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SUBORDINATION THEOREM FOR A FAMILY OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE CONVOLUTION STRUCTURE

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ABSTRACT. We use the familiar convolution structure of analytic functions to introduce new class of analytic functions of complex order. The results investigated in the present paper include, the characterization and subordination properties for this class of analytic functions. Several interesting consequences of our results are also pointed out.

Key words and phrases: Analytic function, Hadamard product(or convolution), Dziok-Srivastava linear operator, Subordination factor sequence, Characterization properties.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disk $\mathbb{U} = \{z;\ z \in \mathbb{C} : |z| < 1\}$. If $f \in \mathcal{A}$ is given by (1.1) and $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined(as usual) by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

In this article we study the class $\mathcal{S}_\gamma(g;\alpha)$ introduced in the following:

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Definition 1.1. For a given function $g(z) \in \mathcal{A}$ defined by (1.2), where $b_k \geq 0$ $(k \geq 2)$. We say that $f(z) \in \mathcal{A}$ is in $\mathcal{S}_\gamma(g;\alpha)$, provided that $(f*g)(z) \neq 0$, and

$$\text{(1.4)} \qquad \operatorname{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z(f*g)'(z)}{(f*g)(z)}-1\right)\right\} > \alpha \qquad (z \in \mathbb{U};\ \gamma \in \mathbb{C}\backslash\{0\};\ 0 \leq \alpha < 1).$$

Note that

$$\mathcal{S}_1\left(\frac{z}{1-z};\ \alpha\right) = \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{S}_1\left(\frac{z}{(1-z)^2};\ \alpha\right) = \mathcal{K}(\alpha),$$

are, respectively, the familiar classes of starlike and convex functions of order α in $\mathbb{U}$ (see, for example, [11]). Also

$$\mathcal{S}_\gamma\left(\frac{z}{1-z};\ 0\right) = \mathcal{S}_\gamma^* \quad \text{and} \quad \mathcal{S}_\gamma\left(\frac{z}{(1-z)^2};\ 0\right) = \mathcal{K}_\gamma,$$

where the classes $\mathcal{S}_\gamma^*$ and $\mathcal{K}_\gamma$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr and Aouf [9] and Wiatrowski [12], respectively (see also [7] and [8]).

Remark 1. When

$$\text{(1.5)} \qquad g(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}(k-1)!} z^k$$

$$\left(\alpha_j \in \mathbb{C}(j=1,2,\ldots,q),\ \ \beta_j \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}\ \ (j=1,2,\ldots,s)\right),$$

with the parameters

$$\alpha_1,\ldots,\alpha_q \quad \text{and} \quad \beta_1,\ldots,\beta_s,$$

being so choosen that the coefficients b_k in (1.2) satisfy the following condition:

$$\text{(1.6)} \qquad b_k = \frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}(k-1)!} \geq 0,$$

then the class $\mathcal{S}_\gamma(g;\alpha)$ is transformed into a (presumbly) new class $\mathcal{S}_\gamma^*(q,s,\alpha)$ defined by

$$\text{(1.7)} \qquad \mathcal{S}_\gamma^*(q,s,\alpha) := \left\{f:\ f \in \mathcal{A} \ \text{ and } \ \operatorname{Re}\left(1+\frac{1}{\gamma}\left(\frac{z(H_s^q[\alpha_1]f)'(z)}{(H_s^q[\alpha_1]f)(z)}-1\right)\right) > \alpha\right\}$$
$$(z \in \mathbb{U};\ q \leq s+1;\ q,s \in \mathbb{N}_0;\ \gamma \in \mathbb{C}\backslash\{0\}).$$

The operator

$$\left(H_s^q[\alpha_1]f\right)(z) := H_s^q(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z),$$

involved in (1.7) is the Dziok-Srivastava linear operator (see for details, [3]) which contains such well known operators as the Hohlov linear operator, Carlson-Shaffer linear operator, Ruscheweyh derivative operator, the Barnardi-Libera-Livingston operator, and the Srivastava-Owa fractional derivative operator. One may refer to the papers [3] to [5] for further details and references for these operators. The Dziok-Srivastava linear operator defined in [3] was further extended by Dziok and Raina [1] (see also [2]).

In our present investigation, we require the following definitions and a related result due to Welf [13].

