journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 10 (2009), Issue 4, Article 112, 6 pp.

ON CERTAIN PROPERTIES OF NEIGHBORHOODS OF MULTIVALENT FUNCTIONS INVOLVING THE GENERALIZED SAITOH OPERATOR

HESAM MAHZOON AND S. LATHA

DEPARTMENT OF STUDIES IN MATHEMATICS
MANASAGANGOTRI UNIVERSITY OF MYSORE - INDIA.
mahzoon_hesam@yahoo.com

DEPARTMENT OF MATHEMATICS
YUVARAJA'S COLLEGE UNIVERSITY OF MYSORE -INDIA.
drlatha@gmail.com

Received 10 June, 2009; accepted 03 November, 2009
Communicated by S.S. Dragomir

ABSTRACT. In this paper, we introduce the generalized Saitoh operator $L_p(a,c,\eta)$ and using this operator, the new subclasses $\mathcal{H}^{p,b}_{n,m}(a,c,\eta)$, $\mathcal{L}^{p,b}_{n,m}(a,c,\eta;\mu)$, $\mathcal{H}^{p,b,\alpha}_{n,m}(a,c,\eta)$ and $\mathcal{L}^{p,b,\alpha}_{n,m}(a,c,\eta;\mu)$ of the class of multivalent functions denoted by $\mathcal{A}_p(n)$ are defined. Further for functions belonging to these classes, certain properties of neighborhoods are studied.

Key words and phrases: Coefficient bounds, (n,δ) - neighborhood and generalized Saitoh operator.

2000 *Mathematics Subject Classification.* 30C45.

1. INTRODUCTION

Let $\mathcal{A}_p(n)$ be the class of normalized functions f of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \qquad (n,\, p \in \mathbb{N}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{T}_p(n)$ be the subclass of $\mathcal{A}_p(n)$, consisting of functions f of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \qquad (a_k \geq 0,\ n,\, p \in \mathbb{N}), \tag{1.2}$$

which are p-valent in $\mathcal{U}$.

The Hadamard product of two power series

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k$$

is defined as

$$(f * g)(z) = z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k.$$

Definition 1.1. For $a \in \mathbb{R},\ c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, where $\mathbb{Z}_0^- = \{..., -2, -1, 0\}$ and $\eta \in \mathbb{R}\,(\eta \geq 0)$, the operator $L_p(a, c, \eta) : \mathcal{A}_p(n) \to \mathcal{A}_p(n),$ is defined as

$$L_p(a, c, \eta) f(z) = \phi_p(a, c, z) * D_\eta f(z), \tag{1.3}$$

where

$$D_\eta f(z) = (1-\eta) f(z) + \frac{\eta}{p} z f'(z), \quad (\eta \geq 0,\ z \in \mathcal{U})$$

and

$$\phi_p(a, c, z) = z^p + \sum_{k=n+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} z^k, \quad z \in \mathcal{U}$$

and $(x)_k$ denotes the Pochammer symbol given by

$$(x)_k = \begin{cases} 1 & \text{if } k = 0, \\ x(x+1)\cdots(x+k-1) & \text{if } k \in \mathbb{N} = \{1, 2, 3, ...\}. \end{cases}$$

In particular, we have, $L_1(a, c, \eta) \equiv L(a, c, \eta)$.

Further, if $f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k$, then

$$L_p(a, c, \eta) f(z) = z^p + \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\eta\right] \frac{(a)_{k-p}}{(c)_{k-p}} a_k z^k.$$

Remark 1. For $\eta = 0$ and $n = 1$, we obtain the Saitoh operator [7] which yields the Carlson - Shaffer operator [1] for $\eta = 0$ and $n = p = 1$.

