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HARDY-HILBERT TYPE INEQUALITIES WITH FRACTIONAL KERNEL IN $\mathbb{R}^n$

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ABSTRACT. The main objective of this paper is some new special Hilbert-type and Hardy-Hilbert-type inequalities in $(\mathbf{R^n})^\mathbf{k}$ with $k \geq 2$ non-conjugate parameters which are obtained by using the well known Selberg's integral formula for fractional integrals in an appropriate form. In such a way we obtain extensions over the whole set of real numbers, of some earlier results, previously known from the literature, where the integrals were taken only over the set of positive real numbers. Also, we obtain the best possible constants in the conjugate case.

Key words and phrases: Inequalities, multiple Hilbert's inequality, multiple Hardy-Hilbert's inequality, equivalent inequalities, non-conjugate parameters, gamma function, Selberg's integral, the best possible constant, symmetric-decreasing function, general rearrangement inequality, hypergeometric function.

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1. INTRODUCTION

In order to obtain our general results, we need to present the definitions of non-conjugate parameters. Let $p_i,\ i = 1, 2, \ldots, k$, be the real parameters which satisfy

$$\sum_{i=1}^{k} \frac{1}{p_i} \geq 1 \quad \text{and} \quad p_i > 1, \quad i = 1, 2, \ldots, k. \tag{1.1}$$

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Further, the parameters ${p_i}'$, $i = 1, 2, \dots, k$ are defined by the equations

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1, \quad i = 1, 2, \dots, k. \tag{1.2}$$

Since $p_i > 1$, $i = 1, 2, \dots, k$, it is obvious that $p_i' > 1$, $i = 1, 2, \dots, k$. We define

$$\lambda := \frac{1}{k-1} \sum_{i=1}^{k} \frac{1}{p_i'}. \tag{1.3}$$

It is easy to deduce that $0 < \lambda \le 1$. Also, we introduce parameters q_i, $i = 1, 2, \dots, k$, defined by the relations

$$\frac{1}{q_i} = \lambda - \frac{1}{p_i'}, \quad i = 1, 2, \dots, k. \tag{1.4}$$

In order to obtain our results we require

$$q_i > 0 \quad i = 1, 2, \dots, k. \tag{1.5}$$

It is easy to see that the above conditions do not automatically imply (1.5). The above conditions were also given by Bonsall (see [2]). It is easy to see that

$$\lambda = \sum_{i=1}^{k} \frac{1}{q_i} \quad \text{and} \quad \frac{1}{q_i} + 1 - \lambda = \frac{1}{p_i}, \quad i = 1, 2, \dots, k.$$

Of course, if $\lambda = 1$, then $\sum_{i=1}^{k} \frac{1}{p_i} = 1$, so the conditions (1.1) – (1.4) reduce to the case of conjugate parameters.

Considering the two-dimensional case of non-conjugate parameters ($k = 2$), Hardy, Littlewood and Pólya, (see [7]), proved that there exists a constant K, dependent only on the parameters p_1 and p_2 such that the following Hilbert-type inequality holds for all non-negative measurable functions $f \in L^{p_1}(\langle 0, \infty \rangle)$ and $g \in L^{p_2}(\langle 0, \infty \rangle)$:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^s} dx dy \le K \left(\int_0^\infty f^{p_1}(x) dx \right)^{\frac{1}{p_1}} \left(\int_0^\infty g^{p_2}(y) dy \right)^{\frac{1}{p_2}}. \tag{1.6}$$

Hardy, Littlewood and Pólya did not give a specific value for the constant K in the previous inequality. An alternative proof by Levin (see [9]) established that $K = B^s \left(\frac{1}{s{p_1}'}, \frac{1}{s{p_2}'} \right)$, where B is the beta function, but the paper did not determine whether this was the best possible constant. This question still remains open. The inequality (1.6) was also generalized by F.F. Bonsall (see [2]).

Hilbert and Hardy-Hilbert type inequalities (see [2]) are very significant weight inequalities which play an important role in many fields of mathematics. Similar inequalities, in operator form, appear in harmonic analysis where one investigates the boundedness properties of such operators. This is the reason why Hilbert's inequality is so popular and is of great interest to numerous mathematicians.

In the last century Hilbert-type inequalities have been generalized in many different directions and numerous mathematicians have reproved them using various techniques. Some possibilities of generalizing such inequalities are, for example, various choices of non-negative measures, kernels, sets of integration, extension to the multi-dimensional case, etc. Several generalizations involve very important notions such as Hilbert's transform, Laplace transform, singular integrals, Weyl operators.

In this paper we refer to a recent paper of Brnetić et al, [4], where a general Hilbert-type and Hardy-Hilbert-type inequalities were obtained for non-conjugate parameters, where $k \ge 2$, with positive $\sigma-$finite measures on Ω. However, we shall keep our attention on a result with Lebesgue measures and a special homogeneous function of degree $-s$. This is contained in:

Theorem 1.1. *Let $k \geq 2$ be an integer, p_i, p_i', q_i, $i = 1, 2, \dots, k$, be real numbers satisfying (1.1) – (1.5) and $\sum_{i=1}^k A_{ij} = 0$, $j = 1, 2, \dots, k$. Then the following inequalities hold and are equivalent:*

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^k f_i(x_i)}{\left(\sum_{j=1}^k x_j\right)^{\lambda s}} dx_1 \dots dx_k < K \prod_{i=1}^k \left[\int_0^\infty x_i^{\frac{p_i}{q_i}(k-1-s)+p_i\alpha_i} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}} \tag{1.7}$$

and

$$\left[\int_0^\infty x_k^{(1-\lambda p_k')(k-1-s)-p_k'\alpha_k} \left(\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{k-1} f_i(x_i)}{\left(\sum_{j=1}^k x_j\right)^{\lambda s}} dx_1 \dots dx_{k-1}\right)^{p_k'} dx_k\right]^{\frac{1}{p_k'}} < K \prod_{i=1}^{k-1} \left[\int_0^\infty x_i^{\frac{p_i}{q_i}(k-1-s)+p_i\alpha_i} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}}, \tag{1.8}$$

where

$$K = \frac{1}{\Gamma(s)^\lambda} \prod_{i=1}^k \Gamma(s-k+1-q_i\alpha_i+q_iA_{ii})^{\frac{1}{q_i}} \prod_{i,j=1, i\neq j}^k \Gamma(q_iA_{ij}+1)^{\frac{1}{q_i}},$$

$\alpha_i = \sum_{j=1}^k A_{ij}$, $A_{ij} > -\frac{1}{q_i}$, $i \neq j$ and $A_{ii} - \alpha_i > \frac{k-s-1}{q_i}$.

Our main objective is to obtain inequalities similar to the inequalities in Theorem 1.1, which will include the integrals taken over the whole set of real numbers.

The techniques that will be used in the proofs are mainly based on classical real analysis, especially on the well known Hölder inequality and on Fubini's theorem.

Conventions. Throughout this paper we suppose that all the functions are non-negative and measurable, so that all integrals converge. Further, the Euclidean norm of the vector $\mathbf{x} \in \mathbf{R}^n$ will be denoted by $|\mathbf{x}|$.

2. PRELIMINARIES

The main results in this paper will be based on the well-known Selberg formula for the fractional integral

$$\int_{(\mathbf{R}^n)^k} |\mathbf{x_k}|^{\alpha_k-n} |\mathbf{x_k}-\mathbf{x_{k-1}}|^{\alpha_{k-1}-n} |\mathbf{x_{k-1}}-\mathbf{x_{k-2}}|^{\alpha_{k-2}-n} \cdots |\mathbf{x_2}-\mathbf{x_1}|^{\alpha_1-n} \cdot |\mathbf{x_1}-\mathbf{y}|^{\alpha_0-n} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_k} = \frac{\prod_{i=0}^k \Gamma_n(\alpha_i)}{\Gamma_n\left(\sum_{i=0}^k \alpha_i\right)} |\mathbf{y}|^{\sum_{i=0}^k \alpha_i - n}, \tag{2.1}$$

for arbitrary $k, n \in \mathbf{N}$ and $0 < \alpha_i < n$ such that $0 < \sum_{i=0}^k \alpha_i < n$. The constant $\Gamma_n(\alpha)$ introduces the $n-$dimensional gamma function and is defined by the formula

$$\Gamma_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}-\frac{\alpha}{2}\right)}, \qquad 0 < \alpha < n, \tag{2.2}$$

where Γ is the well known gamma function. Further, from the definition of the $n-$dimensional gamma function it easily follows that

$$\Gamma_n(n-\alpha) = \frac{(2\pi)^n}{\Gamma_n(\alpha)}, \qquad 0 < \alpha < n. \tag{2.3}$$

In the book [13], Stein derived the formula (2.1) with two parameters using the Riesz potential.

Multiple integrals similar to the one in (2.1) are known as Selberg's integrals and their exact values are useful in representation theory and in mathematical physics. These integrals have only been computed for special cases. For a treatment of Selberg's integral, the reader can consult Section 17.11 of [11].

