

REVERSES OF THE CONTINUOUS TRIANGLE INEQUALITY FOR BOCHNER INTEGRAL OF VECTOR-VALUED FUNCTIONS IN HILBERT SPACES

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Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

Some reverses of the continuous triangle inequality for Bochner integral of vector-valued functions in Hilbert spaces are given. Applications for complex-valued functions are provided as well.

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Contents

1	Introduction	3
2	Reverses for a Unit Vector	5
3	Reverses for Orthonormal Families of Vectors	12
4	Applications for Complex-Valued Functions	18
	References	



Reverses of the Continuous Triangle Inequality for Bochner Integral of Vector-Valued Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 2 of 23

1. Introduction

Let $f : [a, b] \rightarrow \mathbb{K}$, $\mathbb{K} = \mathbb{C}$ or \mathbb{R} be a Lebesgue integrable function. The following inequality, which is the continuous version of the *triangle inequality*,

$$(1.1) \quad \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx,$$

plays a fundamental role in Mathematical Analysis and its applications.

It appears, see [4, p. 492], that the first reverse inequality for (1.1) was obtained by J. Karamata in his book from 1949, [2]. It can be stated as

$$(1.2) \quad \cos \theta \int_a^b |f(x)| dx \leq \left| \int_a^b f(x) dx \right|,$$

provided

$$-\theta \leq \arg f(x) \leq \theta, \quad x \in [a, b]$$

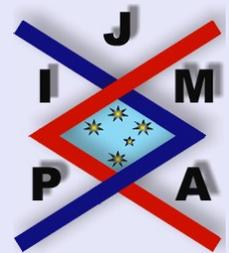
for given $\theta \in (0, \frac{\pi}{2})$.

This integral inequality is the continuous version of a reverse inequality for the generalised triangle inequality

$$(1.3) \quad \cos \theta \sum_{i=1}^n |z_i| \leq \left| \sum_{i=1}^n z_i \right|,$$

provided

$$a - \theta \leq \arg(z_i) \leq a + \theta, \quad \text{for } i \in \{1, \dots, n\},$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 3 of 23

where $a \in \mathbb{R}$ and $\theta \in (0, \frac{\pi}{2})$, which, as pointed out in [4, p. 492], was first discovered by M. Petrovich in 1917, [5], and, subsequently rediscovered by other authors, including J. Karamata [2, p. 300 – 301], H.S. Wilf [6], and in an equivalent form, by M. Marden [3].

The first to consider the problem in the more general case of Hilbert and Banach spaces, were J.B. Diaz and F.T. Metcalf [1] who showed that, in an inner product space H over the real or complex number field, the following reverse of the triangle inequality holds

$$(1.4) \quad r \sum_{i=1}^n \|x_i\| \leq \left\| \sum_{i=1}^n x_i \right\|,$$

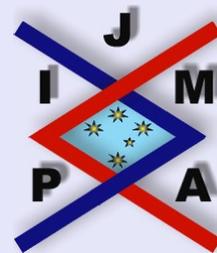
provided

$$0 \leq r \leq \frac{\operatorname{Re} \langle x_i, a \rangle}{\|x_i\|}, \quad i \in \{1, \dots, n\},$$

and $a \in H$ is a unit vector, i.e., $\|a\| = 1$. The case of equality holds in (1.4) if and only if

$$(1.5) \quad \sum_{i=1}^n x_i = r \left(\sum_{i=1}^n \|x_i\| \right) a.$$

The main aim of this paper is to point out some reverses of the triangle inequality for Bochner integrable functions f with values in Hilbert spaces and defined on a compact interval $[a, b] \subset \mathbb{R}$. Applications for Lebesgue integrable complex-valued functions are provided as well.



**Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 4 of 23

2. Reverses for a Unit Vector

We recall that $f \in L([a, b]; H)$, the space of Bochner integrable functions with values in a Hilbert space H , if and only if $f : [a, b] \rightarrow H$ is Bochner measurable on $[a, b]$ and the Lebesgue integral $\int_a^b \|f(t)\| dt$ is finite.

