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ON SOME FENG QI TYPE q-INTEGRAL INEQUALITIES

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ABSTRACT. In this paper, we provide some Feng Qi type q-Integral Inequalities, by using analytic and elementary methods in Quantum Calculus.

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1. INTRODUCTION

In [9], F. Qi studied an interesting integral inequality and proved the following result:

Theorem 1.1. *For a positive integer n and an n^{th} order continuous derivative function f on an interval $[a,b]$ such that $f^{(i)}(a) \geq 0$, $0 \leq i \leq n-1$ and $f^{(n)}(a) \geq n!$, we have*

$$\int_a^b [f(t)]^{n+2} dt \geq \left[\int_a^b f(t)dt\right]^{n+1}. \tag{1.1}$$

Then, he proposed the following open problem:

Under what condition is the inequality (1.1) still true if n is replaced by any positive real number p?

In view of the interest in this type of inequality, much attention has been paid to the problem and many authors have extended the inequality to more general cases (see [1, 8]). In this paper, we shall discuss a q-analogue of the Feng Qi problem and we will generalize the inequalities given in [1], [7] and [8].

This paper is organized as follows: In Section 2, we present definitions and facts from q-calculus necessary for understanding this paper. In Section 3, we discuss some generalizations of the so-called Feng Qi inequality.

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2. NOTATIONS AND PRELIMINARIES

Throughout this paper, we will fix $q \in (0,1)$. For the convenience of the reader, we provide a summary of the mathematical notations and definitions used in this paper (see [3] and [5]). We write for $a \in \mathbb{C}$,

$$[a]_q = \frac{1-q^a}{1-q}, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ n = 1,2,\dots,\infty,$$

$$[0]_q! = 1, \qquad [n]_q! = [1]_q[2]_q ... [n]_q, \quad n = 1,2,\dots$$

and

$$(x-a)_q^n = \begin{cases} 1 & \text{if } n = 0 \\ (x-a)(x-qa)\cdots(x-q^{n-1}a) & \text{if } n \neq 0 \end{cases} \qquad x \in \mathbb{C},\ n \in \mathbb{N}.$$

The q-derivative $D_q f$ of a function f is given by

$$(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad \text{if } x \neq 0, \tag{2.1}$$

$(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

The q-Jackson integral from 0 to a is defined by (see [4])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{2.2}$$

provided the sum converges absolutely.

The q-Jackson integral in a generic interval $[a,b]$ is given by (see [4])

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.3}$$

We recall that for any function f, we have (see [5])

$$D_q\left(\int_a^x f(t)d_qt\right) = f(x). \tag{2.4}$$

Finally, for $b > 0$ and $a = bq^n$, n a positive integer, we write

$$[a,b]_q = \{bq^k :\ 0 \leq k \leq n\} \quad \text{and} \quad (a,b]_q = [q^{-1}a, b]_q.$$

3. q-INTEGRAL INEQUALITIES OF FENG QI TYPE

Let us begin with the following useful result:

Lemma 3.1. *Let $p \geq 1$ be a real number and g be a nonnegative and monotone function on $[a,b]_q$. Then*

$$pg^{p-1}(qx)D_qg(x) \leq D_q[g(x)]^p \leq pg^{p-1}(x)D_qg(x), \quad x \in (a,b]_q.$$

Proof. We have

$$D_q[g^p](x) = \frac{g^p(x)-g^p(qx)}{(1-q)x} = \frac{1}{(1-q)x}p\int_{g(qx)}^{g(x)} t^{p-1}dt. \tag{3.1}$$

Since g is a nonnegative and monotone function, we have

$$g^{p-1}(qx)\left[g(x)-g(qx)\right] \leq \int_{g(qx)}^{g(x)} t^{p-1}dt \leq g^{p-1}(x)\left[g(x)-g(qx)\right].$$

Therefore, according to the relation (3.1), we obtain

$$pg^{p-1}(qx)D_qg(x) \le D_q[g^p](x) \le pg^{p-1}(x)D_qg(x).$$

□

Proposition 3.2. *Let f be a function defined on $[a,b]_q$ satisfying*

$$f(a) \ge 0 \quad \textit{and} \quad D_qf(x) \ge (t-2)(x-a)^{t-3} \quad \textit{for} \quad x \in (a,b]_q \quad \textit{and} \quad t \ge 3.$$

