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ON A SUBORDINATION THEOREM FOR A CLASS OF MEROMORPHIC FUNCTIONS

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ABSTRACT. In this paper, we obtain a subordination result for a class of meromorphic functions.

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1. INTRODUCTION AND MAIN RESULT

Let Σ be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are *analytic* in the punctured unit disc $E = \{z : 0 < |z| < 1\}$. Let $\mathcal{A}$ be the class of all functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ which are analytic in $\Delta = \{z : |z| < 1\}$. The class $\mathcal{P}$

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of *Caratheodory functions* consists of functions $p(z) \in \mathcal{A}$ having positive real part. A function $f(z) \in \Sigma$ is *meromorphic starlike of order* α if $f(z) \neq 0$ and

$$-\Re \frac{zf'(z)}{f(z)} > \alpha, \quad (\alpha < 1; z \in \Delta).$$

Similarly the function $f(z)$ is *meromorphic convex of order* α if $f'(z) \neq 0$ and

$$-\Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (\alpha < 1; z \in \Delta).$$

The classes of these functions are denoted by $MS^*(\alpha)$ and $MC(\alpha)$ respectively. The class $\Sigma^*_\gamma(\alpha)$ of γ*-meromorphic convex of order* α consists of functions $f(z)$ with $f(z)f'(z) \neq 0$ satisfying

$$-\Re \left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \alpha, \quad (\alpha < 1; z \in \Delta).$$

Nunokawa and Ahuja [2] have proved the following:

Theorem 1.1. *Let* $\alpha < 0$. *If*

$$f \in MC\left(\frac{\alpha(3-2\alpha)}{2(1-\alpha)}\right),$$

then $f \in MS^*(\alpha)$.

Theorem 1.2. *Let* $\alpha < 0$ *and* $\gamma \geq 0$. *If*

$$f \in \Sigma^*_\gamma\left(\frac{2\alpha - 2\alpha^2 + \gamma\alpha}{2(1-\alpha)}\right),$$

then $f \in MS^*(\alpha)$.

Our main result is the following generalization of Theorem 1.1 and Theorem 1.2:

Theorem 1.3. *Let* $q(z)$ *be univalent and* $q(z) \neq 0$ *in* $\triangle$ *and*

(1) $zq'(z)/q(z)$ *is starlike univalent in* $\triangle$, *and*
(2) $\Re\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} - \frac{q(z)}{\gamma}\right] > 0$ *for* $z \in \triangle, \gamma \neq 0$.

If $f(z) \in \Sigma$ *and*

$$-\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec q(z) - \gamma\frac{zq'(z)}{q(z)},$$

then

$$-\frac{zf'(z)}{f(z)} \prec q(z)$$

and $q(z)$ *is the best dominant.*

2. PROOF OF THEOREM 1.3

To prove our Theorem 1.3, we need the following result of Miller and Mocanu:

Lemma 2.1. [1, p. 132, Theorem 3.4h] *Let* $q(z)$ *be univalent in the unit disk* $\triangle$ *and* θ *and* ϕ *be analytic in a domain* D *containing* $q(\triangle)$ *with* $\phi(w) \neq 0$ *when* $w \in q(\triangle)$. *Set*

$$Q(z) := zq'(z)\phi(q(z)), \ h(z) := \theta(q(z)) + Q(z).$$

Suppose that either $h(z)$ *is convex, or* $Q(z)$ *is starlike univalent in* $\triangle$. *In addition, assume that*

$$\Re\frac{zh'(z)}{Q(z)} > 0 \quad (z \in \triangle).$$

If $p(z)$ is analytic in Δ, with $p(0) = q(0)$, $p(\Delta) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

By applying Lemma 2.1, we first prove the following:

Lemma 2.2. *Let $q(z) \in \mathcal{A}$ satisfy $q(z) \neq 0$ and the conditions (1) and (2) in Theorem 1.3. If $p(z) \in \mathcal{A}$ satisfies $p(z) \neq 0$ and*

$$p(z) - \gamma\frac{zp'(z)}{p(z)} \prec q(z) - \gamma\frac{zq'(z)}{q(z)}, \tag{2.2}$$

then $p(z) \prec q(z)$ and $q(z)$ is a best dominant.

Proof. Define the functions θ and ϕ by

$$\theta(w) := w \quad \text{and} \quad \phi(w) := -\frac{\gamma}{w}.$$

Then θ and ϕ are analytic in $\mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$. Define the functions $Q(z)$ and $h(z)$ by

$$Q(z) := zq'(z)\phi(q(z)) = -\gamma\frac{zq'(z)}{q(z)}$$

and

$$h(z) := \theta(q(z)) + Q(z) = q(z) - \gamma\frac{zq'(z)}{q(z)}.$$

In view of our assumptions, the functions $Q(z)$ and $h(z)$ satisfy the conditions of Lemma 2.1. Since the subordination (2.2) can be written as the subordination (2.1), the result now follows by an application of Lemma 1.1. □

Proof of Theorem 1.3. Define the function $p(z)$ by

$$p(z) := -\frac{zf'(z)}{f(z)} \quad (z \in \Delta).$$

Then a computation shows that

$$p(z) - \gamma\frac{zp'(z)}{p(z)} = -\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right].$$

The result of Theorem 1.3 now follows from Lemma 2.2. □

3. A SPECIAL CASE

By setting

$$q(z) = \frac{1 + (1-2\alpha)z}{1-z}$$

in Theorem 1.3, we have the following:

Corollary 3.1. *Let $\alpha < 0$, $\gamma \neq 0$. If $f(z) \in \Sigma$ and*

$$-\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{1 + 2[1-\gamma+(\alpha-1)\gamma]z + (1-2\alpha)^2z^2}{1 - 2\alpha z - (1-2\alpha)z^2},$$

then $-\Re\frac{zf'(z)}{f(z)} > \alpha$.

Remark 3.2. Fairly straightforward calculation shows that the image of $|z| < 1$ under

$$w(z) := \frac{1 + 2[1 - \gamma + (\alpha - 1)\gamma]z + (1 - 2\alpha)^2 z^2}{1 - 2\alpha z - (1 - 2\alpha)z^2}$$

contains the half plane $\Re w(z) > \frac{2\alpha - 2\alpha^2 + \gamma\alpha}{2(1-\alpha)}$. Therefore we see that Theorem 1.2 now follows from Corollary 3.1. Theorem 1.1 is indeed a special case of Theorem 1.2 when $\gamma = 1$.

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