The following result holds:

Theorem 2.1. *If $f \in L([a, b]; H)$ is such that there exists a constant $K \geq 1$ and a vector $e \in H$, $\|e\| = 1$ with*

$$(2.1) \quad \|f(t)\| \leq K \operatorname{Re} \langle f(t), e \rangle \quad \text{for a.e. } t \in [a, b],$$

then we have the inequality:

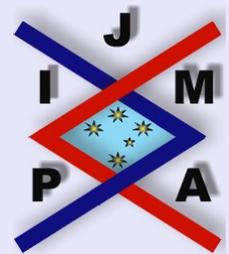
$$(2.2) \quad \int_a^b \|f(t)\| dt \leq K \left\| \int_a^b f(t) dt \right\|.$$

The case of equality holds in (2.2) if and only if

$$(2.3) \quad \int_a^b f(t) dt = \frac{1}{K} \left(\int_a^b \|f(t)\| dt \right) e.$$

Proof. By the Schwarz inequality in inner product spaces, we have

$$(2.4) \quad \begin{aligned} \left\| \int_a^b f(t) dt \right\| &= \left\| \int_a^b f(t) dt \right\| \|e\| \\ &\geq \left| \left\langle \int_a^b f(t) dt, e \right\rangle \right| \end{aligned}$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 5 of 23

$$\begin{aligned} &\geq \left| \operatorname{Re} \left\langle \int_a^b f(t) dt, e \right\rangle \right| \\ &\geq \operatorname{Re} \left\langle \int_a^b f(t) dt, e \right\rangle = \int_a^b \operatorname{Re} \langle f(t), e \rangle dt. \end{aligned}$$

From the condition (2.1), on integrating over $[a, b]$, we deduce

$$(2.5) \quad \int_a^b \operatorname{Re} \langle f(t), e \rangle dt \geq \frac{1}{K} \int_a^b \|f(t)\| dt,$$

and thus, on making use of (2.4) and (2.5), we obtain the desired inequality (2.2).

If (2.3) holds true, then, obviously

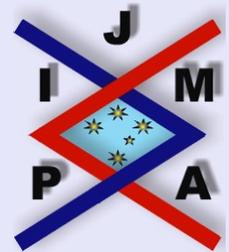
$$K \left\| \int_a^b f(t) dt \right\| = \|e\| \int_a^b \|f(t)\| dt = \int_a^b \|f(t)\| dt,$$

showing that (2.2) holds with equality.

If we assume that the equality holds in (2.2), then by the argument provided at the beginning of our proof, we must have equality in each of the inequalities from (2.4) and (2.5).

Observe that in Schwarz's inequality $\|x\| \|y\| \geq \operatorname{Re} \langle x, y \rangle$, $x, y \in H$, the case of equality holds if and only if there exists a positive scalar μ such that $x = \mu e$. Therefore, equality holds in the first inequality in (2.4) iff $\int_a^b f(t) dt = \lambda e$, with $\lambda \geq 0$.

If we assume that a strict inequality holds in (2.1) on a subset of nonzero Lebesgue measures, then $\int_a^b \|f(t)\| dt < K \int_a^b \operatorname{Re} \langle f(t), e \rangle dt$, and by (2.4) we



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 6 of 23

deduce a strict inequality in (2.2), which contradicts the assumption. Thus, we must have $\|f(t)\| = K \operatorname{Re} \langle f(t), e \rangle$ for a.e. $t \in [a, b]$.

If we integrate this equality, we deduce

$$\begin{aligned} \int_a^b \|f(t)\| dt &= K \int_a^b \operatorname{Re} \langle f(t), e \rangle dt \\ &= K \operatorname{Re} \left\langle \int_a^b f(t) dt, e \right\rangle \\ &= K \operatorname{Re} \langle \lambda e, e \rangle = \lambda K, \end{aligned}$$

giving

$$\lambda = \frac{1}{K} \int_a^b \|f(t)\| dt,$$

and thus the equality (2.3) is necessary.