Then

$$\int_a^b [f(x)]^t d_qx \ge \left(\int_a^b f(qx)d_qx\right)^{t-1}.$$

Proof. Put $g(x) = \int_a^x f(qu)d_qu$ and

$$F(x) = \int_a^x [f(u)]^t d_qu - \left(\int_a^x f(qu)d_qu\right)^{t-1}.$$

We have

$$D_qF(x) = f^t(x) - D_q[g^{t-1}](x).$$

Since f and g increase on $[a,b]_q$, we obtain from Lemma 3.1,

$$\begin{aligned} D_qF(x) &\ge f^t(x) - (t-1)g^{t-2}(x)f(qx) \\ &\ge f^t(x) - (t-1)g^{t-2}(x)f(x) = f(x)h(x), \end{aligned}$$

where $h(x) = f^{t-1}(x) - (t-1)g^{t-2}(x)$.

On the other hand, we have

$$D_qh(x) = D_q[f^{t-1}](x) - (t-1)D_q[g^{t-2}](x).$$

By using Lemma 3.1, we obtain

$$D_qh(x) \ge (t-1)f^{t-2}(qx)D_qf(x) - (t-1)(t-2)g^{t-3}(x)D_qg(x) \tag{3.2}$$

$$\ge (t-1)f(qx)\left[f^{t-3}(qx)D_qf(x) - (t-2)g^{t-3}(x)\right]. \tag{3.3}$$

Since the function f increases, we have

$$\int_a^x f(qu)d_qu \le f(qx)(x-a).$$

Then, from the conditions of the proposition and inequalities (3.2) and (3.3), we get

$$D_qh(x) \ge (t-1)f^{t-2}(qx)\left[D_qf(x) - (t-2)(x-a)^{t-3}\right] \ge 0,$$

and from the fact $h(a) = f^{t-1}(a) \ge 0$, we get $h(x) \ge 0, \quad x \in [a,b]_q$.

From $F(a) = 0$ and $D_qF(x) = f(x)h(x) \ge 0$, it follows that $F(x) \ge 0$ for all $x \in [a,b]_q$, in particular

$$F(b) = \int_a^b [f(u)]^t d_qu - \left(\int_a^b f(qu)d_qu\right)^{t-1} \ge 0.$$

□

Corollary 3.3. *Let n be a positive integer and f be a function defined on $[a,b]_q$ satisfying*

$$f(a) \ge 0 \quad \textit{and} \quad D_qf(x) \ge n(x-a)^{n-1}, \quad x \in (a,b]_q.$$

Then,

$$\int_a^b (f(x))^{n+2} d_qx \ge \left(\int_a^b f(qx)d_qx\right)^{n+1}.$$

Proof. It suffices to take $t = n + 2$ in Proposition 3.2 and the result follows. □

Corollary 3.4. *Let n be a positive integer and f be a function defined on $[a, b]_q$ satisfying*

$$D_q^i f(a) \geq 0, \quad 0 \leq i \leq n-1 \quad \textit{and} \quad D_q^n f(x) \geq n[n-1]_q! \quad x \in (a, b]_q.$$

Then,

$$\int_a^b (f(x))^{n+2} d_q x \geq \left(\int_a^b f(qx) d_q x \right)^{n+1}.$$

Proof. Since $D_q^n f(x) \geq n[n-1]_q!$, then by q-integrating $n-1$ times over $[a, x]$, we get

$$D_q f(x) \geq n(x-a)_q^{n-1} \geq n(x-a)^{n-1}.$$

The result follows from Corollary 3.3. □

Proposition 3.5. *Let $p \geq 1$ be a real number and f be a function defined on $[a, b]_q$ satisfying*

$$f(a) \geq 0, \qquad D_q f(x) \geq p, \ \forall\, x \in (a, b]_q. \tag{3.4}$$

Then we have

$$\int_a^b [f(x)]^{p+2} d_q x \geq \frac{1}{(b-a)^{p-1}} \left[\int_a^b f(qx) d_q x \right]^{p+1}. \tag{3.5}$$

