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MONOTONICITY AND CONVEXITY FOR THE GAMMA FUNCTION

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ABSTRACT. Let a and b be given real numbers with $0 \le a < b < a+1$. Then the function $\theta_{a,b}(x) = [\Gamma(x+b)/\Gamma(x+a)]^{1/(b-a)} - x$ is strictly convex and decreasing on $(-a, \infty)$ with $\theta_{a,b}(\infty) = \frac{a+b-1}{2}$ and $\theta_{a,b}(-a) = a$, where Γ denotes the Euler's gamma function.

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1. INTRODUCTION

Kazarinoff [10] proved that the function $\theta(n)$,

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{1}{\sqrt{\pi(n+\theta(n))}},$$

satisfies

(1.1) $$\frac{1}{4} < \theta(n) < \frac{1}{2}, \quad n \in \mathbb{N}.$$

More generally, set

$$\theta(x) = \left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 - x, \quad x > -\frac{1}{2},$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t \quad (x > 0)$$

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is the Euler's gamma function. Watson [15] proved that the function θ is strictly decreasing on $(-1/2, \infty)$. Applying this result, together with the observation that $\theta(\infty) = 1/4$, $\theta(-1/2) = 1/2$ and $\theta(1) = 4\pi^{-1} - 1$, we obviously imply sharper inequalities:

$$\frac{1}{4} < \theta(x) < \frac{1}{2} \quad \text{for} \quad x > -\frac{1}{2}, \tag{1.2}$$

$$\frac{1}{4} < \theta(x) \leq 4\pi^{-1} - 1 \quad \text{for} \quad x \geq 1. \tag{1.3}$$

In particular, take in (1.3) $x = n$, we get

$$\frac{1}{\sqrt{\pi(n + 4\pi^{-1} - 1)}} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{\pi(n + 1/4)}}, \tag{1.4}$$

and the constants $4\pi^{-1} - 1$ and $1/4$ are the best possible.

The inequality (1.4) is called Wallis' inequality. For more information on Wallis' inequality, please refer to the paper [6] and the references therein.

H. Alzer [2] proved that the function θ is strictly decreasing on $[0, \infty)$. Applying this result, he showed that for all integers $n \geq 1$,

$$\sqrt{\frac{n + A}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n + B}{2\pi}} \tag{1.5}$$

with the best possible constants

$$A = \frac{1}{2} \quad \text{and} \quad B = \frac{\pi}{2} = 0.57079\ldots,$$

where $\Omega_n = \pi^{n/2}/\Gamma(1+n/2)$ denotes the volume of the unit ball in $\mathbf{R}^n$. (1.5) is an improvement of the following result given by Borewardt [5, p. 253]

$$\sqrt{\frac{n}{2\pi}} \leq \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+1}{2\pi}}. \tag{1.6}$$

If we denote by

$$\theta_{a,b}(x) = \left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{\frac{1}{b-a}} - x,$$

then we conclude from the representations [1, p. 257]

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\big(x^{-2}\big) \quad (x \to \infty), \tag{1.7}$$

that

$$\theta_{a,b}(x) = x\left\{\frac{1}{x}\left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(b-a)} - 1\right\} \to \frac{a+b-1}{2} \quad \text{as} \quad x \to \infty. \tag{1.8}$$

Hence it is of interest to investigate the possible monotonic character of the function $x \mapsto \theta_{a,b}(x)$.

Theorem 1.1. *Let a and b be given real numbers with $0 \leq a < b < a+1$. Then the function $\theta_{a,b}(x) = [\Gamma(x+b)/\Gamma(x+a)]^{1/(b-a)} - x$ is strictly convex and decreasing on $(-a, \infty)$.*

Since $\theta_{a,b}(x) = \theta_{b,a}(x)$, it is clear that $x \mapsto \theta_{a,b}(x)$ is strictly convex and decreasing on $(-b, \infty)$ for $0 \leq b < a < b+1$.

From $\theta_{a,b}(\infty) = \frac{a+b-1}{2}$, $\theta_{a,b}(-a) = a$ and the monotonicity of $x \mapsto \theta_{a,b}(x)$, we obtain the following

Corollary 1.2. *Let a and b be given real numbers with $0 \le a < b < a+1$, then for $x > -a$,*

$$\left(x + \frac{a+b-1}{2}\right)^{b-a} < \frac{\Gamma(x+b)}{\Gamma(x+a)} < (x+a)^{b-a}. \tag{1.9}$$

A proof of the theorem above has been shown in [8], here we provide another proof. The ratio of two gamma functions has been investigated intensively by many authors. For example, Gautschi [9] proved the following inequalities

$$x^{1-x} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}, \quad 0 < s < 1,\ x = 1, 2, \dots. \tag{1.10}$$

Kershaw [11] has given some improvements of these inequalities such as

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x - \frac{1}{2} + \sqrt{x + \frac{1}{4}}\right)^{1-s} \tag{1.11}$$

for real $x > 0$ and $0 < s < 1$.