This completes the proof. \square

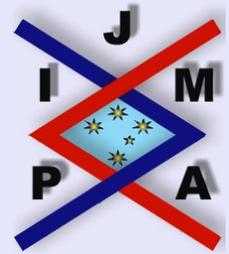
A more appropriate result from an applications point of view is perhaps the following result.

Corollary 2.2. *Let e be a unit vector in the Hilbert space $(H; \langle \cdot, \cdot \rangle)$, $\rho \in (0, 1)$ and $f \in L([a, b]; H)$ so that*

$$(2.6) \quad \|f(t) - e\| \leq \rho \text{ for a.e. } t \in [a, b].$$

Then we have the inequality

$$(2.7) \quad \sqrt{1 - \rho^2} \int_a^b \|f(t)\| dt \leq \left\| \int_a^b f(t) dt \right\|,$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 7 of 23

with equality if and only if

$$(2.8) \quad \int_a^b f(t) dt = \sqrt{1 - \rho^2} \left(\int_a^b \|f(t)\| dt \right) \cdot e.$$

Proof. From (2.6), we have

$$\|f(t)\|^2 - 2 \operatorname{Re} \langle f(t), e \rangle + 1 \leq \rho^2,$$

giving

$$\|f(t)\|^2 + 1 - \rho^2 \leq 2 \operatorname{Re} \langle f(t), e \rangle$$

for a.e. $t \in [a, b]$.

Dividing by $\sqrt{1 - \rho^2} > 0$, we deduce

$$(2.9) \quad \frac{\|f(t)\|^2}{\sqrt{1 - \rho^2}} + \sqrt{1 - \rho^2} \leq \frac{2 \operatorname{Re} \langle f(t), e \rangle}{\sqrt{1 - \rho^2}}$$

for a.e. $t \in [a, b]$.

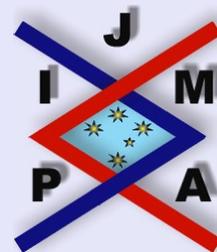
On the other hand, by the elementary inequality

$$\frac{p}{\alpha} + q\alpha \geq 2\sqrt{pq}, \quad p, q \geq 0, \alpha > 0$$

we have

$$(2.10) \quad 2\|f(t)\| \leq \frac{\|f(t)\|^2}{\sqrt{1 - \rho^2}} + \sqrt{1 - \rho^2}$$

for each $t \in [a, b]$.



**Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 8 of 23

Making use of (2.9) and (2.10), we deduce

$$\|f(t)\| \leq \frac{1}{\sqrt{1-\rho^2}} \operatorname{Re} \langle f(t), e \rangle$$

for a.e. $t \in [a, b]$.

Applying Theorem 2.1 for $K = \frac{1}{\sqrt{1-\rho^2}}$, we deduce the desired inequality (2.7). \square

In the same spirit, we also have the following corollary.

Corollary 2.3. *Let e be a unit vector in H and $M \geq m > 0$. If $f \in L([a, b]; H)$ is such that*

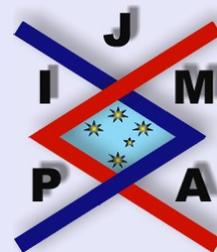
$$(2.11) \quad \operatorname{Re} \langle Me - f(t), f(t) - me \rangle \geq 0 \quad \text{for a.e. } t \in [a, b],$$

or, equivalently,

$$(2.12) \quad \left\| f(t) - \frac{M+m}{2} e \right\| \leq \frac{1}{2} (M-m) \quad \text{for a.e. } t \in [a, b],$$

then we have the inequality

$$(2.13) \quad \frac{2\sqrt{mM}}{M+m} \int_a^b \|f(t)\| dt \leq \left\| \int_a^b f(t) dt \right\|,$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 9 of 23

or, equivalently,

$$(2.14) \quad \begin{aligned} (0 \leq) \int_a^b \|f(t)\| dt - \left\| \int_a^b f(t) dt \right\| \\ \leq \frac{(\sqrt{M} - \sqrt{m})^2}{M + m} \left\| \int_a^b f(t) dt \right\|. \end{aligned}$$

