NEW BOUNDS FOR THE IDENTRIC MEAN OF TWO ARGUMENTS

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Received: 16 April, 2008

Accepted: 27 June, 2008

Communicated by: J. Sandor

2000 AMS Sub. Class.: 26D07, 26D20, 26E60.

Key words: Arithmetic mean, Geometric mean, Identric mean.

Abstract: Given two positive real numbers x and y, let $A(x,y)$, $G(x,y)$, and $I(x,y)$ denote their arithmetic mean, geometric mean, and identric mean, respectively. Also, let $K_p(x,y) = \sqrt[p]{\frac{2}{3}A^p(x,y) + \frac{1}{3}G^p(x,y)}$ for $p > 0$. In this note we prove that $K_p(x,y) < I(x,y)$ for all positive real numbers $x \neq y$ if and only if $p \leq 6/5$, and that $I(x,y) < K_p(x,y)$ for all positive real numbers $x \neq y$ if and only if $p \geq (\ln 3 - \ln 2)/(1 - \ln 2)$. These results, complement and extend similar inequalities due to J. Sándor [2], J. Sándor and T. Trif [3], and H. Alzer and S.-L. Qiu [1].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

In this note we consider several means of two positive real numbers x and y. Recall that the arithmetic mean, the geometric mean and the identric mean are defined by $A(x,y) = \frac{x+y}{2}$, $G(x,y) = \sqrt{xy}$ and

$$I(x,y) = \begin{cases} \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}} & \text{if} \quad x \neq y \\ x & \text{if} \quad x = y \end{cases}$$

We also introduce the family $(K_p(x,y))_{p>0}$ of means of x and y, defined by

$$K_p(x,y) = \sqrt[p]{\frac{2A^p(x,y) + G^p(x,y)}{3}}.$$

Using the fact that, for $\alpha > 1$, the function $t \mapsto t^\alpha$ is strictly convex on $\mathbb{R}_+^*$, and that for $x \neq y$ we have $A(x,y) > G(x,y)$ we conclude that, for $x \neq y$, the function $p \mapsto K_p(x,y)$ is increasing on $\mathbb{R}_+^*$.

In [3] it is proved that $I(x,y) < K_2(x,y)$ for all positive real numbers $x \neq y$. Clearly this implies that $I(x,y) < K_p(x,y)$ for $p \geq 2$ and $x \neq y$ which is the upper (and easy) inequality of Theorem 1.2 of [4].

On the other hand, J. Sándor proved in [2] that $K_1(x,y) < I(x,y)$ for all positive real numbers $x \neq y$, and this implies that $K_p(x,y) < I(x,y)$ for $p \leq 1$ and $x \neq y$.

The aim of this note is to generalize the above-mentioned inequalities by determining exactly the sets

$$\mathcal{L} = \{p > 0 : \forall\,(x,y) \in D,\ K_p(x,y) < I(x,y)\}$$
$$\mathcal{U} = \{p > 0 : \forall\,(x,y) \in D,\ I(x,y) < K_p(x,y)\}$$

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issn: 1443-5756

with $D = \{(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+^* : x \neq y\}$. Clearly, $\mathcal{L}$ and $\mathcal{U}$ are intervals since $p \mapsto K_p(x,y)$ is increasing. And the stated results show that

$$(0,1] \subset \mathcal{L} \subset (0,2) \qquad \text{and} \qquad [2,+\infty) \subset \mathcal{U} \subset (1,+\infty).$$

The following theorem is the main result of this note.

Theorem 1.1. *Let $\mathcal{U}$ and $\mathcal{L}$ be as above, then $\mathcal{L} = (0, p_0]$ and $\mathcal{U} = [p_1, +\infty)$ with*

$$p_0 = \frac{6}{5} = 1.2 \qquad \textit{and} \qquad p_1 = \frac{\ln 3 - \ln 2}{1 - \ln 2} \lessapprox 1.3214.$$

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2. Preliminaries

The following lemmas and corollary pave the way to the proof of Theorem 1.1.

Lemma 2.1. *For* $1 < p < 2$*, let* h *be the function defined on the interval* $I = [1, +\infty)$ *by*

$$h(x) = \frac{(1-p+2x)x^{1-2/p}}{1+(2-p)x},$$

(i) *If* $p \leq \frac{6}{5}$ *then* $h(x) < 1$ *for all* $x > 1$.