Definition 1.2 (Subordination Principal)**.** For two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinated to $g(z)$ in $\mathbb{U}$ and write $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schawarz function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Definition 1.3 (Subordinating Factor Sequence)**.** A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever

$$f(z) = \sum_{k=1}^{\infty} a_k z^k \qquad (a_1 = 1),$$

is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec f(z) \qquad (z \in \mathbb{U}). \tag{1.8}$$

Lemma 1.1 (Wilf, [13])**.** *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \qquad (z \in \mathbb{U}). \tag{1.9}$$

2. CHARACTERIZATION PROPERTIES

In this section we establish two results (Theorem 2.1 and Theorem 2.3) which give the sufficiency conditions for a function $f(z)$ defined by (1.1) and belong to the class $f(z) \in \mathcal{S}_\gamma(g;\alpha)$.

Theorem 2.1. *Let $f(z) \in \mathcal{A}$ such that*

$$\left|\frac{z(f*g)'(z)}{(f*g)(z)} - 1\right| < 1 - \beta \qquad (\beta < 1;\ z \in \mathbb{U}), \tag{2.1}$$

then $f(z) \in \mathcal{S}_\gamma(g;\alpha)$, provided that

$$|\gamma| \geq \frac{1-\beta}{1-\alpha}, \qquad (0 \leq \alpha < 1). \tag{2.2}$$

Proof. In view of (2.1), we write

$$\frac{z(f*g)'(z)}{(f*g)(z)} = 1 + (1-\beta)w(z) \quad \text{where} \quad |w(z)| < 1 \text{ for } z \in \mathbb{U}.$$

Now

$$\begin{aligned}
\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z(f*g)'(z)}{(f*g)(z)} - 1\right)\right\} &= \operatorname{Re}\left\{1 + \frac{1}{\gamma}(1-\beta)w(z)\right\} \\
&= 1 + (1-\beta)\operatorname{Re}\left\{\frac{w(z)}{\gamma}\right\} \\
&\geq 1 - (1-\beta)\left|\frac{w(z)}{\gamma}\right| \\
&> 1 - (1-\beta)\cdot\frac{1}{|\gamma|} \\
&\geq \alpha,
\end{aligned}$$

provided that $|\gamma| \geq \frac{1-\beta}{1-\alpha}$. This completes the proof. □

If we set

$$\beta = 1 - (1-\alpha)|\gamma| \qquad (0 \leq \alpha < 1; \gamma \in \mathbb{C}\backslash\{0\}),$$

in Theorem 2.1, we obtain

Corollary 2.2. *If* $f(z) \in \mathcal{A}$ *such that*

$$\left| \frac{z(f*g)'(z)}{(f*g)(z)} - 1 \right| < (1-\alpha)|\gamma| \qquad (z \in \mathbb{U},\ 0 \leq \alpha < 1;\ \gamma \in \mathbb{C}\backslash\{0\}), \tag{2.3}$$

then $f(z) \in \mathcal{S}_\gamma(g;\alpha)$.

Theorem 2.3. *Let* $f(z) \in \mathcal{A}$ *satisfy the following inequality*

$$\sum_{k=2}^{\infty} b_k[(k-1) + (1-\alpha)|\gamma|\ ]|a_k| \leq (1-\alpha)|\gamma| \tag{2.4}$$

$$(z \in \mathbb{U};\ b_k \geq 0\ (k \geq 2);\ \gamma \in \mathbb{C}\backslash\{0\};\ 0 \leq \alpha < 1),$$

then $f(z) \in \mathcal{S}_\gamma(g;\alpha)$.

Proof. Suppose the inequality (2.4) holds true. Then in view of Corollary 2.2, we have

$$\begin{aligned}
&|z(f*g)'(z) - (f*g)(z)| - (1-\alpha)|\gamma|\,|(f*g)(z)| \\
&= \left| \sum_{k=2}^{\infty} b_k(k-1)a_k\, z^k \right| - (1-\alpha)|\gamma| \left| z + \sum_{k=2}^{\infty} b_k a_k\, z^k \right| \\
&\leq \left\{ \sum_{k=2}^{\infty} b_k(k-1)|a_k| - (1-\alpha)|\gamma| + (1-\alpha)|\gamma| \sum_{k=2}^{\infty} b_k|a_k| \right\} |z| \\
&\leq \left\{ \sum_{k=2}^{\infty} b_k[(k-1) + (1-\alpha)|\gamma|\ ]|a_k| - (1-\alpha)|\gamma| \right\} \\
&\leq 0.
\end{aligned}$$

This completes the proof. □

On specializing the parameters, Theorem 2.1 would yield the following results:

Corollary 2.4. *Let* $f(z) \in \mathcal{A}$ *satisfy the following inequality*

$$\sum_{k=2}^{\infty} (k + |\gamma| - 1)|a_k| \leq |\gamma| \qquad (z \in \mathbb{U},\ \gamma \in \mathbb{C}\backslash\{0\}), \tag{2.5}$$

then $f(z) \in \mathcal{S}^*_\gamma$.

