalent functions, *Far East J. Math. Sci.*, **10** (2003), 353–358.

[3] H. IRMAK AND S. OWA, Certain inequalities for multivalent and meromorphically multivalent starlike functions, *Bull. Inst. Math. Acad. Sinica*, **31** (2003), 11–21.

[4] H. IRMAK , R.K. RAINA AND S. OWA, A certain for multivalent starlike and mermorphically multivalent starlike functions, *Inter. J. Appl. Math.*, **12** (2003), 93–98.

[5] H. IRMAK AND R.K. RAINA, The starlikeness and convexity of multivalent function involving certain inequalities, *Rev. Mat. Complut.*,**16** (2003), 391–398.

[6] H. IRMAK, R.K. RAINA AND S. OWA, Certain results involving inequalities on analytic and univalent functions, *Far East J. Math. Sci.*, **10** (2003), 359–366.

[7] I.S. JACK, Functions starlike and convex of the order α, *J. London Math. Soc.*, **3** (1971), 469–474.

[8] S. OWA, M. NUNOKAWA AND H. SAITOH, Some inequalities involving multivalent functions, *Ann. Polon. Math.*, **60** (1994), 159–162.

[9] S. OWA, M. NUNOKAWA AND S. FUKUI, A criterion for $p-$valent starlike functions, *Intern. J. Math. & Math. Sci.*, **17** (1994), 205–207.

[10] S. OWA, H.M. SRIVASTAVA, F.-Y. REN AND W.-Q. YANG, The starlikeness of a certain class of integral operators, *Complex Variables*, **27** (1995), 185–191.

[11] H.M. SRIVASTAVA AND S. OWA, (EDITORS), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.