The equality holds in (2.13) (or in the second part of (2.14)) if and only if

$$(2.15) \quad \int_a^b f(t) dt = \frac{2\sqrt{mM}}{M + m} \left(\int_a^b \|f(t)\| dt \right) e.$$

Proof. Firstly, we remark that if $x, z, Z \in H$, then the following statements are equivalent

$$(i) \operatorname{Re} \langle Z - x, x - z \rangle \geq 0$$

and

$$(ii) \left\| x - \frac{Z+z}{2} \right\| \leq \frac{1}{2} \|Z - z\|.$$

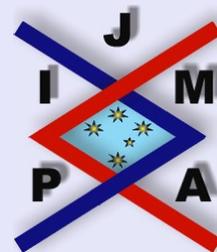
Using this fact, we may simply realise that (2.9) and (2.10) are equivalent.

Now, from (2.9), we obtain

$$\|f(t)\|^2 + mM \leq (M + m) \operatorname{Re} \langle f(t), e \rangle$$

for a.e. $t \in [a, b]$. Dividing this inequality with $\sqrt{mM} > 0$, we deduce the following inequality that will be used in the sequel

$$(2.16) \quad \frac{\|f(t)\|^2}{\sqrt{mM}} + \sqrt{mM} \leq \frac{M + m}{\sqrt{mM}} \operatorname{Re} \langle f(t), e \rangle$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 10 of 23

for a.e. $t \in [a, b]$.

On the other hand

$$(2.17) \quad 2 \|f(t)\| \leq \frac{\|f(t)\|^2}{\sqrt{mM}} + \sqrt{mM},$$

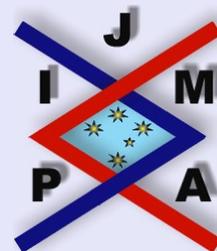
for any $t \in [a, b]$.

Utilising (2.16) and (2.17), we may conclude with the following inequality

$$\|f(t)\| \leq \frac{M+m}{2\sqrt{mM}} \operatorname{Re} \langle f(t), e \rangle,$$

for a.e. $t \in [a, b]$.

Applying Theorem 2.1 for the constant $K := \frac{m+M}{2\sqrt{mM}} \geq 1$, we deduce the desired result. \square



**Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 11 of 23

3. Reverses for Orthonormal Families of Vectors

The following result for orthonormal vectors in H holds.

Theorem 3.1. Let $\{e_1, \dots, e_n\}$ be a family of orthonormal vectors in H , $k_i \geq 0$, $i \in \{1, \dots, n\}$ and $f \in L([a, b]; H)$ such that

$$(3.1) \quad k_i \|f(t)\| \leq \operatorname{Re} \langle f(t), e_i \rangle$$

for each $i \in \{1, \dots, n\}$ and for a.e. $t \in [a, b]$.

Then

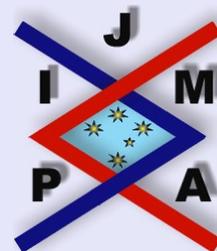
$$(3.2) \quad \left(\sum_{i=1}^n k_i^2 \right)^{\frac{1}{2}} \int_a^b \|f(t)\| dt \leq \left\| \int_a^b f(t) dt \right\|,$$

where equality holds if and only if

$$(3.3) \quad \int_a^b f(t) dt = \left(\int_a^b \|f(t)\| dt \right) \sum_{i=1}^n k_i e_i.$$

Proof. By Bessel's inequality applied for $\int_a^b f(t) dt$ and the orthonormal vectors $\{e_1, \dots, e_n\}$, we have

$$(3.4) \quad \begin{aligned} \left\| \int_a^b f(t) dt \right\|^2 &\geq \sum_{i=1}^n \left| \left\langle \int_a^b f(t) dt, e_i \right\rangle \right|^2 \\ &\geq \sum_{i=1}^n \left[\operatorname{Re} \left\langle \int_a^b f(t) dt, e_i \right\rangle \right]^2 \end{aligned}$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 12 of 23

$$= \sum_{i=1}^n \left[\int_a^b \operatorname{Re} \langle f(t), e_i \rangle dt \right]^2.$$

