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## A NEW PROOF OF THE MONOTONICITY PROPERTY OF POWER MEANS

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Abstract

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## Abstract

If  $M_r$  is the weighted power mean of the numbers  $x_j \in [a, b]$  then  $Q_r(a, b, x) = (a^r + b^r - M_r^r)^{1/r}$  is increasing in  $r$ . A new proof of this fact is given.

*2000 Mathematics Subject Classification:* 26D15.

*Key words:* Convexity, Monotonicity, Power Means.

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# 1. Introduction

Suppose that  $0 < a < b$ ,  $a \leq x_1 \leq \dots \leq x_n \leq b$  and  $w_i$  are positive weights with  $\sum w_i = 1$ . The weighted power means  $M_r(x, w)$  of the numbers  $x_i$  with weights  $w_i$  are defined as

$$M_r(x, w) = \left( \sum w_i x_i^r \right)^{\frac{1}{r}} \quad \text{for } r \neq 0, \quad M_0(x, w) = \exp \left( \sum w_i \log x_i \right).$$

It is well-known (cf. [1, 2, 5]) that  $M_r$  increases with  $r$  unless or  $x_i$  are equal. In [3] Mercer defined another family of functions

$$Q_r(a, b, x) = (a^r + b^r - M_r^r(x, w))^{1/r} \quad \text{for } r \neq 0, \quad Q_0(a, b, x) = ab/M_0$$

and proved the following

**Theorem 1.1.** For  $r < s$   $Q_r(a, b, x) \leq Q_s(a, b, x)$ .

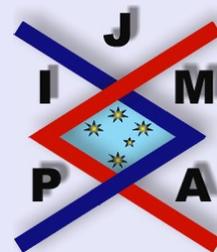
The aim of this note is to give another proof of this theorem. We will use the following version of the Jensen inequality ([4])

**Lemma 1.2.** If  $f$  is convex then

$$(1.1) \quad f \left( a + b - \sum w_i x_i \right) \leq f(a) + f(b) - \sum w_i f(x_i).$$

For concave  $f$  the inequality reverses.

Our proof differs from the original one:



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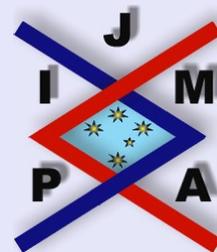
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*Proof.* Let  $x_i = \lambda_i a + (1 - \lambda_i)b$ . Then

$$\begin{aligned} f(a + b - \sum w_i x_i) &= f\left(\sum w_i [(1 - \lambda_i)a + \lambda_i b]\right) \\ &\leq \sum w_i f([(1 - \lambda_i)a + \lambda_i b]) \\ &\leq \sum w_i [(1 - \lambda_i)f(a) + \lambda_i f(b)] \\ &= \sum w_i [f(a) - \lambda_i f(a) + f(b) - (1 - \lambda_i)f(b)] \\ &= f(a) + f(b) + \sum w_i [-\lambda_i f(a) - (1 - \lambda_i)f(b)] \\ &\leq f(a) + f(b) - \sum w_i f(x_i). \end{aligned}$$

□



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## 2. Proof of Theorem 1.1

*Proof.* Let  $\tilde{a} = a^r/Q_r^r$ ,  $\tilde{b} = b^r/Q_r^r$ ,  $\tilde{x}_i = x_i^r/Q_r^r$ . Applying (1.1) to the concave function  $\log x$  we obtain

$$\begin{aligned} 0 &= \log \left( \tilde{a} + \tilde{b} - \sum w_i \tilde{x}_i \right) \geq \log \tilde{a} + \log \tilde{b} - \sum w_i \log \tilde{x}_i \\ &= r \log \frac{Q_0}{Q_r}, \end{aligned}$$

which shows that for  $r > 0$   $Q_{-r} \leq Q_0 \leq Q_r$ .

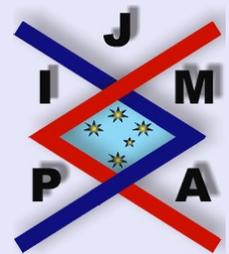
If  $0 < r < s$  then the function  $f(x) = x^{s/r}$  is convex and from (1.1) we have

$$\begin{aligned} 1 &= f \left( \tilde{a} + \tilde{b} - \sum w_i \tilde{x}_i \right) \leq \frac{a^s}{Q_r^s} + \frac{b^s}{Q_r^s} - \sum w_i \frac{x_i^s}{Q_r^s} \\ &= \left( \frac{Q_s}{Q_r} \right)^s, \end{aligned}$$

so  $Q_r \leq Q_s$ .

Finally, for  $r < s < 0$   $f$  is concave and we obtain  $1 \geq \left( \frac{Q_s}{Q_r} \right)^s$  also equivalent to  $Q_r \leq Q_s$ .

Obviously, equality holds if and only if all  $x_i$ 's are equal  $a$  or all are equal  $b$ .  $\square$



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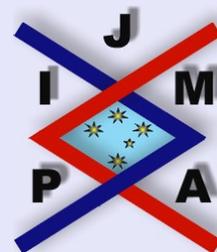
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