(ii) *If* $p > \frac{6}{5}$ *then there exists* x_0 *in* $(1, +\infty)$ *such that* $h(x) > 1$ *for* $1 < x < x_0$*, and* $h(x) < 1$ *for* $x > x_0$.

Proof. Clearly $h(x) > 0$ for $x \geq 1$, so we will consider $H = \ln(h)$.

$$H(x) = \ln(1-p+2x) + \frac{p-2}{p}\ln x - \ln(1+(2-p)x).$$

Now, doing some algebra, we can reduce the derivative of H to the following form,

$$\begin{aligned} H'(x) &= \frac{2}{1-p+2x} - \frac{2-p}{px} - \frac{2-p}{1+(2-p)x} \\ &= -\frac{2(2-p)^2 Q(x)}{px(1-p+2x)(1+(2-p)x)}, \end{aligned}$$

with Q the second degree polynomial given by

$$Q(X) = X^2 - \frac{(p-1)(4-p)}{(2-p)^2}X - \frac{p-1}{4-2p}.$$

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issn: 1443-5756

The key remark here is that, since the product of the zeros of Q is negative, Q must have two real zeros; one of them (say z_-) is negative, and the other (say z_+) is positive. In order to compare z_+ to 1, we evaluate $Q(1)$ to find that,

$$Q(1) = 1 - \frac{(p-1)(4-p)}{(2-p)^2} - \frac{p-1}{4-2p} = \frac{(6-5p)(3-p)}{2(2-p)^2},$$

so we have two cases to consider:

- If $p \le \frac{6}{5}$, then $Q(1) \ge 0$, so we must have $z_+ \le 1$, and consequently $Q(x) > 0$ for $x > 1$. Hence $H'(x) < 0$ for $x > 1$, and H is decreasing on the interval I, but $H(1) = 0$, so that $H(x) < 0$ for $x > 1$, which is equivalent to (i).
- If $p > \frac{6}{5}$, then $Q(1) < 0$ so we must have $1 < z_+$, and consequently, $Q(x) < 0$ for $1 \le x < z_+$ and $Q(x) > 0$ for $x > z_+$. therefore H has the following table of variations:

x	1		z_+		$+\infty$
$H'(x)$		$+$	0	$-$	
$H(x)$	0	$\nearrow$	$\frown$	$\searrow$	$-\infty$

  Hence, the equation $H(x) = 0$ has a unique solution x_0 which is greater than z_+, and $H(x) > 0$ for $1 < x < x_0$, whereas $H(x) < 0$ for $x > x_0$. This proves (ii).

The proof of Lemma 2.1 is now complete. □

Lemma 2.2. *For $1 < p < 2$, let f_p be the function defined on $\mathbb{R}_+^*$ by*

$$f_p(t) = \frac{t}{\tanh t} - 1 - \frac{1}{p} \ln\left(\frac{2\cosh^p t + 1}{3}\right),$$

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(i) *If* $p \le \frac{6}{5}$ *then* f_p *is increasing on* $\mathbb{R}_+^*$.

(ii) *If* $p > \frac{6}{5}$ *then there exists* t_p *in* $\mathbb{R}_+^*$ *such that* f_p *is decreasing on* $(0, t_p]$, *and increasing on* $[t_p, +\infty)$.

Proof. First we note that

$$f_p'(t) = \frac{1}{\sinh^2 t}\left(\sinh t \cosh t - t - \frac{2\sinh^3 t}{(2+\cosh^{-p} t)\cosh t}\right),$$

so if we define the function g on $\mathbb{R}_+^*$ by

$$g(t) = \sinh t \cosh t - t - \frac{2\sinh^3 t}{(2+\cosh^{-p} t)\cosh t},$$

we find that

$$\begin{aligned} g'(t) &= 2\sinh^2 t - \frac{6\sinh^2 t}{2+\cosh^{-p} t} + \frac{2\sinh^4 t(2+(1-p)\cosh^{-p} t)}{(2+\cosh^{-p} t)^2\cosh^2 t} \\ &= \frac{2\tanh^2 t\ \left((1+(2-p)\cosh^p t)\cosh^2 t - (1-p+2\cosh^p t)\cosh^p t\right)}{(1+2\cosh^p t)^2} \\ &= \frac{2\sinh^2 t\,(1+(2-p)\cosh^p t)}{(1+2\cosh^p t)^2}\left(1 - \frac{(1-p+2\cosh^p t)\cosh^p t}{(1+(2-p)\cosh^p t)\cosh^2 t}\right) \\ &= \frac{2\sinh^2 t\,(1+(2-p)\cosh^p t)}{(1+2\cosh^p t)^2}\,(1 - h(\cosh^p t)) \end{aligned}$$

where h is the function defined in Lemma 2.1. This allows us to conclude, as follows:

- If $p \le \frac{6}{5}$, then using Lemma 2.1, we conclude that $h(\cosh^p t) < 1$ for $t > 0$, so g' is positive on $\mathbb{R}_+^*$. Now, by the fact that $g(0) = 0$ and that g is increasing

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on $\mathbb{R}_+^*$ we conclude that $g(t)$ is positive for $t > 0$, therefore f_p is increasing on $\mathbb{R}_+^*$. This proves (i).

- If $p > \frac{6}{5}$, then using Lemma 2.1, and the fact that $t \mapsto \cosh^p t$ defines an increasing bijection from $\mathbb{R}_+^*$ onto $(1, +\infty)$, we conclude that g has the following table of variations:

t	0		t_0		$+\infty$
$g'(t)$		$-$	0	$+$	
$g(t)$	0	$\searrow$	$\smile$	$\nearrow$	$+\infty$

  with $t_0 = \operatorname{arg\,cosh} \sqrt[p]{x_0}$. Hence, the equation $g(t) = 0$ has a unique positive solution t_p, and $g(t) < 0$ for $0 < t < t_p$, whereas $g(t) > 0$ for $t > t_p$, and (ii) follows.

This achieves the proof of Lemma 2.2. $\square$

Now, using the fact that

$$\lim_{t\to 0} f_p(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} f_p(t) = \ln\left(\frac{2}{e}\sqrt[p]{\frac{3}{2}}\right),$$

the following corollary follows.

Corollary 2.3. *For $1 < p < 2$, let f_p be the function defined in Lemma 2.2.*

(i) If $p \le \frac{6}{5}$, then f_p has the following table of variations:

t	0		$+\infty$
$f_p(t)$	0	$\nearrow$	$\ln\left(\frac{2}{e}\sqrt[p]{\frac{3}{2}}\right)$

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(ii) If $p > \frac{6}{5}$ then f_p has the following table of variations:

t	0				$+\infty$
$f_p(t)$	0	$\searrow$	$\smile$	$\nearrow$	$\ln\left(\frac{2}{e}\sqrt[p]{\frac{3}{2}}\right)$

In particular, for $1 < p < 2$, we have proved the following statements.

$$(\forall\, t > 0,\ f_p(t) > 0) \iff p \leq p_0, \tag{2.1}$$

$$(\forall\, t > 0,\ f_p(t) < 0) \iff \ln\left(\frac{2}{e}\sqrt[p]{\frac{3}{2}}\right) \leq 0 \iff p \geq p_1 \tag{2.2}$$

where p_0 and p_1 are defined in the statement of Theorem 1.1.

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3. Proof of Theorem 1.1

Proof. In what follows, we use the notation of the preceding corollary.

- First, consider some p in $\mathcal{L}$, then for all (x,y) in D we have $K_p(x,y) < I(x,y)$. This implies that

$$\forall t > 0. \quad \ln(K_p(e^t, e^{-t})) < \ln(I(e^t, e^{-t})),$$

  but $I(e^t, e^{-t}) = \exp\left(\frac{t}{\tanh t} - 1\right)$ and $A(e^t, e^{-t}) = \cosh t$, so we have

$$\forall t > 0, \quad \frac{t}{\tanh t} - 1 - \frac{1}{p}\ln\left(\frac{2\cosh^p t + 1}{3}\right) > 0,$$

  Now, if $p > 1$, this proves that $f_p(t) > 0$ for every positive t, so we deduce from (2.1) that $p \leq p_0$. Hence $\mathcal{L} \subset (0, p_0]$.