Integrating (3.1), we get for each $i \in \{1, \dots, n\}$

$$0 \leq k_i \int_a^b \|f(t)\| dt \leq \int_a^b \operatorname{Re} \langle f(t), e_i \rangle dt,$$

implying

$$(3.5) \quad \sum_{i=1}^n \left[\int_a^b \operatorname{Re} \langle f(t), e_i \rangle dt \right]^2 \geq \sum_{i=1}^n k_i^2 \left(\int_a^b \|f(t)\| dt \right)^2.$$

On making use of (3.4) and (3.5), we deduce

$$\left\| \int_a^b f(t) dt \right\|^2 \geq \sum_{i=1}^n k_i^2 \left(\int_a^b \|f(t)\| dt \right)^2,$$

which is clearly equivalent to (3.2).

If (3.3) holds true, then

$$\begin{aligned} \left\| \int_a^b f(t) dt \right\| &= \left(\int_a^b \|f(t)\| dt \right) \left\| \sum_{i=1}^n k_i e_i \right\| \\ &= \left(\int_a^b \|f(t)\| dt \right) \left[\sum_{i=1}^n k_i^2 \|e_i\|^2 \right]^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^n k_i^2 \right)^{\frac{1}{2}} \int_a^b \|f(t)\| dt, \end{aligned}$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 13 of 23

showing that (3.2) holds with equality.

Now, suppose that there is an $i_0 \in \{1, \dots, n\}$ for which

$$k_{i_0} \|f(t)\| < \operatorname{Re} \langle f(t), e_{i_0} \rangle$$

on a subset of nonzero Lebesgue measures enclosed in $[a, b]$. Then obviously

$$k_{i_0} \int_a^b \|f(t)\| dt < \int_a^b \operatorname{Re} \langle f(t), e_{i_0} \rangle dt,$$

and using the argument given above, we deduce

$$\left(\sum_{i=1}^n k_i^2 \right)^{\frac{1}{2}} \int_a^b \|f(t)\| dt < \left\| \int_a^b f(t) dt \right\|.$$

Therefore, if the equality holds in (3.2), we must have

$$(3.6) \quad k_i \|f(t)\| = \operatorname{Re} \langle f(t), e_i \rangle$$

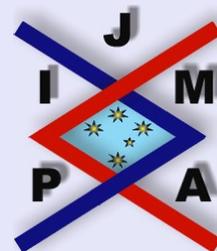
for each $i \in \{1, \dots, n\}$ and a.e. $t \in [a, b]$.

Also, if the equality holds in (3.2), then we must have equality in all inequalities (3.4), this means that

$$(3.7) \quad \int_a^b f(t) dt = \sum_{i=1}^n \left\langle \int_a^b f(t) dt, e_i \right\rangle e_i$$

and

$$(3.8) \quad \operatorname{Im} \left\langle \int_a^b f(t) dt, e_i \right\rangle = 0 \text{ for each } i \in \{1, \dots, n\}.$$



**Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 14 of 23

Using (3.6) and (3.8) in (3.7), we deduce

$$\begin{aligned}
 \int_a^b f(t) dt &= \sum_{i=1}^n \operatorname{Re} \left\langle \int_a^b f(t) dt, e_i \right\rangle e_i \\
 &= \sum_{i=1}^n \int_a^b \operatorname{Re} \langle f(t), e_i \rangle e_i dt \\
 &= \sum_{i=1}^n \left(\int_a^b \|f(t)\| dt \right) k_i e_i \\
 &= \int_a^b \|f(t)\| dt \sum_{i=1}^n k_i e_i,
 \end{aligned}$$

and the condition (3.3) is necessary.