Now, by using the integral equality (2.1), we can easily compute the integral

$$\int_{(\mathbf{R}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} |\mathbf{x_i}|^{-\beta_i}}{\left|\sum_{i=1}^{k} \mathbf{x_i}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_{k-1}},$$

where $0 < \beta_i < n$, $0 < s < n$ and

$$(k-1)n - \sum_{i=1}^{k-1} \beta_i < s < kn - \sum_{i=1}^{k-1} \beta_i.$$

Such an integral will be more suitable for our computations. Namely, by using the substitution $\mathbf{x_1} = \mathbf{t_1} - \mathbf{x_k}$ and $\mathbf{x_i} = \mathbf{t_i} - \mathbf{t_{i-1}}$, $i = 2, 3, \dots, k-1$ (see also [5]), one obtains the formula

$$\begin{aligned} \int_{(\mathbf{R}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} |\mathbf{x_i}|^{-\beta_i}}{\left|\sum_{i=1}^{k} \mathbf{x_i}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_{k-1}} \\ = \frac{\Gamma_n(n-s)\prod_{i=1}^{k-1}\Gamma_n(n-\beta_i)}{\Gamma_n\left(kn - s - \sum_{i=1}^{k-1}\beta_i\right)} |\mathbf{x_k}|^{(k-1)n-s-\sum_{i=1}^{k-1}\beta_i}, \qquad \mathbf{x_k} \neq \mathbf{0} \end{aligned} \tag{2.4}$$

where $0 < \beta_i < n$, $0 < s < n$ and $(k-1)n - \sum_{i=1}^{k-1} \beta_i < s < kn - \sum_{i=1}^{k-1} \beta_i$. Obviously, if $0 < \beta_i < n$ and $0 < s < n$ then the condition $s < kn - \sum_{i=1}^{k-1} \beta_i$ is trivially satisfied.

We shall use the relation (2.4) in the next section, to obtain generalizations of the multiple Hilbert inequality, over the set of real numbers.

3. BASIC RESULT

As we have already mentioned, we shall obtain some extensions of the multiple Hilbert inequality on the whole set of real numbers. We also obtain the equivalent inequality, usually called the Hardy-Hilbert inequality. For more details about equivalent inequalities the reader can consult [7]. To obtain our results we introduce the real parameters A_{ij}, $i, j = 1, 2, \dots, k$ satisfying

$$\sum_{i=1}^{k} A_{ij} = 0, \quad j = 1, 2, \dots, k. \tag{3.1}$$

We also define

$$\alpha_i = \sum_{j=1}^{k} A_{ij}, \quad i = 1, 2, \dots, k. \tag{3.2}$$

The main result of this paper is as follows:

Theorem 3.1. *Let $k \geq 2$ be an integer and p_i, p_i', q_i, $i = 1, 2, \ldots, k$, be real numbers satisfying (1.1) – (1.5). Further, let $A_{ij},\ i, j = 1, 2, \ldots, k$ be real parameters defined by (3.1) and (3.2). Then the following inequalities hold and are equivalent:*

$$\int_{(\mathbf{R}^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2} \ldots d\mathbf{x_k} \leq K \prod_{i=1}^k \left[\int_{\mathbf{R}^n} |\mathbf{x_i}|^{\frac{p_i(k-1)n - p_i s}{q_i} + p_i\alpha_i} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{p_i}} \tag{3.3}$$

and

$$\left\{\int_{\mathbf{R}^n} |\mathbf{x_k}|^{-\frac{p_k'}{q_k}[(k-1)n-s] - p_k'\alpha_k} \cdot \left[\int_{(\mathbf{R}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} f_i(\mathbf{x_i})}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2} \ldots d\mathbf{x_{k-1}}\right]^{p_k'} d\mathbf{x_k}\right\}^{\frac{1}{p_k'}} \leq K \prod_{i=1}^{k-1} \left[\int_{\mathbf{R}^n} |\mathbf{x_i}|^{\frac{p_i(k-1)n - p_i s}{q_i} + p_i\alpha_i} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{p_i}}, \tag{3.4}$$

for any $0 < s < n$, $A_{ij} \in \left(-\frac{n}{q_i}, 0\right)$, $\alpha_i - A_{ii} < \frac{s-(k-1)n}{q_i}$, where the constant K is given by the formula

$$K = \frac{1}{\Gamma_n^{\lambda}(s)} \prod_{i,j=1, i\neq j}^k \Gamma_n(n + q_i A_{ij})^{\frac{1}{q_i}} \prod_{i=1}^k \Gamma_n(s - (k-1)n - q_i\alpha_i + q_i A_{ii})^{\frac{1}{q_i}}.$$

Proof. We start with the inequality (3.3). The left-hand side of the inequality (3.3) can easily be transformed in the following way

$$\begin{aligned}\int_{(\mathbf{R}^n)^k} &\frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2} \ldots d\mathbf{x_k} \\ &= \int_{(\mathbf{R}^n)^k} \prod_{i=1}^k \left[\frac{|\mathbf{x_i}|^{p_i A_{ii}} \prod_{j=1, j\neq i}^k |\mathbf{x_j}|^{q_i A_{ij}}}{\left|\sum_{j=1}^k \mathbf{x_j}\right|^s} F_i^{p_i - q_i}(\mathbf{x_i}) f_i^{p_i}(\mathbf{x_i})\right]^{\frac{1}{q_i}} \\ &\qquad \cdot \left[\prod_{i=1}^k |\mathbf{x_i}|^{p_i A_{ii}} (F_i f_i)^{p_i}(\mathbf{x_i})\right]^{1-\lambda} d\mathbf{x_1} d\mathbf{x_2} \ldots d\mathbf{x_k},\end{aligned}$$

where

$$F_i(\mathbf{x_i}) = \left[\int_{(\mathbf{R}^n)^{k-1}} \frac{\prod_{j=1, j\neq i}^k |\mathbf{x_j}|^{q_i A_{ij}}}{\left|\sum_{j=1}^k \mathbf{x_j}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \ldots d\mathbf{x_{i-1}} d\mathbf{x_{i+1}} \ldots d\mathbf{x_n}\right]^{\frac{1}{q_i}}.$$

Now by using Selberg's integral formula (2.4) it follows easily that

$$F_i(\mathbf{x_i}) = \left[\frac{\prod_{j=1, j\neq i}^k \Gamma_n(n + q_i A_{ij}) \Gamma_n(n-s)}{\Gamma_n(kn + q_i\alpha_i - q_i A_{ii} - s)}\right]^{\frac{1}{q_i}} |\mathbf{x_i}|^{\frac{(k-1)n-s}{q_i} + \alpha_i - A_{ii}}. \tag{3.5}$$

Further, since $\sum_{i=1}^k \frac{1}{q_i} + 1 - \lambda = 1$, $q_i > 0$ and $0 < \lambda \le 1$, we can apply Hölder's inequality with conjugate parameters $q_1, q_2, \dots, q_k$ and $\frac{1}{1-\lambda}$, on the above transformation. In such a way, we obtain the inequality

$$\int_{(\mathbf{R}^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_k}$$
$$\le \prod_{i=1}^k \left[\int_{\mathbf{R}^n} |\mathbf{x_i}|^{p_i A_{ii}} (F_i f_i)^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{q_i}} \prod_{i=1}^k \left[\int_{\mathbf{R}^n} |\mathbf{x_i}|^{p_i A_{ii}} (F_i f_i)^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{1-\lambda}$$
$$= \prod_{i=1}^k \left[\int_{\mathbf{R}^n} |\mathbf{x_i}|^{p_i A_{ii}} (F_i f_i)^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{p_i}},$$

since $\frac{1}{q_i} + 1 - \lambda = \frac{1}{p_i}$. Finally, by using definition (3.5) of the functions F_i, $i = 1, 2, \dots, k$, one obtains the inequality (3.3).

Let us show that the inequalities (3.3) and (3.4) are equivalent. Suppose that the inequality (3.3) is valid. If we put the function $f_n : \mathbf{R}^n \mapsto \mathbf{R}$, defined by

$$f_k(\mathbf{x_k}) = |\mathbf{x_k}|^{-\frac{p_k'}{q_k}[(k-1)n-s] - p_k'\alpha_k} \left[\int_{(\mathbf{R}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} f_i(\mathbf{x_i})}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_{k-1}}\right]^{\frac{p_k'}{p_k}}$$

in the inequality (3.3), we obtain

$$I(\mathbf{x_k})^{p_k'} \le K \prod_{i=1}^{k-1} \left[\int_{\mathbf{R}^n} |\mathbf{x_i}|^{\frac{p_i(k-1)n - p_i s}{q_i} + p_i \alpha_i} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{p_i}} I(\mathbf{x_k})^{\frac{p_k'}{p_k}},$$

where $I(\mathbf{x_k})$ denotes the left-hand side of the inequality (3.4). This gives the inequality (3.4).