Inequalities for the ratio $\Gamma(x+1)/\Gamma(1+\lambda)$ $(x > 0; \lambda \in (0,1))$ have a remarkable application, they can be used to obtain estimates for ultraspherical polynomials. The ultraspherical polynomials are defined by

$$P_n^{(\lambda)}(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda)\Gamma(k+1)\Gamma(n-2k+1)} (2x)^{n-2k},$$

where $n \ge 0$ is an integer and $\lambda > 0$ is a real number.

In 1931, S. Bernstein [4] proved the following inequality for ultraspherical polynomials: If $0 < \lambda < 1,\ n \ge 1$, and $0 < \theta < \pi$, then

$$(\sin\theta)^{\lambda} \left| P_n^{(\lambda)} \cos\theta \right| < \frac{2^{1-\lambda}}{\Gamma(\lambda)} n^{\lambda-1}, \tag{1.12}$$

where the constant $2^{1-\lambda}/\Gamma(\lambda)$ cannot be replaced by a smaller one.

We note that inequality (1.12) with $\lambda = 1/2$ leads to a well-known inequality for the Legendre polynomials $P_n = P_n^{(1/2)}$:

$$(\sin\theta)^{1/2} |P_n \cos\theta| < \left(\frac{2}{\pi}\right)^{1/2} n^{-1/2}. \tag{1.13}$$

Several authors presented remarkable refinements of (1.12). They proved that in (1.12) the factor $n^{\lambda-1}$ can be replaced by smaller expressions. The left-hand inequality of (1.11) was also considered in 1984 by L. Lorch [13]. He obtained the following results for integer $x > 0$:

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} > \left(x + \frac{s}{2}\right)^{1-s} \quad \text{for} \quad 0 < s < 1 \quad \text{or} \quad s > 2, \tag{1.14}$$

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} > \left(x + \frac{s}{2}\right)^{1-s} \quad \text{for} \quad 1 < s < 2. \tag{1.15}$$

Lorch [13] used (1.14) to prove a sharpened inequality for ultraspherical polynomials:

$$(\sin\theta)^{\lambda} |P_n^{(\lambda)} \cos\theta| < \frac{2^{1-\lambda}}{\Gamma(\lambda)} (n+\lambda)^{\lambda-1}. \tag{1.16}$$

In 1992, A. Laforgia [12] proved in (1.12) that the term $n^{\lambda-1}$ can be replaced by $\Gamma(n+\lambda)/\Gamma(n+1)$. Since $\Gamma(n+\lambda)/\Gamma(n+1) < n^{\lambda-1}$, see [9], this provides another refinement of Bernstein's inequality. In 1994, Y. Chow et al. [7] showed that (1.12) holds with $\Gamma(n+2\lambda)(\Gamma(n+1))^{-1}(n+\lambda)^{-\lambda}$ instead of $n^{\lambda-1}$. This sharpens Lorch's result for all $\lambda \in (0, 1/2)$, since the inequality

$$\frac{\Gamma(n+2\lambda)}{\Gamma(n+1)(n+\lambda)^{\lambda}} < (n+\lambda)^{\lambda-1}$$

is valid for all $\lambda \in (0, 1/2)$. In 1997, H. Alzer [3] showed the following inequality

$$(\sin\theta)^{\lambda}|P_n^{(\lambda)}\cos\theta| < \frac{2^{1-\lambda}}{\Gamma(\lambda)} \cdot \frac{\Gamma\left(n+\frac{3}{2}\lambda\right)}{\Gamma\left(n+1+\frac{1}{2}\lambda\right)}. \tag{1.17}$$

Inequality (1.17) refines the results given by S. Bernstein, L. Lorch and A. Laforgia.