This completes the proof. \square

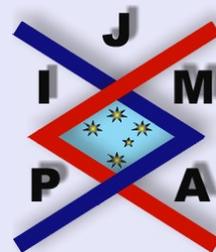
The following two corollaries are of interest.

Corollary 3.2. *Let $\{e_1, \dots, e_n\}$ be a family of orthonormal vectors in H , $\rho_i \in (0, 1)$, $i \in \{1, \dots, n\}$ and $f \in L([a, b]; H)$ such that:*

$$(3.9) \quad \|f(t) - e_i\| \leq \rho_i \text{ for } i \in \{1, \dots, n\} \text{ and a.e. } t \in [a, b].$$

Then we have the inequality

$$\left(n - \sum_{i=1}^n \rho_i^2 \right)^{\frac{1}{2}} \int_a^b \|f(t)\| dt \leq \left\| \int_a^b f(t) dt \right\|,$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 15 of 23

with equality if and only if

$$\int_a^b f(t) dt = \int_a^b \|f(t)\| dt \left(\sum_{i=1}^n (1 - \rho_i^2)^{\frac{1}{2}} e_i \right).$$

Proof. From the proof of Theorem 2.1, we know that (3.3) implies the inequality

$$\sqrt{1 - \rho_i^2} \|f(t)\| \leq \operatorname{Re} \langle f(t), e_i \rangle, \quad i \in \{1, \dots, n\}, \quad \text{for a.e. } t \in [a, b].$$

Now, applying Theorem 3.1 for $k_i := \sqrt{1 - \rho_i^2}$, $i \in \{1, \dots, n\}$, we deduce the desired result. \square

Corollary 3.3. Let $\{e_1, \dots, e_n\}$ be a family of orthonormal vectors in H , $M_i \geq m_i > 0$, $i \in \{1, \dots, n\}$ and $f \in L([a, b]; H)$ such that

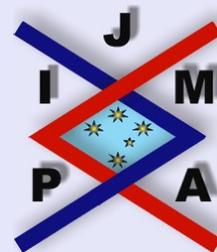
$$(3.10) \quad \operatorname{Re} \langle M_i e_i - f(t), f(t) - m_i e_i \rangle \geq 0$$

or, equivalently,

$$\left\| f(t) - \frac{M_i + m_i}{2} e_i \right\| \leq \frac{1}{2} (M_i - m_i)$$

for $i \in \{1, \dots, n\}$ and a.e. $t \in [a, b]$. Then we have the reverse of the generalised triangle inequality

$$\left[\sum_{i=1}^n \frac{4m_i M_i}{(m_i + M_i)^2} \right]^{\frac{1}{2}} \int_a^b \|f(t)\| dt \leq \left\| \int_a^b f(t) dt \right\|,$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 16 of 23

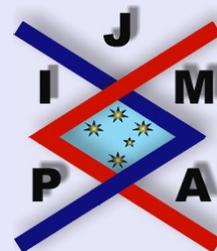
with equality if and only if

$$\int_a^b f(t) dt = \int_a^b \|f(t)\| dt \left(\sum_{i=1}^n \frac{2\sqrt{m_i M_i}}{m_i + M_i} e_i \right).$$

Proof. From the proof of Corollary 2.3, we know (3.10) implies that

$$\frac{2\sqrt{m_i M_i}}{m_i + M_i} \|f(t)\| \leq \operatorname{Re} \langle f(t), e_i \rangle, \quad i \in \{1, \dots, n\} \quad \text{and a.e. } t \in [a, b].$$

Now, applying Theorem 3.1 for $k_i := \frac{2\sqrt{m_i M_i}}{m_i + M_i}$, $i \in \{1, \dots, n\}$, we deduce the desired result. \square



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 17 of 23

4. Applications for Complex-Valued Functions

Let $e = \alpha + i\beta$ ($\alpha, \beta \in \mathbb{R}$) be a complex number with the property that $|e| = 1$, i.e., $\alpha^2 + \beta^2 = 1$.