Corollary 2.5. *Let* $f(z) \in \mathcal{A}$ *satisfy the following inequality*

$$\sum_{k=2}^{\infty} k(k + |\gamma| - 1)|a_k| \leq |\gamma| \qquad (z \in \mathbb{U},\ \gamma \in \mathbb{C}\backslash\{0\}), \tag{2.6}$$

then $f(z) \in \mathcal{K}_\gamma$.

Corollary 2.6. *Let* $f(z) \in \mathcal{A}$ *satisfy the following inequality*

$$\sum_{k=2}^{\infty} \frac{[(k-1) + (1-\alpha)|\gamma|](\alpha_1)_{k-1} \cdots (\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ \cdots\ (\beta_s)_{k-1}(k-1)!} |a_k| \leq (1-\alpha)|\gamma| \tag{2.7}$$

$$(z \in \mathbb{U};\ q \leq s+1;\ q, s \in \mathbb{N}_0;\ \gamma \in \mathbb{C}\backslash\{0\};\ 0 \leq \alpha < 1),$$

then $f(z) \in \mathcal{S}^*_\gamma(q, s, \alpha)$.

3. SUBORDINATION THEOREM

Theorem 3.1. *Let the function $f(z) \in \mathcal{A}$ satisfy the inequality (2.4), and $\mathcal{K}$ denote the familiar class of functions $h(z) \in \mathcal{A}$ which are univalent and convex in $\mathbb{U}$. Then for every $\psi \in \mathcal{K}$, we have*

$$\frac{[1+(1-\alpha)|\gamma|]b_2}{2[b_2+(1-\alpha)(b_2+1)|\gamma|]}(f*\psi)(z) \prec \psi(z) \tag{3.1}$$

$$(z \in \mathbb{U};\ b_k \geq b_2 > 0\ (k \geq 2);\ \gamma \in \mathbb{C}\backslash\{0\};\ 0 \leq \alpha < 1),$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{[b_2+(1-\alpha)(b_2+1)|\gamma|]}{[1+(1-\alpha)|\gamma|]b_2} \qquad (z \in \mathbb{U}). \tag{3.2}$$

The following constant factor

$$\frac{[1+(1-\alpha)|\gamma|]b_2}{2[b_2+(1-\alpha)(b_2+1)|\gamma|]}$$

in the subordination result (3.1) is the best dominant.

Proof. Let $f(z)$ satisfy the inequality (2.4) and let $\psi(z) = \sum_{k=0}^{\infty} c_k z^{k+1} \in \mathcal{K}$, then

$$\begin{aligned}\frac{[1+(1-\alpha)|\gamma|]b_2}{2[b_2+(1-\alpha)(b_2+1)|\gamma|]}&(f*\psi)(z) \\ &= \frac{[1+(1-\alpha)|\gamma|]b_2}{2[b_2+(1-\alpha)(b_2+1)|\gamma|]}\left(z+\sum_{k=2}^{\infty} a_k c_k z^k\right).\end{aligned} \tag{3.3}$$

By invoking Definition 1.3, the subordination (3.1) of our theorem will hold true if the sequence

$$\left\{\frac{[1+(1-\alpha)|\gamma|]b_2}{2[b_2+(1-\alpha)(b_2+1)|\gamma|]}a_k\right\}_{k=1}^{\infty}, \tag{3.4}$$

is a subordination factor sequence. By virtue of Lemma 1.1, this is equivalent to the inequality

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{[1+(1-\alpha)|\gamma|]b_2}{[b_2+(1-\alpha)(b_2+1)|\gamma|]}a_k z^k\right\} > 0 \qquad (z \in \mathbb{U}). \tag{3.5}$$

Since $b_k \geq b_2 > 0$ for $k \geq 2$, we have

$$\begin{aligned}&\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{[1+(1-\alpha)|\gamma|]b_2}{[b_2+(1-\alpha)(b_2+1)|\gamma|]}a_k z^k\right\} \\ &= \operatorname{Re}\left\{1+\frac{[1+(1-\alpha)|\gamma|]b_2}{[b_2+(1-\alpha)(b_2+1)|\gamma|]}z+\frac{1}{[b_2+(1-\alpha)(b_2+1)|\gamma|]}\sum_{k=2}^{\infty}[1+(1-\alpha)|\gamma|]b_2 a_k z^k\right\} \\ &\geq 1-\frac{[1+(1-\alpha)|\gamma|]b_2}{[b_2+(1-\alpha)(b_2+1)|\gamma|]}r-\frac{1}{[b_2+(1-\alpha)(b_2+1)|\gamma|]}\sum_{k=2}^{\infty}[(k-1)+(1-\alpha)|\gamma|]b_k|a_k|r^k \\ &> 1-\frac{[1+(1-\alpha)|\gamma|]b_2}{[b_2+(1-\alpha)(b_2+1)|\gamma|]}r-\frac{(1-\alpha)|\gamma|}{[b_2+(1-\alpha)(b_2+1)|\gamma|]}r > 0 \qquad (|z| = r < 1).\end{aligned}$$