2. PROOF OF THEOREM 1.1

Easy computation yields

$$\theta'_{a,b}(x) = \frac{1}{b-a}[\psi(x+b) - \psi(x+a)](\theta_{a,b}(x)+x) - 1,$$
$$\frac{(b-a)\theta''_{a,b}(x)}{\theta_{a,b}(x)+x} = \psi'(x+b) - \psi'(x+a) + \frac{1}{b-a}[\psi(x+b) - \psi(x+a)]^2.$$

Using the representations [14, p. 16]

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-xt}}{1-e^{-t}}\,\mathrm{d}t,$$
$$\psi'(x) = \int_0^{\infty} \frac{t}{1-e^{-t}} e^{-xt}\,\mathrm{d}t$$

for $x > 0$, $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant, it follows that

$$\frac{(b-a)\theta''_{a,b}(x)}{\theta_{a,b}(x)+x} = -\int_0^{\infty} t\delta(t)e^{-(x+a)t}\,\mathrm{d}t + \frac{1}{b-a}\left(\int_0^{\infty}\delta(t)e^{-(x+a)t}\,\mathrm{d}t\right)^2,$$

where

$$\delta(t) = \frac{1-e^{-(b-a)t}}{1-e^{-t}} \quad \text{and} \quad \delta(0) = b-a.$$

By using the convolution theorem for Laplace transforms, we have

$$\begin{aligned}\frac{(b-a)\theta''_{a,b}(x)}{\theta_{a,b}(x)+x} &= -\int_0^{\infty} t\delta(t)e^{-(x+a)t}\,\mathrm{d}t \\ &\quad + \frac{1}{b-a}\int_0^{\infty}\left[\int_0^t \delta(s)\delta(t-s)\,\mathrm{d}s\right]e^{-(x+a)t}\,\mathrm{d}t \\ &= \int_0^{\infty} e^{-(x+a)t}\omega(t)\,\mathrm{d}t,\end{aligned} \tag{2.1}$$

where

$$\omega(t) = \int_0^t \left[\frac{1}{b-a}\delta(s)\delta(t-s) - \delta(t)\right]\mathrm{d}s. \tag{2.2}$$

Now we are in a position to prove that

$$\frac{1}{b-a}\delta(s)\delta(t-s) - \delta(t) > 0 \quad \text{for} \quad t > s > 0. \tag{2.3}$$

Define for $t > s > 0$,

$$\varphi(t) = \ln\frac{\delta(s)}{b-a} + \ln\delta(t-s) - \ln\delta(t).$$

Elementary calculations reveal that

$$\varphi'(t) = \frac{\delta'(t-s)}{\delta(t-s)} - \frac{\delta'(t)}{\delta(t)},$$
$$\left(\frac{\delta'(t)}{\delta(t)}\right)' = (\ln\delta(t))'' = \frac{e^t}{(e^t-1)^2} - \frac{(b-a)^2 e^{(b-a)t}}{\left[e^{(b-a)t}-1\right]^2}.$$

Defined for $r \in (0,1)$,

$$g(r) = \frac{r^2 e^{rt}}{(e^{rt}-1)^2} = \frac{1}{t^2}\left(\frac{rt/2}{\sinh(rt/2)}\right)^2.$$

Since $x \mapsto \frac{\sinh x}{x}$ is strictly increasing with $x \in (0,\infty)$, we get g is strictly decreasing with $r \in (0,1)$. This implies that $\left(\frac{\delta'(t)}{\delta(t)}\right)' < 0$ for $t > 0$ and $0 < b-a < 1$, and then, $\varphi'(t) > 0$ and $\varphi(t) > \varphi(s) = 0$. This means (2.3) holds, and thus, $\theta''_{a,b}(x) > 0$ $(x > -a)$ follows from (2.1), (2.2) and (2.3).

From the representations (1.7) and

$$\psi(x) = \ln x - \frac{1}{2x} + O\big(x^{-2}\big) \quad (x \to \infty),$$

(see [1, p. 259]), we conclude that

$$\lim_{x\to\infty} x^{a-b}\frac{\Gamma(x+b)}{\Gamma\big(x+a\big)} = 1, \tag{2.4}$$

$$\lim_{x\to\infty} x\left[\psi(x+b) - \psi\Big(x+a\Big)\right] = \frac{1}{b-a}. \tag{2.5}$$

From (2.4), (2.5) and the monotonicity of the function $x \mapsto \theta'_{a,b}(x)$, we imply

$$\begin{aligned}\theta'_{a,b}(x) &< \lim_{x\to\infty}\theta'_{a,b}(x)\\ &= \lim_{x\to\infty}\frac{1}{b-a}\left[x^{a-b}\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{\frac{1}{b-a}} x\left[\psi(x+b)-\psi(x+a)\right] - 1\\ &= 0.\end{aligned}$$

The proof is complete.

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