It remains to prove that the inequality (3.3) is a consequence of the inequality (3.4). For this purpose, let us suppose that the inequality (3.4) is valid. Then the left-hand side of the inequality (3.3) can be transformed in the following way:

$$\int_{(\mathbf{R}^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_k} = \int_{\mathbf{R}^n} |\mathbf{x_k}|^{\frac{(k-1)n-s}{q_k} + \alpha_k} f_k(\mathbf{x_k})$$
$$\cdot \left[|\mathbf{x_k}|^{-\frac{(k-1)n-s}{q_k} - \alpha_k} \int_{(\mathbf{R}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} f_i(\mathbf{x_i})}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_{k-1}}\right] d\mathbf{x_k}.$$

Applying Hölder's inequality with conjugate parameters p_k and p_k' to the above transformation, we have

$$\int_{(\mathbf{R}^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_k} \le \left[\int_{\mathbf{R}^n} |\mathbf{x_k}|^{\frac{p_k(k-1)n - p_k s}{q_k} + p_k \alpha_k} f_k^{p_k}(\mathbf{x_k}) d\mathbf{x_k}\right]^{\frac{1}{p_k}} \cdot I(\mathbf{x_k}),$$

and the result follows from (3.4). Hence, we have shown that the inequalities (3.3) and (3.4) are equivalent. Since the first inequality is valid, the second one is also valid. This completes the proof. □

Clearly, by putting $n = 1$ in Theorem 3.1, we obtain inequalities which are similar to the inequalities in Theorem 1.1. The integrals are taken over the whole set of real numbers, the weight functions are the same and the constant is of the same form as in Theorem 1.1, where the ordinary gamma function is replaced with $\Gamma_1(\alpha)$.

Remark 1. Observe that equality in the inequality (3.3) holds if and only if it holds in Hölder's inequality. By using the notation from Theorem 3.1, it means that the functions

$$|\mathbf{x_i}|^{p_iA_{ii}}\prod_{j=1,j\neq i}^{k}|\mathbf{x_j}|^{q_iA_{ij}}\left|\sum_{j=1}^{k}\mathbf{x_j}\right|^{-s}F_i^{p_i-q_i}(\mathbf{x_i})f_i^{p_i}(\mathbf{x_i}),\qquad i=1,2,\dots,k$$

and

$$\prod_{i=1}^{k}|\mathbf{x_i}|^{p_iA_{ii}}\left(F_if_i\right)^{p_i}(\mathbf{x_i})$$

are effectively proportional. So, if we suppose that the functions f_i, $i = 1, 2, \dots, k$ are not equal to zero, straightforward computation (see also [4, Remark 1]) leads to the condition

$$\left|\sum_{i=1}^{k}\mathbf{x_i}\right|^{-s}=C\prod_{i=1}^{k}|\mathbf{x_i}|^{(k-1)n-s+q_i(\alpha_i-A_{ii})},$$

where C is an appropriate constant, and that is a contradiction. So equality in Theorem 3.1 holds if and only if at least one of the functions f_i is identically equal to zero. Otherwise, for non-negative and non-zero functions, the inequalities (3.3) and (3.4) are strict.

Remark 2. If the parameters p_i, $i = 1, 2, \dots, k$ are chosen in such a way that

(3.6) $$q_j > 0,\ \text{for some } j\in\{1,2,\dots n\},\quad q_i<0, i\neq j\quad\text{and}\quad \lambda<1$$

or

(3.7) $$q_i<0,\quad i=1,2,\dots,n$$

then the exponents from the proof of Theorem 3.1 fulfill the conditions for the reverse Hölder inequality (for details see e.g. [12, Chapter V]), which gives the reverse of the inequalities (3.3) and (3.4).

4. THE BEST POSSIBLE CONSTANTS IN THE CONJUGATE CASE

In this section we shall focus on the case of the conjugate exponent, to obtain the best possible constants in Theorem 3.1, for some general cases. It seems to be a difficult problem to obtain the best possible constant in the case of non-conjugate parameters.

It follows easily that the constant K from the previous theorem, in the conjugate case ($\lambda = 1$, $p_i = q_i$), takes the form

$$K=\frac{1}{\Gamma_n(s)}\prod_{i,j=1,i\neq j}^{k}\Gamma_n(n+p_iA_{ij})^{\frac{1}{p_i}}\prod_{i=1}^{k}\Gamma_n(s-(k-1)n-p_i\alpha_i+p_iA_{ii})^{\frac{1}{p_i}}.$$

However, we shall deal with an appropriate form of the inequalities obtained in the previous section in the conjugate case. The main idea is to simplify the above constant K, i.e. to obtain the constant without exponents. For this sake, it is natural to consider real parameters A_{ij} satisfying the following constraint

(4.1) $$s-(k-1)n+p_iA_{ii}-p_i\alpha_i=n+p_jA_{ji},\quad j\neq i,\quad i,j\in\{1,2,\dots,k\}.$$

In this case, the above constant K takes the form

$$K^* = \frac{1}{\Gamma_n(s)} \prod_{i=1}^{k} \Gamma_n(n + \widetilde{A_i}), \tag{4.2}$$

where

$$\widetilde{A_i} = p_j A_{ji}, \quad j \neq i \quad \text{and} \quad -n < \widetilde{A_i} < 0. \tag{4.3}$$

It is easy to see that the parameters $\widetilde{A_i}$ satisfy the relation

$$\sum_{i=1}^{k} \widetilde{A_i} = s - kn. \tag{4.4}$$

Further, the inequalities (3.3) and (3.4) with the parameters A_{ij}, satisfying (4.1), become

$$\int_{(\mathbf{R}^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_k} \leq K^* \prod_{i=1}^{k} \left[\int_{\mathbf{R}^n} |\mathbf{x_i}|^{-n-p_i\widetilde{A_i}} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i} \right]^{\frac{1}{p_i}} \tag{4.5}$$

and

$$\left\{ \int_{\mathbf{R}^n} |\mathbf{x_k}|^{(1-p_k')(-n-p_k\widetilde{A_k})} \cdot \left[\int_{(\mathbf{R}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} f_i(\mathbf{x_i})}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_{k-1}} \right]^{p_k{}'} d\mathbf{x_k} \right\}^{\frac{1}{p_k{}'}} \leq K^* \prod_{i=1}^{k-1} \left[\int_{\mathbf{R}^n} |\mathbf{x_i}|^{-n-p_i\widetilde{A_i}} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i} \right]^{\frac{1}{p_i}}. \tag{4.6}$$

We shall see that the constant K^* in (4.5) and (4.6) is the best possible in the sense that we cannot replace the constant K^* in inequalities (4.5) and (4.6) with the smaller constant, so that inequalities are fulfilled for all non-negative measurable functions. Before we prove the facts we have to establish the following two lemmas:

Lemma 4.1. *Let $k \geq 2$ be an integer, $\mathbf{x_k} \in \mathbf{R}^n$, and $\mathbf{x_k} \neq \mathbf{0}$. We define*

$$I_1^\varepsilon(\mathbf{x_k}) = \int_{\mathbf{K}^n(\varepsilon)} |\mathbf{x_1}|^{\widetilde{A_1}} \left[\int_{(\mathbf{R}^n)^{k-2}} \frac{\prod_{i=2}^{k-1} |\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^s} d\mathbf{x_2} \dots d\mathbf{x_{k-1}} \right] d\mathbf{x_1},$$

where $\varepsilon > 0$, $\mathbf{K}^n(\varepsilon)$ is the closed $n-$dimensional ball of radius ε and parameters $\widetilde{A_i}$, $i = 1, 2, \dots, k$ are defined by (4.3). Then there exists a positive constant C_k such that

$$I_1^\varepsilon(\mathbf{x_k}) \leq C_k \varepsilon^{n+\widetilde{A_1}} |\mathbf{x_k}|^{-2n-\widetilde{A_1}-\widetilde{A_k}}, \quad \textit{when} \quad \varepsilon \to 0. \tag{4.7}$$

Proof. We treat two cases. If $k = 2$ we have

$$I_1^\varepsilon(\mathbf{x_2}) = \int_{\mathbf{K}^n(\varepsilon)} \frac{|\mathbf{x_1}|^{\widetilde{A_1}}}{|\mathbf{x_1} + \mathbf{x_2}|^s} d\mathbf{x_1}.$$

By letting $\varepsilon \to 0$, we easily conclude that there exists a positive constant c_2 such that

$$I_1^\varepsilon(\mathbf{x_2}) \leq c_2 |\mathbf{x_2}|^{-s} \int_{\mathbf{K}^n(\varepsilon)} |\mathbf{x_1}|^{\widetilde{A_1}} d\mathbf{x_1}.$$