Proof. Put $g(t) = \int_a^t f(qx) d_q x$ and

$$H(t) = \int_a^t [f(x)]^{p+2} d_q x - \frac{1}{(b-a)^{p-1}} \left[\int_a^t f(qx) d_q x \right]^{p+1}, \quad t \in [a, b]_q. \tag{3.6}$$

We have

$$D_q H(t) = [f(t)]^{p+2} - \frac{1}{(b-a)^{p-1}} D_q [g^{p+1}](t), \quad t \in (a, b]_q.$$

Since f and g increase on $[a, b]_q$, we obtain, according to Lemma 3.1, for $t \in (a, b]_q$,

$$\begin{aligned} D_q H(t) &\geq [f(t)]^{p+2} - \frac{1}{(b-a)^{p-1}} (p+1) g^p(t) f(qt) \\ &\geq [f(t)]^{p+2} - \frac{1}{(b-a)^{p-1}} (p+1) g^p(t) f(t) \\ &\geq \left([f(t)]^{p+1} - \frac{1}{(b-a)^{p-1}} (p+1) g^p(t) \right) f(t) = h(t) f(t), \end{aligned}$$

where

$$h(t) = [f(t)]^{p+1} - \frac{1}{(b-a)^{p-1}} (p+1) g^p(t).$$

On the other hand, we have

$$D_q h(t) = D_q [f^{p+1}](t) - \frac{1}{(b-a)^{p-1}} (p+1) D_q [g^p](t).$$

By using Lemma 3.1, we obtain

$$\begin{aligned} D_q h(t) &\geq (p+1) f^p(qt) D_q f(t) - \frac{(p+1)p}{(b-a)^{p-1}} g^{p-1}(t) f(qt) \\ &\geq (p+1) f(qt) \left[f^{p-1}(qt) D_q f(t) - \frac{p}{(b-a)^{p-1}} g^{p-1}(t) \right]. \end{aligned}$$

Since f increases, then for $t \in [a,b]_q$,

$$\int_a^t f(qx)d_qx \leq (b-a)f(qt), \tag{3.7}$$

therefore,

$$D_q h(t) \geq (p+1)f^p(qt)[D_q f(t) - p]. \tag{3.8}$$

We deduce, from the relation (3.4), that h increases on $[a,b]_q$.

Finally, since $h(a) = f^{p+1}(a) \geq 0$, then H increases and $H(b) \geq H(a) \geq 0$, which completes the proof. □

Corollary 3.6. *Let $p \geq 1$ be a real number and f be a nonnegative function on $[0,1]$ such that $D_q f(x) \geq 1$. Then*

$$\int_0^1 [f(x)]^{p+2} d_qx \geq \frac{1}{p}\left[\int_0^1 f(qx)d_qx\right]^{p+1}. \tag{3.9}$$

Proof. Replacing, in the previous proposition, $f(x)$ by $pf(x)$, b by 1 and a by q^N ($N = 1,2,\dots$), we obtain then the result by tending N to ∞. □

In what follows, we will adopt the terminology of the following definition.

Definition 3.1. Let $b > 0$ and $a = bq^n$, where n is a positive integer. For each real number r, we denote by $E_{q,r}([a,b])$ the set of functions defined on $[a,b]_q$ such that

$$f(a) \geq 0 \qquad \text{and} \qquad D_q f(x) \geq [r]_q, \quad \forall x \in (a,b]_q.$$

Proposition 3.7. *Let $f \in E_{q,2}([a,b])$. Then for all $p > 0$, we have*

$$\int_a^b [f(x)]^{2p+1} d_qx > \left[\int_a^b (f(x))^p d_qx\right]^2. \tag{3.10}$$

Proof. For $t \in [a,b]_q$, we put

$$F(t) = \int_a^t [f(x)]^{2p+1} d_qx - \left[\int_a^t (f(x))^p d_qx\right]^2 \quad \text{and} \quad g(t) = \int_a^t [f(x)]^p d_qx.$$

Then, we have for $t \in [a,b]_q$,

$$\begin{aligned} D_q F(t) &= [f(t)]^{2p+1} - [f(t)]^p (g(t) + g(qt)) \\ &= [f(t)]^p \left([f(t)]^{p+1} - [g(t) + g(qt)]\right) \\ &= [f(t)]^p G(t), \end{aligned}$$

where $G(t) = [f(t)]^{p+1} - [g(t) + g(qt)]$.