This establishes the inequality (3.5), and consequently the subordination relation (3.1) of Theorem 3.1 is proved. The assertion (3.2) follows readily from (3.1) when the function $\psi(z)$ is selected as

$$\psi(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \in \mathcal{K}. \tag{3.6}$$

The sharpness of the multiplying factor in (3.1) can be established by considering a function $h(z)$ defined by

$$h(z) = z - \frac{(1-\alpha)|\gamma|}{[1+(1-\alpha)|\gamma|]} z^2 \qquad (z \in \mathbb{U};\ \gamma \in \mathbb{C}\backslash\{0\};\ 0 \leq \alpha < 1),$$

which belongs to the class $\mathcal{S}_\gamma(g;\alpha)$. Using (3.1), we infer that

$$\frac{[1+(1-\alpha)|\gamma|]b_2}{2[b_2+(1-\alpha)(b_2+1)|\gamma|]} h(z) \prec \frac{z}{1-z}.$$

It can easily be verified that

$$\min_{|z|\leq 1} \left[\frac{[1+(1-\alpha)|\gamma|]b_2}{2[b_2+(1-\alpha)(b_2+1)|\gamma|]} h(z) \right] = -\frac{1}{2}, \tag{3.7}$$

which shows that the constant

$$\frac{[1+(1-\alpha)|\gamma|]b_2}{2[b_2+(1-\alpha)(b_2+1)|\gamma|]}$$

is the best estimate. □

Before concluding this paper, we consider some useful consequences of the subordination Theorem 3.1.

Corollary 3.2. *Let the function $f(z)$ defined by (1.1) satisfy the inequality (2.5). Then for every $\psi \in \mathcal{K}$, we have*

$$\frac{(1+|\gamma|)}{2(1+2|\gamma|)} (f * \psi)(z) \prec \psi(z) \qquad (z \in \mathbb{U}), \tag{3.8}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{(1+2|\gamma|)}{(1+|\gamma|)} \qquad (z \in \mathbb{U}). \tag{3.9}$$

The constant factor

$$\frac{(1+|\gamma|)}{2(1+2|\gamma|)},$$

in the subordination result (3.8) is the best dominant.

Corollary 3.3. *Let the function $f(z)$ defined by (1.1) satisfy the inequality (2.6). Then for every $\psi \in \mathcal{K}$, we have*

$$\frac{(1+|\gamma|)}{(2+3|\gamma|)} (f * \psi)(z) \prec \psi(z) \qquad (z \in \mathbb{U}), \tag{3.10}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{2(2+3|\gamma|)}{(1+|\gamma|)} \qquad (z \in \mathbb{U}). \tag{3.11}$$

The constant factor

$$\frac{(1+|\gamma|)}{(2+3|\gamma|)},$$

in the subordination result (3.10) is the best dominant.

Corollary 3.4. *Let the function $f(z)$ defined by (1.1) satisfy the inequality (2.7). Then for every $\psi \in \mathcal{K}$, we have*

$$\frac{[1+(1-\alpha)|\gamma|]c_2}{2[c_2+(1-\alpha)(c_2+1)|\gamma|]}(f*\psi)(z) \prec \psi(z) \qquad (z \in \mathbb{U}), \tag{3.12}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{[c_2+(1-\alpha)(c_2+1)|\gamma|]}{[1+(1-\alpha)|\gamma|]c_2} \qquad (z \in \mathbb{U}). \tag{3.13}$$

The constant factor

$$\frac{[1+(1-\alpha)|\gamma|]c_2}{2[c_2+(1-\alpha)(c_2+1)|\gamma|]},$$

in the subordination result (3.12) is the best dominant, where c_2 is given by

$$c_2 = \frac{\alpha_1 \ \cdots \ \alpha_q}{\beta_1 \ \cdots \ \beta_s}.$$

Remark 2. On setting $\gamma = 1$ in Corollaries 3.2 and 3.3, we obtain results that correspond to those of Frasin [6, p. 5, Corollary 2.4; p. 6 , Corollary 2.7] (see also, Singh [10, p. 434, Corollary 2.2]).

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