The previous integral can be calculated by using $n-$dimensional spherical coordinates. More precisely, we have

$$\begin{aligned}(4.8)\qquad &\int_{\mathbf{K}^n(\varepsilon)} |\mathbf{x_1}|^{\widetilde{A_1}} d\mathbf{x_1} \\ &= \int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}\int_0^{\varepsilon} r^{n+\widetilde{A_1}-1}\sin^{n-2}\theta_{n-1}\sin^{n-3}\theta_{n-2}\cdots\sin\theta_2 dr d\theta_1\dots d\theta_{n-1} \\ &= \int_0^{\varepsilon} r^{n+\widetilde{A_1}-1}dr\int_{S_n} dS = \frac{|S_n|\varepsilon^{n+\widetilde{A_1}}}{n+\widetilde{A_1}},\end{aligned}$$

where $|S_n| = 2\pi^{\frac{n}{2}}\Gamma^{-1}(\frac{n}{2})$ is the Lebesgue measure of the unit sphere in $\mathbf{R}^n$. Consequently,

$$I_1^{\varepsilon}(\mathbf{x_2}) \le \frac{c_2|S_n|\varepsilon^{n+\widetilde{A_1}}}{n+\widetilde{A_1}}|\mathbf{x_2}|^{-s},$$

so the inequality holds when $\varepsilon \to 0$, since $-2n - \widetilde{A_1} - \widetilde{A_2} = -s$ holds for $k = 2$. Further, if $k > 2$, then by letting $\varepsilon \to 0$, since $|\mathbf{x_1}| \to 0$, we easily conclude that there exists a positive constant c_k such that

$$(4.9)\qquad I_1^{\varepsilon}(\mathbf{x_k}) \le c_k\left[\int_{\mathbf{K}^n(\varepsilon)} |\mathbf{x_1}|^{\widetilde{A_1}} d\mathbf{x_1}\right]\left[\int_{(\mathbf{R}^n)^{k-2}} \frac{\prod_{i=2}^{k-1}|\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=2}^{k}\mathbf{x_i}\right|^s} d\mathbf{x_2}\dots d\mathbf{x_{k-1}}\right].$$

We have already calculated the first integral in the inequality (4.9), and the second one is the Selberg integral. Namely, by using the formulas (2.3), (2.4) and (4.4) we have

$$\begin{aligned}(4.10)\qquad &\int_{(\mathbf{R}^n)^{k-2}} \frac{\prod_{i=2}^{k-1}|\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=2}^{k}\mathbf{x_i}\right|^s} d\mathbf{x_2}\dots d\mathbf{x_{k-1}} \\ &\qquad = \frac{\Gamma_n(2n+\widetilde{A_1}+\widetilde{A_k})\prod_{i=2}^{k-1}\Gamma_n(n+\widetilde{A_i})}{\Gamma_n(s)}|\mathbf{x_k}|^{-2n-\widetilde{A_1}-\widetilde{A_k}}.\end{aligned}$$

Finally, by using (4.8), (4.9) and (4.10), we obtain the inequality (4.7) and the proof is completed. □

Similarly, we have

Lemma 4.2. *Let $k \ge 2$ be an integer and $\mathbf{x_k} \in \mathbf{R}^n$. We define*

$$I_1^{\varepsilon^{-1}}(\mathbf{x_k}) = \int_{\mathbf{R}^n\setminus\mathbf{K}^n(\varepsilon^{-1})} |\mathbf{x_1}|^{\widetilde{A_1}}\left[\int_{(\mathbf{R}^n)^{k-2}} \frac{\prod_{i=2}^{k-1}|\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=1}^{k}\mathbf{x_i}\right|^s} d\mathbf{x_2}\dots d\mathbf{x_{k-1}}\right] d\mathbf{x_1},$$

where $\varepsilon > 0$ and parameters $\widetilde{A_i}$, $i = 1,2,\dots,k$ are defined by (4.3). Then there exists a positive constant D_k such that

$$(4.11)\qquad I_1^{\varepsilon^{-1}}(\mathbf{x_k}) \le D_k\varepsilon^{n+\widetilde{A_k}}, \quad \textit{when} \quad \varepsilon \to 0.$$

Proof. We treat again two cases. If $k = 2$ we have

$$I_1^{\varepsilon^{-1}}(\mathbf{x_2}) = \int_{\mathbf{R}^n\setminus\mathbf{K}^n(\varepsilon^{-1})} \frac{|\mathbf{x_1}|^{\widetilde{A_1}}}{|\mathbf{x_1}+\mathbf{x_2}|^s} d\mathbf{x_1}.$$

If $\varepsilon \to 0$, then $|\mathbf{x_1}| \to \infty$, so we easily conclude that there exists a positive constant d_2 such that

$$I_1^{\varepsilon^{-1}}(\mathbf{x_2}) \le d_2 \int_{\mathbf{R}^n \setminus \mathbf{K}^n(\varepsilon^{-1})} |\mathbf{x_1}|^{\widetilde{A_1}-s} d\mathbf{x_1},$$

and by using spherical coordinates for calculating the integral on the right-hand side of the previous inequality, we obtain

$$I_1^{\varepsilon^{-1}}(\mathbf{x_2}) \le \frac{d_2|S_n|}{n+\widetilde{A_2}} \varepsilon^{n+\widetilde{A_2}}.$$

Further, if $k > 2$, then by using (2.3), (2.4) and (4.4), we have

$$\int_{(\mathbf{R}^n)^{k-2}} \frac{\prod_{i=2}^{k-1} |\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^s} d\mathbf{x_2} \dots d\mathbf{x_{k-1}} = \frac{\Gamma_n(2n+\widetilde{A_1}+\widetilde{A_k}) \prod_{i=2}^{k-1} \Gamma_n(n+\widetilde{A_i})}{\Gamma_n(s)} |\mathbf{x_1}+\mathbf{x_k}|^{-2n-\widetilde{A_1}-\widetilde{A_k}}. \tag{4.12}$$

So, we get

$$I_1^{\varepsilon^{-1}}(\mathbf{x_k}) = \frac{\Gamma_n(2n+\widetilde{A_1}+\widetilde{A_k}) \prod_{i=2}^{k-1} \Gamma_n(n+\widetilde{A_i})}{\Gamma_n(s)} \cdot \int_{\mathbf{R}^n \setminus \mathbf{K}^n(\varepsilon^{-1})} |\mathbf{x_1}|^{\widetilde{A_1}} |\mathbf{x_1}+\mathbf{x_k}|^{-2n-\widetilde{A_1}-\widetilde{A_k}} d\mathbf{x_1}. \tag{4.13}$$

By letting $\varepsilon \to 0$, then $|\mathbf{x_1}| \to \infty$, so there exists a positive constant d_k such that

$$I_1^{\varepsilon^{-1}}(\mathbf{x_k}) \le d_k \int_{\mathbf{R}^n \setminus \mathbf{K}^n(\varepsilon^{-1})} |\mathbf{x_1}|^{-2n-\widetilde{A_k}} d\mathbf{x_1}.$$

Since,

$$\int_{\mathbf{R}^n \setminus \mathbf{K}^n(\varepsilon^{-1})} |\mathbf{x_1}|^{-2n-\widetilde{A_k}} d\mathbf{x_1} = \frac{|S_n|\varepsilon^{n+\widetilde{A_k}}}{n+\widetilde{A_k}},$$

the inequality (4.11) holds. □

Now, we are able to obtain the main result, i.e. the best possible constants in the inequalities (4.5) and (4.6). Clearly, inequalities (4.5) and (4.6) do not contain parameters A_{ij}, $i,j = 1,2,\dots,k$, so we can regard these inequalities with $\widetilde{A_i}$, $i = 1,2,\dots,k$, as primitive parameters. More precisely, we have

Theorem 4.3. *Suppose $\widetilde{A_i}$, $i = 1,2,\dots,k$, are real parameters fulfilling constraint (4.4) and $-n < \widetilde{A_i} < 0$, $i = 1,2,\dots,k$. Then, the constant K^* is the best possible in both inequalities (4.5) and (4.6).*

Proof. Let us denote by $\mathbf{K}^n(\varepsilon)$ the closed $n-$dimensional ball of radius ε with the center in $\mathbf{0}$.

Let $0 < \varepsilon < 1$. We define the functions $\widetilde{f_i} : \mathbf{R}^n \mapsto \mathbf{R}$, $i = 1,2,\dots,k$ in the following way

$$\widetilde{f_i}(\mathbf{x_i}) = \begin{cases} |\mathbf{x_i}|^{\widetilde{A_i}}, & \mathbf{x_i} \in \mathbf{K}^n(\varepsilon^{-1}) \setminus \mathbf{K}^n(\varepsilon), \\ 0, & \text{otherwise}. \end{cases}$$

If we put defined functions in the inequality (4.5), then the right-hand side of the inequality (4.5) becomes

$$K^* \prod_{i=1}^k \left(\int_{\mathbf{K}^n(\varepsilon^{-1}) \setminus \mathbf{K}^n(\varepsilon)} |\mathbf{x_i}|^{-n} d\mathbf{x_i} \right)^{\frac{1}{p_i}} = K^* \int_{\mathbf{K}^n(\varepsilon^{-1}) \setminus \mathbf{K}^n(\varepsilon)} |\mathbf{x_i}|^{-n} d\mathbf{x_i}.$$