- Conversely, consider a pair (x,y) from D, and define t as $\ln\left(\frac{\max(x,y)}{\sqrt{xy}}\right)$. Now, using (2.1) we conclude that $f_{p_0}(t) > 0$, and this is equivalent to $K_{p_0}(x,y) < I(x,y)$. Therefore, $p_0 \in \mathcal{L}$ and consequently $(0, p_0] \subset \mathcal{L}$. This achieves the proof of the first equality, that is $\mathcal{L} = (0, p_0]$.

- Second, consider some p in $\mathcal{U}$, then for all (x,y) in D we have $I(x,y) < K_p(x,y)$. This implies that

$$\forall t > 0, \quad \ln(K_p(e^t, e^{-t})) > \ln(I(e^t, e^{-t})),$$

  so we have

$$\forall t > 0, \quad \frac{t}{\tanh t} - 1 - \frac{1}{p}\ln\left(\frac{2\cosh^p t + 1}{3}\right) < 0,$$

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Now, if $p < 2$, this proves that $f_p(t) < 0$ for every positive t, so we deduce from (2.2) that $p \geq p_1$. Hence $\mathcal{U} \subset [p_1, \infty)$.

- Conversely, consider a pair (x, y) from D, and as before define $t = \ln\left(\frac{\max(x,y)}{\sqrt{xy}}\right)$. Now, using (2.2) we obtain $f_{p_1}(t) < 0$, and this is equivalent to $I(x, y) < K_{p_1}(x, y)$. Therefore, $p_1 \in \mathcal{U}$ and consequently $[p_1, \infty) \subset \mathcal{U}$. This achieves the proof of the second equality, that is $\mathcal{U} = [p_1, \infty)$.

This concludes the proof of the main Theorem 1.1. □

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4. Remarks

Remark 1. The same approach, as in the proof of Theorem 1.1 can be used to prove that for $\lambda \le 2/3$ and $p \le \frac{3-\lambda-\sqrt{(1-\lambda)(3\lambda+1)}}{(1-\lambda)^2+1}$ we have

$$\sqrt[p]{\lambda A^p(x,y) + (1-\lambda)G^p(x,y)} < I(x,y)$$

for all positive real numbers $x \ne y$. Similarly, we can also prove that for $\lambda \ge 2/3$ and $p \ge \frac{\ln \lambda}{\ln 2 - 1}$ we have

$$I(x,y) < \sqrt[p]{\lambda A^p(x,y) + (1-\lambda)G^p(x,y)}.$$

for all positive real numbers $x \ne y$. We leave the details to the interested reader.

Remark 2. The inequality $I(x,y) < \sqrt{\frac{2}{3}A^2(x,y) + \frac{1}{3}G^2(x,y)}$ was proved in [3] using power series. Another proof can be found in [4] using the Gauss quadrature formula. It can also be seen as a consequence of our main theorem. Here, we will show that this inequality can be proved elementarily as a consequence of Jensen's inequality.

Let us recall that $\ln(I(x,y))$ can be expressed as follows

$$\ln(I(x,y)) = \int_0^1 \ln(tx + (1-t)y)\,dt = \int_0^1 \ln((1-t)x + ty)\,dt.$$

Therefore,

$$2\ln(I(x,y)) = \int_0^1 \ln\big((tx + (1-t)y)((1-t)x + ty)\big)\,dt,$$

but

$$(tx + (1-t)y)((1-t)x + ty) = (1-(2t-1)^2)A^2(x,y) + (2t-1)^2 G^2(x,y),$$

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so that, by $u \leftarrow 2t - 1$, we obtain,

$$2\ln(I(x,y)) = \frac{1}{2}\int_{-1}^{1} \ln((1-u^2)A^2(x,y) + u^2G^2(x,y))\,du$$
$$= \int_{0}^{1} \ln((1-u^2)A^2(x,y) + u^2G^2(x,y))\,du.$$

Hence,

$$I^2(x,y) = \exp\left(\int_0^1 \ln((1-u^2)A^2(x,y) + u^2G^2(x,y))\,du\right)$$

Now, the function $t \mapsto e^t$ is strictly convex, and the integrand is a continuous non-constant function when $x \neq y$, so using Jensen's inequality we obtain

$$I^2(x,y) < \int_0^1 \exp\big(\ln((1-u^2)A^2(x,y)+u^2G^2(x,y))\big)\,du = \frac{2}{3}A^2(x,y)+\frac{1}{3}G^2(x,y).$$

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issn: 1443-5756

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issn: 1443-5756