The following proposition holds.

Proposition 4.1. *If $f : [a, b] \rightarrow \mathbb{C}$ is a Lebesgue integrable function with the property that there exists a constant $K \geq 1$ such that*

$$(4.1) \quad |f(t)| \leq K [\alpha \operatorname{Re} f(t) + \beta \operatorname{Im} f(t)]$$

for a.e. $t \in [a, b]$, where $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$ are given, then we have the following reverse of the continuous triangle inequality:

$$(4.2) \quad \int_a^b |f(t)| dt \leq K \left| \int_a^b f(t) dt \right|.$$

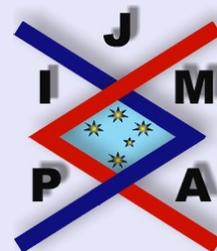
The case of equality holds in (2.2) if and only if

$$\int_a^b f(t) dt = \frac{1}{K} (\alpha + i\beta) \int_a^b |f(t)| dt.$$

The proof is obvious by Theorem 2.1, and we omit the details.

Remark 1. *If in the above Proposition 4.1 we choose $\alpha = 1, \beta = 0$, then the condition (4.1) for $\operatorname{Re} f(t) > 0$ is equivalent to*

$$[\operatorname{Re} f(t)]^2 + [\operatorname{Im} f(t)]^2 \leq K^2 [\operatorname{Re} f(t)]^2$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 18 of 23

or with the inequality:

$$\frac{|\operatorname{Im} f(t)|}{\operatorname{Re} f(t)} \leq \sqrt{K^2 - 1}.$$

Now, if we assume that

$$(4.3) \quad |\arg f(t)| \leq \theta, \quad \theta \in \left(0, \frac{\pi}{2}\right),$$

then, for $\operatorname{Re} f(t) > 0$,

$$|\tan [\arg f(t)]| = \frac{|\operatorname{Im} f(t)|}{\operatorname{Re} f(t)} \leq \tan \theta,$$

and if we choose $K = \frac{1}{\cos \theta} > 1$, then

$$\sqrt{K^2 - 1} = \tan \theta,$$

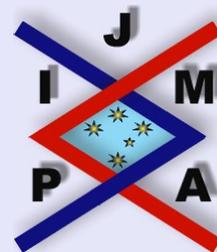
and by Proposition 4.1, we deduce

$$(4.4) \quad \cos \theta \int_a^b |f(t)| dt \leq \left| \int_a^b f(t) dt \right|,$$

which is exactly the Karamata inequality (1.2) from the Introduction.

Obviously, the result from Proposition 4.1 is more comprehensive since for other values of $(\alpha, \beta) \in \mathbb{R}^2$ with $\alpha^2 + \beta^2 = 1$ we can get different sufficient conditions for the function f such that the inequality (4.2) holds true.

A different sufficient condition in terms of complex disks is incorporated in the following proposition.



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 19 of 23

Proposition 4.2. Let $e = \alpha + i\beta$ with $\alpha^2 + \beta^2 = 1$, $r \in (0, 1)$ and $f : [a, b] \rightarrow \mathbb{C}$ be a Lebesgue integrable function such that

$$(4.5) \quad f(t) \in \bar{D}(e, r) := \{z \in \mathbb{C} \mid |z - e| \leq r\} \quad \text{for a.e. } t \in [a, b].$$

Then we have the inequality

$$(4.6) \quad \sqrt{1 - r^2} \int_a^b |f(t)| dt \leq \left| \int_a^b f(t) dt \right|.$$

The case of equality holds in (4.6) if and only if

$$\int_a^b f(t) dt = \sqrt{1 - r^2} (\alpha + i\beta) \int_a^b |f(t)| dt.$$

The proof follows by Corollary 2.2 and we omit the details.

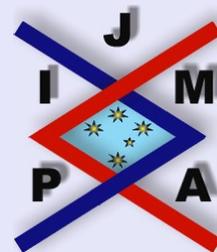
Finally, we may state the following proposition as well.