By using $n-$dimensional spherical coordinates we obtain for the above integral

$$\int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_i}|^{-n} d\mathbf{x_i} = \int_{\varepsilon}^{\varepsilon^{-1}} r^{-1} dr \int_{S_n} dS = |S_n| \ln \frac{1}{\varepsilon^2},$$

where $|S_n| = 2\pi^{\frac{n}{2}}\Gamma^{-1}(\frac{n}{2})$ is the Lebesgue measure of the unit sphere in $\mathbf{R}^n$. So for the above choice of functions f_i the right-hand side of the inequality (4.5) becomes

$$K^*|S_n| \ln \frac{1}{\varepsilon^2}. \tag{4.14}$$

Now let J denote the left-hand side of the inequality (4.5). By using Fubini's theorem, for the above choice of functions f_i, we have

$$\begin{aligned} J &= \int_{(\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon))^k} \frac{\prod_{i=1}^k |\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_k} \\ &= \int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{\widetilde{A_k}} \\ &\qquad \cdot \left[\int_{(\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon))^{k-1}} \frac{\prod_{i=1}^{k-1} |\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_{k-1}}\right] d\mathbf{x_k}. \end{aligned}$$

Note that the integral J can be transformed in the following way: $J = J_1 - J_2 - J_3$, where

$$\begin{aligned} J_1 &= \int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{\widetilde{A_k}} \left[\int_{(\mathbf{R}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} |\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_{k-1}}\right] d\mathbf{x_k}, \\ J_2 &= \int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{\widetilde{A_k}} \sum_{j=1}^{k-1} I_j^{\varepsilon}(\mathbf{x_k}) d\mathbf{x_k}, \\ J_3 &= \int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{\widetilde{A_k}} \sum_{j=1}^{k-1} I_j^{\varepsilon^{-1}}(\mathbf{x_k}) d\mathbf{x_k}. \end{aligned}$$

Here, for $j = 1, 2, \dots, k-1$, the integrals $I_j^{\varepsilon}(\mathbf{x_k})$ and $I_j^{\varepsilon^{-1}}(\mathbf{x_k})$ are defined by

$$I_j^{\varepsilon}(\mathbf{x_k}) = \int_{\mathbf{P}_j} \frac{\prod_{i=1}^{k-1} |\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_{k-1}},$$

satisfying $\mathbf{P}_j = \{(U_1, U_2, \dots, U_{k-1})\,; U_j = \mathbf{K}^n(\varepsilon), U_l = \mathbf{R}^n, l \neq j\}$, and

$$I_j^{\varepsilon^{-1}}(\mathbf{x_k}) = \int_{\mathbf{Q}_j} \frac{\prod_{i=1}^{k-1} |\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_{k-1}},$$

satisfying $\mathbf{Q}_j = \{(U_1, U_2, \dots, U_{k-1})\,; U_j = \mathbf{R}^n \setminus \mathbf{K}^n(\varepsilon^{-1}), U_l = \mathbf{R}^n, l \neq j\}$.

Now, the main idea is to find the lower bound for J. The first part J_1 can easily be computed. Namely by using Selberg's integral formula (2.4) and since the relation (4.4) holds for parameters $\widetilde{A_i}$, it easily follows that

$$\int_{(\mathbf{R}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} |\mathbf{x_i}|^{\widetilde{A_i}}}{\left|\sum_{i=1}^k \mathbf{x_i}\right|^s} d\mathbf{x_1} d\mathbf{x_2} \dots d\mathbf{x_{k-1}} = K^* |\mathbf{x_k}|^{-\widetilde{A_k} - n},$$

and consequently, by using $n-$dimensional spherical coordinates, as we did for computing the right-hand side of the inequality (4.5), we obtain that

$$J_1 = K^*|S_n| \ln \frac{1}{\varepsilon^2}. \tag{4.15}$$

Now we shall show that the parts J_2 and J_3 converge when $\varepsilon \to 0$. For that sake, without loss of generality, it is enough to estimate the integrals

$$\int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{\widetilde{A_k}} I_1^{\varepsilon}(\mathbf{x_k}) d\mathbf{x_k} \quad \text{and} \quad \int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{\widetilde{A_k}} I_1^{\varepsilon^{-1}}(\mathbf{x_k}) d\mathbf{x_k}.$$

By using Lemma 4.1 and $n-$dimensional spherical coordinates we obtain

$$\begin{aligned}\int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{\widetilde{A_k}} I_1^{\varepsilon}(\mathbf{x_k}) d\mathbf{x_k} &\leq C_k \varepsilon^{n+\widetilde{A_1}} \int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{-2n-\widetilde{A_1}} d\mathbf{x_k} \\ &= C_k |S_n| \varepsilon^{n+\widetilde{A_1}} \int_{\varepsilon}^{\varepsilon^{-1}} r^{-n-\widetilde{A_1}-1} dr \\ &= \frac{C_k|S_n|}{n+\widetilde{A_1}} \left(1-\varepsilon^{2(n+\widetilde{A_1})}\right).\end{aligned}$$

Further, we use Lemma 4.2 to estimate the second integral

$$\int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{\widetilde{A_k}} I_1^{\varepsilon^{-1}}(\mathbf{x_k}) d\mathbf{x_k}.$$

Similarly to before, by using spherical coordinates we obtain the inequality

$$\begin{aligned}\int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{\widetilde{A_k}} I_1^{\varepsilon^{-1}}(\mathbf{x_k}) d\mathbf{x_k} &\leq D_k \varepsilon^{n+\widetilde{A_k}} \int_{\mathbf{K}^n(\varepsilon^{-1})\setminus\mathbf{K}^n(\varepsilon)} |\mathbf{x_k}|^{\widetilde{A_k}} d\mathbf{x_k} \\ &= \frac{D_k|S_n|}{n+\widetilde{A_k}} \left(1-\varepsilon^{2(n+\widetilde{A_k})}\right).\end{aligned}$$

Now, since $n+\widetilde{A_i} > 0$, $i = 1, 2, \ldots, k$, the above computation shows that $J_2 + J_3 \leq O(1)$ when $\varepsilon \to 0$. Hence, for the right-hand side of the inequality (4.5), by using (4.15), we obtain

$$J \geq K^*|S_n| \ln \frac{1}{\varepsilon^2} - O(1), \quad \text{when} \quad \varepsilon \to 0. \tag{4.16}$$

Now, let us suppose that the constant K^* is not the best possible. That means that there exists a smaller positive constant L^*, $0 < L^* < K^*$, such that the inequality (4.5) holds, if we replace K^* with L^*. In that case, for the above choice of functions $\widetilde{f_i}$, the right hand-side of the inequality (4.5) becomes $L^*|S_n| \ln \frac{1}{\varepsilon^2}$. Since $L^*|S_n| \ln \frac{1}{\varepsilon^2} \geq J$, by using (4.16), we obtain the inequality

$$(K^* - L^*)\,|S_n| \ln \frac{1}{\varepsilon^2} \leq O(1), \quad \text{when} \quad \varepsilon \to 0. \tag{4.17}$$

Now, by letting $\varepsilon \to 0$, we obtain from (4.17) a contradiction, since the left hand side of the inequality goes to infinity. This contradiction shows that the constant K^* is the best possible in the inequality (4.5).

Finally, the equivalence of the inequalities (4.5) and (4.6) means that the constant K^* is also the best possible in the inequality (4.6). That completes the proof. □

Remark 3. In the papers [3] and [8] we have also obtained the best possible constants, but only for $n = 1$ and for the inequalities which involve the integrals taken over the set of non-negative real numbers.

5. SOME APPLICATIONS

In this section we shall consider some special choices of real parameters A_{ij}, $i,j=1,2,\dots,k$, in Theorem 3.1. In such a way, we shall obtain some extensions (on the set of real numbers) of the numerous versions of multiple Hilbert's and Hardy-Hilbert's inequality, previously known from the literature. Further, in the conjugate case we shall obtain the best possible constants in some cases.

To begin with, let us define real parameters A_{ij}, $i,j=1,2,\dots,k$, by $A_{ii}=(nk-s)\frac{\lambda q_i-1}{q_i^2}$ and $A_{ij}=(s-nk)\frac{1}{q_iq_j}$, $i\neq j$, $i,j=1,2,\dots,k$. Then we have

$$\sum_{i=1}^{k}A_{ij}=\sum_{i\neq j}\frac{s-nk}{q_iq_j}+(nk-s)\left(\frac{\lambda q_j-1}{{q_j}^2}\right)=\frac{s-nk}{q_j}\left(\sum_{i=1}^{k}\frac{1}{q_i}-\lambda\right)=0,$$

for $j=1,2,\dots,k$. Clearly, the parameters A_{ij} are symmetric and it directly follows that $\alpha_i=\sum_{j=1}^{n}A_{ij}=0$, for $j=1,2,\dots,k$. In such a way we obtain the following result:

Corollary 5.1. *Let $k\geq 2$ be an integer and p_i,p_i',q_i, $i=1,2,\dots,k$, be real numbers satisfying (1.1) – (1.5). Then the following inequalities hold and are equivalent:*

$$\int_{(\mathbf{R}^n)^k}\frac{\prod_{i=1}^{k}f_i(\mathbf{x_i})}{\left|\sum_{i=1}^{k}\mathbf{x_i}\right|^{\lambda s}}d\mathbf{x_1}d\mathbf{x_2}\dots d\mathbf{x_k}\leq L\prod_{i=1}^{k}\left[\int_{\mathbf{R}^n}|\mathbf{x_i}|^{\frac{p_i(k-1)n-p_is}{q_i}}f_i^{p_i}(\mathbf{x_i})d\mathbf{x_i}\right]^{\frac{1}{p_i}} \tag{5.1}$$

and

$$\left\{\int_{\mathbf{R}^n}|\mathbf{x_k}|^{-\frac{p_k'}{q_k}[(k-1)n-s]}\left[\int_{(\mathbf{R}^n)^{k-1}}\frac{\prod_{i=1}^{k-1}f_i(\mathbf{x_i})}{\left|\sum_{i=1}^{k}\mathbf{x_i}\right|^{\lambda s}}d\mathbf{x_1}d\mathbf{x_2}\dots d\mathbf{x_{k-1}}\right]^{p_k'}d\mathbf{x_k}\right\}^{\frac{1}{p_k'}}$$
$$\leq L\prod_{i=1}^{k-1}\left[\int_{\mathbf{R}^n}|\mathbf{x_i}|^{\frac{p_i(k-1)n-p_is}{q_i}}f_i^{p_i}(\mathbf{x_i})d\mathbf{x_i}\right]^{\frac{1}{p_i}},$$

where $0<nk-s<n\min\{p_i,q_j,i,j=1,2,\dots,k\}$ and the constant L is defined by the formula

$$L=\frac{1}{\Gamma_n^{\lambda}(s)}\prod_{i=1}^{k}\Gamma_n\left(n-\frac{nk-s}{q_i}\right)^{\frac{1}{p_i'}}\prod_{i=1}^{k}\Gamma_n\left(n-\frac{nk-s}{p_i}\right)^{\frac{1}{q_i}}.$$

The equality in both inequalities holds if and only if at least one of the functions f_i, $i=1,2,\dots,k$, is equal to zero.

Remark 4. Straightforward computation shows that parameters A_{ij} from Corollary 5.1, in the conjugate case, satisfy equation (4.1). Hence, the constant L from the previous corollary becomes

$$L=\frac{1}{\Gamma_n(s)}\prod_{i=1}^{k}\Gamma_n\left(n-\frac{nk-s}{p_i}\right),$$

and that is the best possible constant in the conjugate case.

Remark 5. Similar to the previous corollary, if we define the parameters A_{ij} by $A_{ii} = \frac{n(\lambda q_i - 1)}{\lambda q_i^2}$ and $A_{ij} = -\frac{n}{\lambda q_i q_j}$, $i \neq j$, $i, j \in \{1, 2, \dots, k\}$, then we have

$$\sum_{i=1}^{k} A_{ij} = \sum_{i \neq j} -\frac{n}{\lambda q_i q_j} + \frac{n(\lambda q_j - 1)}{\lambda {q_j}^2} = -\frac{n}{\lambda q_j}\left(\sum_{i=1}^{k} \frac{1}{q_i} - \lambda\right) = 0,$$

for $j = 1, 2, \dots, k$. Since the parameters A_{ij} are symmetric one obtains $\alpha_i = \sum_{j=1}^{n} A_{ij} = 0$, for $j = 1, 2, \dots, k$. So, by putting these parameters in Theorem 3.1 we obtain the same inequalities as those in Corollary 5.1, with the constant L replaced by

$$L' = \frac{1}{\Gamma_n^{\lambda}(s)} \prod_{i=1}^{k} \Gamma_n\left(\frac{n}{\lambda p_i'}\right)^{\lambda - \frac{1}{q_i}} \prod_{i=1}^{k} \Gamma_n\left(s + \frac{n}{\lambda p_i'} - (k-1)n\right)^{\frac{1}{q_i}},$$

where $(k-1)n - s < \frac{n}{\lambda p_i'} < n$, $i = 1, 2, \dots, k$.

It is important to mention that the results in this section, as well as Theorem 3.1, are extensions of our papers [3] and [4], obtained by using Selberg's integral formula.

6. TRILINEAR VERSION OF A STANDARD BETA INTEGRAL

As we know, Selberg's integral formula is the $k-$fold generalization of a standard beta integral on $\mathbf{R}^n$. A few years ago, by using a Fourier transform (see [6]), the following trilinear version of a standard beta integral was obtained:

$$\int_{\mathbf{R}^n} \frac{|\mathbf{t}|^{\alpha+\beta-2n}}{|\mathbf{x}-\mathbf{t}|^{\alpha}|\mathbf{y}-\mathbf{t}|^{\beta}} d\mathbf{t} = B(\alpha, \beta, n) \frac{|\mathbf{x}-\mathbf{y}|^{n-\alpha-\beta}}{|\mathbf{x}|^{n-\beta}|\mathbf{y}|^{n-\alpha}}, \tag{6.1}$$

where $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, $\mathbf{x} \neq \mathbf{y} \neq \mathbf{0}$, $0 < \alpha, \beta < n$, $\alpha + \beta > n$ and

$$B(\alpha, \beta, n) = \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta-n}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(n - \frac{\alpha+\beta}{2}\right)}.$$

By using the definition (2.2) of the $n-$dimensional gamma function we easily obtain that

$$B(\alpha, \beta, n) = \frac{\Gamma_n(n-\alpha)\Gamma_n(n-\beta)}{\Gamma_n(2n-\alpha-\beta)}. \tag{6.2}$$

We also define

$$B^*(\alpha, \beta, n) = \frac{\Gamma_n(\alpha)\Gamma_n(\beta)}{\Gamma_n(\alpha+\beta)}. \tag{6.3}$$

It is still unclear whether or not there is a corresponding $k-$fold analogue of (6.1). In spite of that, we shall use the trilinear formula (6.1) to obtain a $2-$fold inequality of Hilbert type for the kernel $K(\mathbf{x}, \mathbf{y}) = |\mathbf{x}-\mathbf{y}|^{\alpha-n}|\mathbf{x}+\mathbf{y}|^{\beta-n}$, where $0 < \alpha, \beta < n$, $\alpha + \beta < n$.

In the $2-$dimensional case we denote non-conjugate exponents in the following way: $p_1 = p$, $p_2 = q$, $p_1' = p'$ and $p_2' = q'$. So, with the above notation, we have the following result:

Theorem 6.1. *Let α and β be real parameters satisfying $0 < \alpha, \beta < n$ and $\alpha + \beta < n$. Then, the following inequalities hold and are equivalent*

$$\int_{(\mathbf{R}^n)^2} \frac{f(\mathbf{x})g(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\lambda(n-\alpha)}|\mathbf{x}+\mathbf{y}|^{\lambda(n-\beta)}} d\mathbf{x}d\mathbf{y} \tag{6.4}$$
$$\leq N \left[\int_{\mathbf{R}^n} |\mathbf{x}|^{(p-1)(\alpha+\beta+n)-pn\lambda} f^p(\mathbf{x}) d\mathbf{x}\right]^{\frac{1}{p}} \left[\int_{\mathbf{R}^n} |\mathbf{y}|^{(q-1)(\alpha+\beta+n)-qn\lambda} g^q(\mathbf{y}) d\mathbf{y}\right]^{\frac{1}{q}}$$

and

$$\left\{\int_{\mathbf{R}^n} |\mathbf{y}|^{n(\lambda q'-1)-\alpha-\beta}\left[\int_{(\mathbf{R}^n)} \frac{f(\mathbf{x})d\mathbf{x}}{|\mathbf{x}-\mathbf{y}|^{\lambda(n-\alpha)}|\mathbf{x}+\mathbf{y}|^{\lambda(n-\beta)}}\right]^{q'} d\mathbf{y}\right\}^{\frac{1}{q'}} \le N\left[\int_{\mathbf{R}^n} |\mathbf{x}|^{(p-1)(\alpha+\beta+n)-pn\lambda} f^p(\mathbf{x})d\mathbf{x}\right]^{\frac{1}{p}}, \tag{6.5}$$

where the constant N is defined by $N = 2^{\lambda(\alpha+\beta-n)}B^*(\alpha,\beta,n)^{\lambda}$.

Proof. The main idea is the same as in Theorem 3.1, i.e. to reduce the case of non-conjugate exponents to the case of conjugate exponents. Note that the right-hand side of the first inequality (6.4) can be transformed in the following way:

$$\int_{(\mathbf{R}^n)^2} \frac{f(\mathbf{x})g(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\lambda(n-\alpha)}|\mathbf{x}+\mathbf{y}|^{\lambda(n-\beta)}}d\mathbf{x}d\mathbf{y} = \int_{(\mathbf{R}^n)^2} {P_1}^{\frac{1}{q'}}{P_2}^{\frac{1}{p'}}{P_3}^{1-\lambda}d\mathbf{x}d\mathbf{y},$$

where

$$\begin{aligned} P_1 &= \frac{|\mathbf{x}|^{(p-1)(\alpha+\beta)}|\mathbf{y}|^{-\alpha-\beta}}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}}f^p(\mathbf{x}), \\ P_2 &= \frac{|\mathbf{y}|^{(q-1)(\alpha+\beta)}|\mathbf{x}|^{-\alpha-\beta}}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}}g^q(\mathbf{y}), \\ P_3 &= |\mathbf{x}|^{(p-1)(\alpha+\beta)}|\mathbf{y}|^{(q-1)(\alpha+\beta)}f^p(\mathbf{x})g^q(\mathbf{y}). \end{aligned}$$

Therefore, respectively, by applying Hölder's inequality with conjugate exponents q', p', $\frac{1}{1-\lambda}$ and Fubini's theorem, we obtain the inequality (6.4).