On the other hand, we have

$$\begin{aligned} D_q G(t) &= \frac{f^{p+1}(t) - f^{p+1}(qt)}{(1-q)t} - f^p(t) - qf^p(qt) \\ &= f^p(t)\frac{f(t) - (1-q)t}{(1-q)t} - f^p(qt)\frac{f(qt) + q(1-q)t}{(1-q)t}. \end{aligned}$$

By using the relation $D_q f(t) \geq [2]_q$, we obtain $f(t) \geq f(qt) + (1-q^2)t$, therefore

$$D_q G(t) \geq \frac{f^p(t) - f^p(qt)}{(1-q)t}[f(qt) + q(1-q)t] > 0, \quad t \in (a,b]_q. \tag{3.11}$$

Hence, G is strictly increasing on $[a,b]_q$. Moreover, we have

$$G(a) = [f(a)]^{p+1} + (1-q)af(a) \geq 0,$$

for all $t \in (a,b]_q$, $G(t) > G(a) \geq 0$, which proves that $D_q F(t) > 0$, for all $t \in (a,b]_q$. Thus, F is strictly increasing on $[a,b]_q$. In particular, $F(b) > F(a) = 0$. □

Corollary 3.8. *Let $\alpha > 0$ and $f \in E_{q,2}([a,b])$. Then for all positive integers m, we have*

$$\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x > \left[\int_a^b [f(x)]^{\alpha} d_q x\right]^{2^m}. \tag{3.12}$$

Proof. We suggest here a proof by induction. For this purpose, we put:

$$p_m(\alpha) = (\alpha+1)2^m - 1.$$

We have

$$p_m(\alpha) > 0 \quad \text{and} \quad p_{m+1}(\alpha) = 2p_m(\alpha) + 1. \tag{3.13}$$

From Proposition 3.7, we deduce that the inequality (3.12) is true for $m = 1$.

Suppose that (3.12) holds for an integer m and let us prove it for $m+1$.

By using the relation (3.13) and Proposition 3.7, we obtain

$$\int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x\right]^2. \tag{3.14}$$

And, by assumption, we have

$$\int_a^b [f(x)]^{(\alpha+1)2^m-1} > \left[\int_a^b [f(x)]^{\alpha} d_q x\right]^{2^m}. \tag{3.15}$$

Finally, the relations (3.14) and (3.15) imply that the inequality (3.12) is true for $m+1$. This completes the proof. □

Corollary 3.9. *Let $f \in E_{q,2}([a,b])$ and $\alpha > 0$. For $m \in \mathbb{N}$, we have*

$$\left[\int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x\right]^{\frac{1}{2^{m+1}}} > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x\right]^{\frac{1}{2^m}}. \tag{3.16}$$

Proof. Since, from Proposition 3.7,

$$\int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x\right]^2, \tag{3.17}$$

then

$$\left[\int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x\right]^{\frac{1}{2^{m+1}}} > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x\right]^{\frac{1}{2^m}}. \tag{3.18}$$

□

Corollary 3.10. *Let $f \in E_{q,2}([a,b])$. For all integers $m \geq 2$, we have*

$$\int_a^b [f(x)]^{2^{m+1}-1} d_q x > \left[\int_a^b [f(x)]^3 d_q x\right]^{2^{m-1}} \tag{3.19}$$

$$> \left[\int_a^b f(x) d_q x\right]^{2^m}. \tag{3.20}$$

Proof. By using Proposition 3.7 and the two previous corollaries for $\alpha = 1$, we obtain the required result. □

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