For any function $f \in \mathcal{T}_p(n)$ and $\delta \geq 0$, the (n, δ)-neighborhood of f is defined as,

$$\mathcal{N}_{n,\delta}(f) = \left\{ g \in \mathcal{T}_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k|a_k - b_k| \leq \delta \right\}. \tag{1.4}$$

For the function $h(z) = z^p$, $(p \in \mathbb{N})$ we have,

$$\mathcal{N}_{n,\delta}(h) = \left\{ g \in \mathcal{T}_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k|b_k| \leq \delta \right\}. \tag{1.5}$$

The concept of neighborhoods was first introduced by Goodman [2] and then generalized by Ruscheweyh [6] .

Definition 1.2. A function $f \in \mathcal{T}_p(n)$ is said to be in the class $\mathcal{H}_{n,m}^{p,b}(a, c, \eta)$ if

$$\left| \frac{1}{b} \left(\frac{z\,(L_p(a, c, \eta) f(z))^{(m+1)}}{(L_p(a, c, \eta) f(z))^{(m)}} - (p - m) \right) \right| < 1, \tag{1.6}$$

where $p \in \mathbb{N},\ m \in \mathbb{N}_0,\ a > 0,\ \eta \geq 0,\ p > m,\ b \in \mathbb{C} \setminus \{0\}$ and $z \in \mathcal{U}$.

Definition 1.3. A function $f \in \mathcal{T}_p(n)$ is said to be in the class $\mathcal{L}_{n,m}^{p,b}(a, c, \eta; \mu)$ if

$$\left| \frac{1}{b} \left[p(1-\mu) \left(\frac{L_p(a, c, \eta) f(z)}{z} \right)^{(m)} + \mu\,(L_p(a, c, \eta) f(z))^{(m+1)} - (p - m) \right] \right| < p - m, \tag{1.7}$$

where $p \in \mathbb{N},\ m \in \mathbb{N}_0,\ a > 0,\ \eta \geq 0,\ p > m,\ \mu \geq 0,\ b \in \mathbb{C} \setminus \{0\}$ and $z \in \mathcal{U}$.

2. COEFFICIENT BOUNDS

In this section, we determine the coefficient inequalities for functions to be in the subclasses $\mathcal{H}_{n,m}^{p,b}(a,c,\eta)$ and $\mathcal{L}_{n,m}^{p,b}(a,c,\eta;\mu)$.

Theorem 2.1. *Let* $f \in \mathcal{T}_p(n)$. *Then,* $f \in \mathcal{H}_{n,m}^{p,b}(a,c,\eta)$ *if and only if*

$$\sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\eta\right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} (k + |b| - p)\, a_k \le |b| \binom{p}{m}. \tag{2.1}$$

Proof. Let $f \in \mathcal{H}_{n,m}^{p,b}(a,c,\eta)$. Then, by (1.6) and (1.7) we can write,

$$\Re\left\{ \frac{\sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\eta\right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m}(p-k) a_k z^{k-p}}{\binom{p}{m} - \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\eta\right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} a_k z^{k-p}} \right\} > -|b|, \quad (z \in \mathcal{U}). \tag{2.2}$$

Taking $z = r,\ (0 \le r < 1)$ in (2.2), we see that the expression in the denominator on the left hand side of (2.2), is positive for $r = 0$ and for all $r,\ 0 \le r < 1$. Hence, by letting $r \mapsto 1^-$ through real values, expression (2.2) yields the desired assertion (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we obtain,

$$\begin{aligned}
&\left| \frac{z\left(L_p(a,c,\eta)f(z)\right)^{(m+1)}}{\left(L_p(a,c,\eta)f(z)\right)^{(m)}} - (p-m) \right| \\
&\qquad = \left| \frac{\sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\eta\right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m}(p-k) a_k z^{k-m}}{\binom{p}{m} z^{p-m} - \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\eta\right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} a_k z^{k-m}} \right| \\
&\qquad \le \frac{|b| \left[\binom{p}{m} - \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\eta\right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} a_k\right]}{\binom{p}{m} - \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\eta\right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k}{m} a_k} \\
&\qquad = |b|.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f \in \mathcal{H}_{n,m}^{p,b}(a,c,\eta)$. □

On similar lines, we can prove the following theorem.