Let us show that the inequalities (6.4) and (6.5) are equivalent. To this aim, suppose that the inequality (6.4) is valid. If we put the function

$$g(\mathbf{y}) = |\mathbf{y}|^{n(\lambda q'-1)-\alpha-\beta}\left[\int_{\mathbf{R}^n} \frac{f(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{\lambda(n-\alpha)}|\mathbf{x}+\mathbf{y}|^{\lambda(n-\beta)}}d\mathbf{x}\right]^{\frac{q'}{q}}$$

in the inequality (6.4), then the left-hand side of (6.4) becomes J, where J is the left-hand side of the inequality (6.5). Also, the second factor on the right-hand side inequality (6.4) becomes $J^{\frac{1}{q}}$, so (6.5) follows easily.

It remains to prove that (6.4) is a consequence of (6.5). For this purpose, let's suppose that the inequality (6.5) is valid. Then the left-hand side of the inequality (6.4) can be transformed in the following way:

$$\begin{aligned} &\int_{(\mathbf{R}^n)^2} \frac{f(\mathbf{x})g(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\lambda(n-\alpha)}|\mathbf{x}+\mathbf{y}|^{\lambda(n-\beta)}}d\mathbf{x}d\mathbf{y} \\ &\qquad = \int_{\mathbf{R}^n} |\mathbf{y}|^{\frac{\alpha+\beta+n}{q'}-n\lambda}g(\mathbf{y})\left[|\mathbf{y}|^{-\frac{\alpha+\beta+n}{q'}+n\lambda}\int_{\mathbf{R}^n} \frac{f(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{\lambda(n-\alpha)}|\mathbf{x}+\mathbf{y}|^{\lambda(n-\beta)}}d\mathbf{x}\right]d\mathbf{y}. \end{aligned}$$

Finally, by applying Holder's inequality with conjugate exponents q and q' on the previous transformation, and by using the inequality (6.5) one easily obtains (6.4). Hence, the inequalities are equivalent and the proof is completed. □

Real parameters α and β in (6.4) and (6.5) satisfy the condition $\alpha+\beta < n$. In what follows we shall obtain similar inequalities which are, in some way, complementary to the inequalities (6.4) and (6.5). The first step is to consider the case when the function $g \in L^q(\mathbf{R}^n)$ is symmetric-decreasing, that is, $g(\mathbf{x}) \geq g(\mathbf{y})$ whenever $|\mathbf{x}| \leq |\mathbf{y}|$. Since $q > 1$, for such a function and $\mathbf{y} \in \mathbf{R}^n$, $\mathbf{y} \neq \mathbf{0}$, we have

$$g^q(\mathbf{y}) \leq \frac{1}{|B(|\mathbf{y}|)|} \int_{B(|\mathbf{y}|)} g^q(\mathbf{x})\, d\mathbf{x} \leq \frac{1}{|B(|\mathbf{y}|)|} \int_{\mathbf{R}^n} g^q(\mathbf{x})\, d\mathbf{x} = \frac{n}{|S_n|} |\mathbf{y}|^{-n} \|g\|_q^q, \tag{6.6}$$

where $B(|\mathbf{y}|)$ denotes the ball of radius $|\mathbf{y}|$ in $\mathbf{R}^n$, centered at the origin, and $|B(|\mathbf{y}|)| = |\mathbf{y}|^n \frac{|S_n|}{n}$ is its volume.

Theorem 6.2. *Let α and β be real parameters satisfying $0 < \alpha < n$, $0 < \beta < n$, $\alpha + \beta = n\left(\frac{1}{p} + \frac{1}{q}\right) > n$. If f and g are nonnegative functions such that $f \in L^p(\mathbf{R}^n)$, $g \in L^q(\mathbf{R}^n)$, then the following inequalities hold and are equivalent*

$$\int_{(\mathbf{R}^n)^2} \frac{f(\mathbf{x})g(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}} d\mathbf{x}d\mathbf{y} \leq \left(\frac{n}{|S_n|}\right)^{1-\lambda} C(p,q;\alpha,\beta;n)\|f\|_p\|g\|_q, \tag{6.7}$$

and

$$\left\{\int_{\mathbf{R}^n} \left[\int_{\mathbf{R}^n} \frac{f(\mathbf{x})d\mathbf{x}}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}}\right]^{q'} d\mathbf{y}\right\}^{\frac{1}{q'}} \leq \left(\frac{n}{|S_n|}\right)^{1-\lambda} C(p,q;\alpha,\beta;n)\|f\|_p, \tag{6.8}$$

with the constant

$$C(p,q;\alpha,\beta;n) = \int_{\mathbf{R}^n} \frac{|\mathbf{x}|^{-\frac{n}{q}} d\mathbf{x}}{|\mathbf{e}_1-\mathbf{x}|^{n-\alpha}|\mathbf{e}_1+\mathbf{x}|^{n-\beta}}, \tag{6.9}$$

where $\mathbf{e}_1 = (1,0,\dots,0) \in \mathbf{R}^n$ and $|S_n|$ is the Lebesgue measure of the unit sphere in $\mathbf{R}^n$.

Proof. Since we shall use a general rearrangement inequality (see e.g. [10]) it is enough to prove the inequality for symmetric-decreasing functions f and g. First, using Hölder's inequality with parameters q', p' and $\frac{1}{1-\lambda}$, we have

$$\int_{\mathbf{R}^{2n}} \frac{f(\mathbf{x})g(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}} d\mathbf{x}d\mathbf{y} \leq I_1^{\frac{1}{q'}} I_2^{\frac{1}{p'}} I_3^{1-\lambda}, \tag{6.10}$$

where

$$I_1 = \int_{\mathbf{R}^{2n}} \frac{|\mathbf{x}|^{\frac{n}{p'}}|\mathbf{y}|^{-\frac{n}{q}}}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}} f^p(\mathbf{x})\, d\mathbf{x}d\mathbf{y},$$

$$I_2 = \int_{\mathbf{R}^{2n}} \frac{|\mathbf{x}|^{-\frac{n}{p}}|\mathbf{y}|^{\frac{n}{q'}}}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}} g^q(\mathbf{y})\, d\mathbf{x}d\mathbf{y},$$

$$I_3 = \int_{\mathbf{R}^{2n}} \frac{|\mathbf{x}|^{\frac{n}{p'}}|\mathbf{y}|^{\frac{n}{q'}}}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}} f^p(\mathbf{x})g^q(\mathbf{y})\, d\mathbf{x}d\mathbf{y}.$$

Further, using the substitution $\mathbf{y} = |\mathbf{x}|\mathbf{u}$ (so $d\mathbf{y} = |\mathbf{x}|^n d\mathbf{u}$) and rotational invariance of the Lebesgue integral in $\mathbf{R}^n$ we easily get:

$$\begin{aligned} I_1 &= \int_{\mathbf{R}^n} |\mathbf{x}|^{\frac{n}{p'}} f^p(\mathbf{x}) \int_{\mathbf{R}^n} \frac{|\mathbf{y}|^{-\frac{n}{q}}}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}} d\mathbf{y}d\mathbf{x} \\ &= \int_{\mathbf{R}^n} |\mathbf{x}|^{\frac{n}{p'}-\frac{n}{q}+\alpha+\beta-n} f^p(\mathbf{x}) \int_{\mathbf{R}^n} \frac{|\mathbf{u}|^{-\frac{n}{q}}}{\left|\frac{\mathbf{x}}{|\mathbf{x}|}-\mathbf{u}\right|^{n-\alpha}\left|\frac{\mathbf{x}}{|\mathbf{x}|}+\mathbf{u}\right|^{n-\beta}} d\mathbf{u}d\mathbf{x} \\ &= \int_{\mathbf{R}^n} \frac{|\mathbf{u}|^{-\frac{n}{q}} d\mathbf{u}}{|\mathbf{e}_1-\mathbf{u}|^{n-\alpha}\,|\mathbf{e}_1+\mathbf{u}|^{n-\beta}} \|f\|_p^p. \end{aligned}$$

Analogously,

$$I_2 = \int_{\mathbf{R}^n} \frac{|\mathbf{u}|^{-\frac{n}{p}} d\mathbf{u}}{|\mathbf{e}_1-\mathbf{u}|^{n-\alpha}\,|\mathbf{e}_1+\mathbf{u}|^{n-\beta}} \|g\|_q^q,$$

and, by (6.6),

$$I_3 \le \frac{n}{|S_n|} \int_{\mathbf{R}^n} \frac{|\mathbf{u}|^{-\frac{n}{q}} d\mathbf{u}}{|\mathbf{e}_1-\mathbf{u}|^{n-\alpha}\,|\mathbf{e}_1+\mathbf{u}|^{n-\beta}} \|f\|_p^p\,\|g\|_q^q.$$