Proposition 4.3. Let $e = \alpha + i\beta$ with $\alpha^2 + \beta^2 = 1$ and $M \geq m > 0$. If $f : [a, b] \rightarrow \mathbb{C}$ is such that

$$(4.7) \quad \operatorname{Re} \left[(Me - f(t)) \left(\overline{f(t)} - m\bar{e} \right) \right] \geq 0 \quad \text{for a.e. } t \in [a, b],$$

or, equivalently,

$$(4.8) \quad \left| f(t) - \frac{M+m}{2} e \right| \leq \frac{1}{2} (M - m) \quad \text{for a.e. } t \in [a, b],$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 20 of 23

then we have the inequality

$$(4.9) \quad \frac{2\sqrt{mM}}{M+m} \int_a^b |f(t)| dt \leq \left| \int_a^b f(t) dt \right|,$$

or, equivalently,

$$(4.10) \quad (0 \leq) \int_a^b |f(t)| dt - \left| \int_a^b f(t) dt \right| \leq \frac{(\sqrt{M} - \sqrt{m})^2}{M+m} \left| \int_a^b f(t) dt \right|.$$

The equality holds in (4.9) (or in the second part of (4.10)) if and only if

$$\int_a^b f(t) dt = \frac{2\sqrt{mM}}{M+m} (\alpha + i\beta) \int_a^b |f(t)| dt.$$

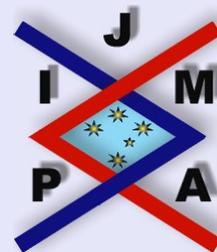
The proof follows by Corollary 2.3 and we omit the details.

Remark 2. Since

$$\begin{aligned} Me - f(t) &= M\alpha - \operatorname{Re} f(t) + i[M\beta - \operatorname{Im} f(t)], \\ \overline{f(t)} - m\bar{e} &= \operatorname{Re} f(t) - m\alpha - i[\operatorname{Im} f(t) - m\beta] \end{aligned}$$

hence

$$(4.11) \quad \begin{aligned} \operatorname{Re} \left[(Me - f(t)) \left(\overline{f(t)} - m\bar{e} \right) \right] \\ = [M\alpha - \operatorname{Re} f(t)] [\operatorname{Re} f(t) - m\alpha] \\ + [M\beta - \operatorname{Im} f(t)] [\operatorname{Im} f(t) - m\beta]. \end{aligned}$$



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 21 of 23

It is obvious that, if

$$(4.12) \quad m\alpha \leq \operatorname{Re} f(t) \leq M\alpha \quad \text{for a.e. } t \in [a, b],$$

and

$$(4.13) \quad m\beta \leq \operatorname{Im} f(t) \leq M\beta \quad \text{for a.e. } t \in [a, b],$$

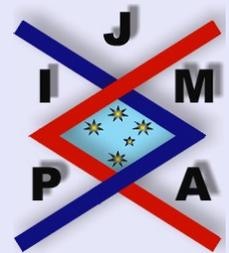
then, by (4.11),

$$\operatorname{Re} \left[(Me - f(t)) \left(\overline{f(t)} - m\bar{e} \right) \right] \geq 0 \quad \text{for a.e. } t \in [a, b],$$

and then either (4.9) or (4.12) hold true.

We observe that the conditions (4.12) and (4.13) are very easy to verify in practice and may be useful in various applications where reverses of the continuous triangle inequality are required.

Remark 3. Similar results may be stated for functions $f : [a, b] \rightarrow \mathbb{R}^n$ or $f : [a, b] \rightarrow H$, with H particular instances of Hilbert spaces of significance in applications, but we leave them to the interested reader.



Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

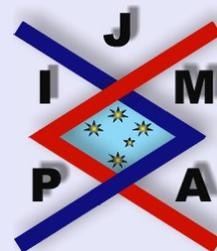
Close

Quit

Page 22 of 23

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Reverses of the Continuous
Triangle Inequality for Bochner
Integral of Vector-Valued
Functions in Hilbert Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 23 of 23