Theorem 2.2. *A function* $f \in \mathcal{L}_{n,m}^{p,b}(a,c,\eta;\mu)$ *if and only if*

$$\begin{aligned}
\sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\eta\right] \frac{(a)_{k-p}}{(c)_{k-p}} \binom{k-1}{m} \left[\mu(k-1) + 1\right] a_k& \\
\le (p-m) \left[\frac{|b| - 1}{m!} + \binom{p}{m}\right]&.
\end{aligned} \tag{2.3}$$

3. INCLUSION RELATIONSHIPS INVOLVING (n,δ)-NEIGHBORHOODS

In this section, we prove certain inclusion relationships for functions belonging to the classes $\mathcal{H}_{n,m}^{p,b}(a,c,\eta)$ and $\mathcal{L}_{n,m}^{p,b}(a,c,\eta;\mu)$.

Theorem 3.1. *If*

$$\delta = \frac{(n+p)|b|\binom{p}{m}}{(n+|b|)\left(1 + \frac{n}{p}\eta\right) \frac{(a)_n}{(c)_n} \binom{n+p}{m}}, \quad (p > |b|), \tag{3.1}$$

then $\mathcal{H}^{p,b}_{n,m}(a,c,\eta) \subset \mathcal{N}_{n,\delta}(h)$.

Proof. Let $f \in \mathcal{H}^{p,b}_{n,m}(a,c,\eta)$. By Theorem 2.1, we have,

$$(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}\sum_{k=n+p}^{\infty} a_k \le |b|\binom{p}{m},$$

which implies

$$\sum_{k=n+p}^{\infty} a_k \le \frac{|b|\binom{p}{m}}{(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}}. \tag{3.2}$$

Using (2.1) and (3.2), we have,

$$\begin{aligned}
&\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}\sum_{k=n+p}^{\infty} k a_k \\
&\le |b|\binom{p}{m} + (p-|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}\sum_{k=n+p}^{\infty} a_k \\
&\le |b|\binom{p}{m} + (p-|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}\frac{|b|\binom{p}{m}}{(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}} \\
&= |b|\binom{p}{m}\frac{n+p}{n+|b|}.
\end{aligned}$$

That is,

$$\sum_{k=n+p}^{\infty} k a_k \le \frac{|b|(n+p)\binom{p}{m}}{(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}} = \delta, \quad (p > |b|).$$

Thus, by the definition given by (1.5), $f \in \mathcal{N}_{n,\delta}(h)$. □

Similarly, we prove the following theorem.

Theorem 3.2. *If*

$$\delta = \frac{(p-m)(n+p)\left[\frac{|b|-1}{m!}+\binom{p}{m}\right]}{[\mu(n+p-1)+1]\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}}, \quad (\mu > 1) \tag{3.3}$$

then $\mathcal{L}^{p,b}_{n,m}(a,c,\eta;\mu) \subset \mathcal{N}_{n,\delta}(h)$.

4. FURTHER NEIGHBORHOOD PROPERTIES

In this section, we determine the neighborhood properties of functions belonging to the subclasses $\mathcal{H}^{p,b,\alpha}_{n,m}(a,c,\eta)$ and $\mathcal{L}^{p,b,\alpha}_{n,m}(a,c,\eta;\mu)$.

For $0 \le \alpha < p$ and $z \in \mathcal{U}$, a function f is said to be in the class $\mathcal{H}^{p,b,\alpha}_{n,m}(a,c,\eta)$ if there exists a function $g \in \mathcal{H}^{p,b}_{n,m}(a,c,\eta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha. \tag{4.1}$$

For $0 \le \alpha < p$ and $z \in \mathcal{U}$, a function f is said to be in the class $\mathcal{L}^{p,b,\alpha}_{n,m}(a,c,\eta;\mu)$ if there exists a function $g \in \mathcal{L}^{p,b}_{n,m}(a,c,\eta;\mu)$ such that the inequality (4.1) holds true.