It remains to prove that

$$\int_{\mathbf{R}^n} \frac{|\mathbf{x}|^{-\frac{n}{p}} d\mathbf{x}}{|\mathbf{e}_1-\mathbf{x}|^{n-\alpha}|\mathbf{e}_1+\mathbf{x}|^{n-\beta}} = \int_{\mathbf{R}^n} \frac{|\mathbf{x}|^{-\frac{n}{q}} d\mathbf{x}}{|\mathbf{e}_1-\mathbf{x}|^{n-\alpha}|\mathbf{e}_1+\mathbf{x}|^{n-\beta}}.$$

We transform the left integral in polar coordinates using $\mathbf{x} = t\theta$, $t \ge 0$, $\theta \in S_n$ and the substitution $t = \frac{1}{u}$ to obtain:

$$\begin{aligned} &\int_{\mathbf{R^n}} \frac{|\mathbf{x}|^{-\frac{\mathbf{n}}{\mathbf{p}}} \mathbf{dx}}{|\mathbf{e_1}-\mathbf{x}|^{\mathbf{n}-\alpha}|\mathbf{e_1}+\mathbf{x}|^{\mathbf{n}-\beta}} \\ &\qquad = \int_{S_n} d\theta \int_{\mathbf{0}}^{\infty} \frac{\mathbf{t}^{-\frac{\mathbf{n}}{\mathbf{p}}}\mathbf{t}^{\mathbf{n}-\mathbf{1}}\mathbf{dt}}{|\mathbf{e_1}-\mathbf{t}\theta|^{\mathbf{n}-\alpha}|\mathbf{e_1}+\mathbf{t}\theta|^{\mathbf{n}-\beta}} \\ &\qquad = \int_{S_n} d\theta \int_0^{\infty} \frac{t^{-\frac{n}{p}} t^{n-1} dt}{(1+t^2-2t\langle \mathbf{e}_1,\theta\rangle)^{\frac{n-\alpha}{2}}(1+t^2+2t\langle \mathbf{e}_1,\theta\rangle)^{\frac{n-\beta}{2}}} \\ &\qquad = \int_{S_n} d\theta \int_0^{\infty} \frac{u^{\frac{n}{p}-\alpha-\beta} u^{n-1} du}{(1+u^2-2u\langle \mathbf{e}_1,\theta\rangle)^{\frac{n-\alpha}{2}}(1+u^2+2u\langle \mathbf{e}_1,\theta\rangle)^{\frac{n-\beta}{2}}} \\ &\qquad = \int_{\mathbf{R}^n} \frac{|\mathbf{x}|^{-\frac{n}{q}} d\mathbf{x}}{|\mathbf{e}_1-\mathbf{x}|^{n-\alpha}|\mathbf{e}_1+\mathbf{x}|^{n-\beta}}. \end{aligned}$$

To complete the proof, we need to consider the general case, that is, for arbitrary nonnegative functions f and g. Since $\mathbf{x} \mapsto |\mathbf{x}|^{n-\alpha}$, $\mathbf{x} \mapsto |\mathbf{x}|^{n-\beta}$ are symmetric-decreasing functions vanishing at infinity, the general rearrangement inequality implies that

$$\int_{\mathbf{R}^{2n}} \frac{f(\mathbf{x})g(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}} d\mathbf{x}d\mathbf{y} \le \int_{\mathbf{R}^{2n}} \frac{f^*(\mathbf{x})g^*(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}} d\mathbf{x}d\mathbf{y}. \tag{6.11}$$

Clearly, by (6.10), the right-hand side of (6.11) is not greater than

$$\left(\frac{n}{|S_n|}\right)^{1-\lambda} C(p,q;\alpha,\beta;n)\|f^*\|_p\|g^*\|_q = \left(\frac{n}{|S_n|}\right)^{1-\lambda} C(p,q;\alpha,\beta;n)\|f\|_p\|g\|_q, \tag{6.12}$$

where $C(p,q;\alpha,\beta;n)$ is the constant from the right-hand side of (6.7). To achieve equality in (6.12), we used the fact that the symmetric-decreasing rearrangement is norm preserving.

On the other hand, by putting the function

$$g(\mathbf{y}) = \left[\int_{\mathbf{R}^n} \frac{f(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}|\mathbf{x}+\mathbf{y}|^{n-\beta}} d\mathbf{x}\right]^{\frac{q'}{q}}$$

in the inequality (6.7) we obtain (6.8). The equivalence of the inequalities (6.7) and (6.8) can be shown in the same way as in Theorem 6.1. □

The case $n = 1$ of the previous theorem is interesting as for that case the constant $C(p, q; \alpha, \beta; n)$ can be expressed in terms of the hypergeometric function. More precisely, using the definition of hypergeometric functions (for more details see [1]) it is easy to see that the following identity holds for $0 < d_1, d_2, d_3 < 1,\ d_1 + d_2 + d_3 > 1$:

$$\begin{aligned}
&\int_{\mathbf{R}} |t|^{-d_2}|1-t|^{-d_3}|1+t|^{-d_1} dt \\
&= B(1-d_2, 1-d_3)F(d_1, 1-d_2; 2-d_2-d_3; -1) \\
&\qquad + B(1-d_2, 1-d_1)F(d_3, 1-d_2; 2-d_2-d_1; -1) \\
&\qquad + B(d_1+d_2+d_3-1, 1-d_3)F(d_1, d_1+d_2+d_3-1; d_1+d_2; -1) \\
&\qquad + B(d_1+d_2+d_3-1, 1-d_1)F(d_3, d_1+d_2+d_3-1; d_3+d_2; -1).
\end{aligned}$$

Hence, for $n = 1$ we have

Corollary 6.3. *Let α and β be real parameters satisfying $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta = \frac{1}{p} + \frac{1}{q} > 1$. If f and g are nonnegative functions such that $f \in L^p(\mathbf{R})$, $g \in L^q(\mathbf{R})$, then the following inequalities hold and are equivalent*

$$\int_{\mathbf{R}^2} \frac{f(x)g(y)}{|x-y|^{1-\alpha}|x+y|^{1-\beta}} dxdy \le 2^{\lambda-1} C(p, q; \alpha, \beta)\|f\|_p\|g\|_q, \tag{6.13}$$

and

$$\left\{\int_{\mathbf{R}} \left[\int_{\mathbf{R}} \frac{f(x)dx}{|x-y|^{1-\alpha}|x+y|^{1-\beta}}\right]^{q'} dy\right\}^{\frac{1}{q'}} \le 2^{\lambda-1} C(p, q; \alpha, \beta)\|f\|_p, \tag{6.14}$$

where

$$\begin{aligned}
C(p, q; \alpha, \beta) &= B\left(\frac{1}{q'}, \alpha\right) F\left(1-\beta, \frac{1}{q'}; \frac{1}{q'}+\alpha; -1\right) \\
&\quad + B\left(\frac{1}{q'}, \beta\right) F\left(1-\alpha, \frac{1}{q'}; \frac{1}{q'}+\beta; -1\right) \\
&\quad + B\left(\frac{1}{p'}, \alpha\right) F\left(1-\beta, \frac{1}{p'}; \frac{1}{p'}+\alpha; -1\right) \\
&\quad + B\left(\frac{1}{p'}, \beta\right) F\left(1-\alpha, \frac{1}{p'}; \frac{1}{p'}+\beta; -1\right),
\end{aligned} \tag{6.15}$$

$B(\cdot,\cdot)$ is the usual (one-dimensional) beta function and $F(d_1, d_2; d_3; z)$ is the hypergeometric function.

The following corollary should be compared with Theorem 6.1.

Corollary 6.4. *If f and g are nonnegative functions such that $f \in L^p(\mathbf{R})$ and $g \in L^q(\mathbf{R})$, then the following inequalities hold and are equivalent*

$$\int_{\mathbf{R}^2} \frac{f(x)g(y)dxdy}{|x^2-y^2|^{\frac{\lambda}{2}}} \le 2^{\lambda-1}\left[B\left(1-\frac{\lambda}{2}, \frac{1}{2p'}\right) + B\left(1-\frac{\lambda}{2}, \frac{1}{2q'}\right)\right]\|f\|_p\|g\|_q. \tag{6.16}$$

and

$$\left\{\int_{\mathbf{R}}\left[\int_{\mathbf{R}}\frac{f(x)dx}{|x^2-y^2|^{\frac{\lambda}{2}}}\right]^{q'}dy\right\}^{\frac{1}{q'}} \le 2^{\lambda-1}\left[B\left(1-\frac{\lambda}{2},\frac{1}{2p'}\right)+B\left(1-\frac{\lambda}{2},\frac{1}{2q'}\right)\right]\|f\|_p, \tag{6.17}$$

Proof. Set $\alpha=\beta=1-\frac{\lambda}{2}$ in the previous corollary. □

Note that inequalities (6.16) and (6.17) could not be obtained from Theorem 6.1. In other words there are no such α and β for which the kernel in inequalities (6.4) and (6.5) reduces to $|x^2-y^2|^{-\frac{\lambda}{2}}$.

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