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NEW INEQUALITIES ABOUT CONVEX FUNCTIONS

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ABSTRACT. If f is a convex function and $x_1, \dots, x_n$ or $a_1, \dots, a_n$ lie in its domain the following inequalities are proved

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \frac{n-1}{n}\left[f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right) + f\left(\frac{x_n+x_1}{2}\right)\right]$$

and

$$(n-1)\left[f(b_1) + \cdots + f(b_n)\right] \leq n\left[f(a_1) + \cdots + f(a_n) - f(a)\right],$$

where $a = \frac{a_1+\cdots+a_n}{n}$ and $b_i = \frac{na-a_i}{n-1}$, $i = 1, \dots, n$.

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1. MAIN THEOREMS

The well-known Jensen's inequality is given as follows [1]:

Theorem 1.1. *Let f be a convex function on an interval I and let $w_1, \dots, w_n$ be nonnegative real numbers whose sum is 1. Then for all $x_1, \dots, x_n \in I$,*

$$w_1 f(x_1) + \cdots + w_n f(x_n) \geq f(w_1 x_1 + \cdots + w_n x_n). \tag{1.1}$$

Recall that a function f is said to be convex if for any $t \in [0,1]$ and any x, y in the domain of f,

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y). \tag{1.2}$$

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The aim of the present note is to establish new inequalities similar to the following known inequalities:
(Via Titu Andreescu (see [2, p. 6]))

$$f(x_1)+f(x_2)+f(x_3)+f\left(\frac{x_1+x_2+x_3}{3}\right) \geq \frac{4}{3}\left[f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_3+x_1}{2}\right)\right],$$

where f is a convex function and x_1, x_2, x_3 lie in its domain,
(Popoviciu inequality [3])

$$\sum_{i=1}^{n} f(x_i)+\frac{n}{n-2} f\left(\frac{x_1+\cdots+x_n}{n}\right) \geq \frac{2}{n-2} \sum_{i<j} f\left(\frac{x_i+x_j}{2}\right),$$

where f is a convex function on I and $x_1, \ldots, x_n \in I$, and
(Generalized Popoviciu inequality)

$$(n-1)\left[f(b_1)+\cdots+f(b_n)\right] \leq f(a_1)+\cdots+f(a_n)+n(n-2) f(a),$$

where $a=\frac{a_1+\cdots+a_n}{n}$ and $b_i=\frac{na-a_i}{n-1},\ i=1, \ldots, n$, and $a_1, \ldots, a_n \in I$.
Our main results are given in the following theorems:

Theorem 1.2. *If f is a convex function and $x_1, x_2, \ldots, x_n$ lie in its domain, then*

$$\sum_{i=1}^{n} f(x_i)-f\left(\frac{x_1+\cdots+x_n}{n}\right) \geq \frac{n-1}{n}\left[f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right)\right]. \tag{1.3}$$

Proof. Using (1.2) with $t=\frac{1}{2}$, we obtain

$$f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right) \leq f(x_1)+f(x_2)+\cdots+f(x_n). \tag{1.4}$$

In the summation on the right side of (1.4), the expression $\sum_{i=1}^{n} f(x_i)$ can be written as

$$\sum_{i=1}^{n} f(x_i)=\frac{n}{n-1} \sum_{i=1}^{n} f(x_i)-\frac{1}{n-1} \sum_{i=1}^{n} f(x_i),$$

$$\sum_{i=1}^{n} f(x_i)=\frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i)-\sum_{i=1}^{n} \frac{1}{n} f(x_i)\right].$$

Replacing $\sum_{i=1}^{n} f(x_i)$ with the equivalent expression in (1.4),

$$f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right) \leq \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i)-\sum_{i=1}^{n} \frac{1}{n} f(x_i)\right].$$

Hence, applying Jensen's inequality (1.1) to the right hand side of the above resulting inequality we get

$$f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right) \leq \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i)-f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right],$$

and this concludes the proof. □

Remark 1.3. Now we consider the simplest case of Theorem 1.2 for $n = 3$ to obtain the following variant of via Titu Andreescu [2]:

$$f(x_1)+f(x_2)+f(x_3)-f\left(\frac{x_1+x_2+x_3}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_3+x_1}{2}\right)\right].$$

The variant of the generalized Popovicui inequality is given in the following theorem.

Theorem 1.4. *If f is a convex function and $a_1, \ldots, a_n$ lie in its domain, then*

$$(n-1)\left[f(b_1)+\cdots+f(b_n)\right] \leq n\left[f(a_1)+\cdots+f(a_n)-f(a)\right], \tag{1.5}$$

where $a = \frac{a_1+\cdots+a_n}{n}$ and $b_i = \frac{na-a_i}{n-1}$, $i = 1, \ldots, n$.

Proof. By using the Jensen inequality (1.1),

$$f(b_1)+\cdots+f(b_n) \leq f(a_1)+\cdots+f(a_n),$$

and so,

$$f(b_1)+\cdots+f(b_n) \leq \frac{n}{n-1}\left[f(a_1)+\cdots+f(a_n)\right]-\frac{1}{n-1}\left[f(a_1)+\cdots+f(a_n)\right],$$

or

$$f(b_1)+\cdots+f(b_n) \leq \frac{n}{n-1}\left[f(a_1)+\cdots+f(a_n)\right]-\frac{n}{n-1}\left[\frac{1}{n}f(a_1)+\cdots+\frac{1}{n}f(a_n)\right],$$

and so

$$f(b_1)+\cdots+f(b_n) \leq \frac{n}{n-1}\left[f(a_1)+\cdots+f(a_n)-\left(\frac{1}{n}f(a_1)+\cdots+\frac{1}{n}f(a_n)\right)\right]. \tag{1.6}$$

Hence, applying Jensen's inequality (1.1) to the right hand side of (1.6) we get

$$f(b_1)+\cdots+f(b_n) \leq \frac{n}{n-1}\left[f(a_1)+\cdots+f(a_n)-f\left(\frac{a_1+\cdots+a_n}{n}\right)\right],$$

and this concludes the proof. □

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