Theorem 4.1. *If* $g \in \mathcal{H}^{p,b}_{n,m}(a,c,\eta)$ *and*

$$\alpha = p - \frac{\delta(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}}{(n+p)\left[(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m} - |b|\binom{p}{m}\right]}, \tag{4.2}$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{H}^{p,b,\alpha}_{n,m}(a,c,\eta)$.

Proof. Let $f \in \mathcal{N}_{n,\delta}(g)$. Then,

$$\sum_{k=n+p}^{\infty} k|a_k - b_k| \leq \delta, \tag{4.3}$$

which yields the coefficient inequality,

$$\sum_{k=n+p}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+p}, \quad (n \in \mathbb{N}). \tag{4.4}$$

Since $g \in \mathcal{H}^{p,b}_{n,m}(a,c,\eta)$, by (3.2) we have,

$$\sum_{k=n+p}^{\infty} b_k \leq \frac{|b|\binom{p}{m}}{(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}} \tag{4.5}$$

so that,

$$\begin{aligned}\left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{k=n+p}^{\infty}|a_k - b_k|}{1 - \sum_{k=n+p}^{\infty} b_k} \\ &\leq \frac{\delta}{n+p}\,\frac{(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m}}{\left[(n+|b|)\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p}{m} - |b|\binom{p}{m}\right]} \\ &= p - \alpha.\end{aligned}$$

Thus, by definition, $f \in \mathcal{H}^{p,b,\alpha}_{n,m}(a,c,\eta)$ for α given by (4.2). □

On similar lines, we prove the following theorem.

Theorem 4.2. *If* $g \in \mathcal{L}^{p,b}_{n,m}(a,c,\eta;\mu)$ *and*

$$\alpha = p - \frac{\delta[\mu(n+p-1)+1]\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p-1}{m}}{(n+p)\left[\{\mu(n+p-1)+1\}\left(1+\frac{n}{p}\eta\right)\frac{(a)_n}{(c)_n}\binom{n+p-1}{m} - (p-m)\left(\frac{|b|-1}{m!} + \binom{p}{m}\right)\right]}, \tag{4.6}$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{L}^{p,b,\alpha}_{n,m}(a,c,\eta;\mu)$.

REFERENCES

[1] B.C. CARLSON AND D.B. SHAFFER, Starlike and prestarlike hypergeometric functions, *SIAM. J. Math. Anal.*,**15**(4) (1984), 737–745.

[2] A.W. GOODMAN, Univalent functions and non-analytic curves, *Proc. Amer. Math. Soc.*,**8** (1957), 598–601.

[3] G. MURUGUSUNDARAMOORTHY AND H.M. SRIVASTAVA, Neighborhoods of certain classes of analytic functions of complex order, *J. Inequal. Pure and Appl. Math.*, **5**(2) (2004), Art. 24. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=374`]

[4] M.A. NASR AND M.K. AOUF, Starlike function of complex order, *J. Natur. Sci. Math.*, **25** (1985), 1–12.

[5] R.K. RAINA AND H.M. SRIVASTAVA, Inclusion and neighborhood properties of some analytic and multivalent functions, *J. Inequal. Pure and Appl. Math.*,**7**(1) (2006), Art. 5. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=640`]

[6] S. RUSCHEWEYH, Neighbohoods of univalent functions, *Proc. Amer. Math. Soc.*, **81** (1981), 521–527.

[7] H. SAITOH, A linear operator and its applications of first order differential subordinations, *Math. Japan*, **44** (1996), 31–38.

[8] H.M. SRIVASTAVA AND S. OWA, *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, 1992.

[9] P. WIATROWSKI, On the coefficients of some family of holomorphic functions, *Zeszyty Nauk. Uniw. Lodz Nauk. Mat. - Przyrod.*, (2), **39** (1970), 75–85.