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AN INTEGRAL INEQUALITY FOR 3-CONVEX FUNCTIONS

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ABSTRACT. In this paper, an integral inequality and an application of it, that imply the Chebyshev functional for two 3-convex (3-concave) functions, are given.

Key words and phrases: Chebyshev functional, Convex functions, Integral inequality.

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1. INTRODUCTION

For two Lebesgue functions $f, g : [a, b] \to \mathbb{R}$, consider the Chebyshev functional

$$C(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \cdot \int_a^b g(x)dx.$$

In 1972, A. Lupaş [2] showed that if f, g are convex functions on the interval $[a, b]$, then

$$C(f,g) \geq \frac{12}{(b-a)^4}\int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) g(x)dx, \tag{1.1}$$

with equality when at least one of the functions f, g is a linear function on $[a, b]$. He proved this result using the following lemma:

Lemma 1.1. *If $f, g : [a, b] \to \mathbb{R}$ are convex functions on the interval $[a, b]$, then*

$$\begin{aligned}[F(e^2) - F(e)^2]F(fg) - F(e^2)F(f)F(g) \\ \geq F(ef)F(eg) - F(e)[F(f)F(eg) + F(ef)F(g)],\end{aligned} \tag{1.2}$$

where F is an isotonic positive linear functional, defined by one of the following relations:

$$F(f) := \frac{1}{b-a}\int_a^b f(x)dx, \quad F(f) := \frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}, \quad F(f) := \sum_{i=1}^n p_i f(x_i) \tag{1.3}$$

$(x_i \in [a, b]; p_i \geq 0, i = 1, 2, \ldots, n, \sum_{i=1}^n p_i = 1)$, $p : [a, b] \to \mathbb{R}$ is a positive, integrable function on $[a, b]$ and $e(x) = x$, $x \in [a, b]$. If f or g is a linear function, then the equality holds in (1.2).

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In this note, we provide a lower bound for the Chebyshev functional in the case of two 3-convex (3-concave) functions f and g.

2. RESULTS

Note that the inequality (1.2) can be written in the form:

$$\begin{vmatrix} 1 & F(e) & F(g) \\ F(e) & F(e^2) & F(eg) \\ F(f) & F(ef) & F(fg) \end{vmatrix} \geq 0. \tag{2.1}$$

The following lemma holds.

Lemma 2.1. *If $f, g : [a,b] \to \mathbb{R}$ are 3-convex (3-concave) functions on the interval $[a,b]$, then*

$$\begin{vmatrix} 1 & F(e) & F(e^2) & F(g) \\ F(e) & F(e^2) & F(e^3) & F(eg) \\ F(e^2) & F(e^3) & F(e^4) & F(e^2g) \\ F(f) & F(ef) & F(e^2f) & F(fg) \end{vmatrix} \geq 0, \tag{2.2}$$

where $e^i(x) = x^i, x \in [a,b], i = \overline{1,4}$ and F is defined by (1.3).

If f is 3-convex (3-concave) and g is 3-concave (3-convex) then the reverse of the inequality in (2.2) *holds.*

If f or g is a polynomial function of degree at most two, then the equality holds in (2.2).

Proof. Let $[x,y,z,t;f]$ be the divided difference of a certain function f. If f and g are 3-convex (3-concave) on the interval $[a,b]$, then we have

$$[x,y,z,t;f] \cdot [x,y,z,t;g] \geq 0, \tag{2.3}$$

for all distinct points x,y,z,t from $[a,b]$.

When f is 3-convex (3-concave) and g is 3-concave (3-convex) then the reverse of the inequality in (2.3) holds.

In the following we prove (2.2) in the case when both functions f and g are 3-convex (3-concave). The inequality (2.3) is equivalent to

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ x^2 & y^2 & z^2 & t^2 \\ f(x) & f(y) & f(z) & f(t) \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ x^2 & y^2 & z^2 & t^2 \\ g(x) & g(y) & g(z) & g(t) \end{vmatrix} \geq 0, \tag{2.4}$$

with true equality holding when at least one of f and g is a polynomial function of degree at most two.

Note that the function F defined by (1.3) has the property $F(1) = 1$. In order to put in evidence the variable u, we write F_u instead of F.

Now, using the fact that F is a linear positive functional, by applying successively on (2.4) the functionals F_x, F_y, F_z and then F_t, we obtain the inequality (2.2). For instance, if

$$A = A(x,y,z,t,f,g) := \begin{vmatrix} 1 & 1 & 1 \\ y & z & t \\ y^2 & z^2 & t^2 \end{vmatrix}^2 \cdot f(x)g(x),$$

then

$$F_x(A) = \begin{vmatrix} 1 & 1 & 1 \\ y & z & t \\ y^2 & z^2 & t^2 \end{vmatrix}^2 \cdot F(fg),$$

$$F_tF_zF_yF_x(A) = 6 \cdot \begin{vmatrix} 1 & F(e) & F(e^2) \\ F(e) & F(e^2) & F(e^3) \\ F(e^2) & F(e^3) & F(e^4) \end{vmatrix} \cdot F(fg)$$

and if

$$B = B(x,y,z,t,f,g) := \begin{vmatrix} 1 & 1 & 1 \\ y & z & t \\ y^2 & z^2 & t^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ x & z & t \\ x^2 & z^2 & t^2 \end{vmatrix} \cdot f(x)g(y),$$

then

$$F_x(B) = \begin{vmatrix} 1 & 1 & 1 \\ y & z & t \\ y^2 & z^2 & t^2 \end{vmatrix} \cdot g(y) \cdot \begin{vmatrix} F(f) & 1 & 1 \\ F(ef) & z & t \\ F(e^2f) & z^2 & t^2 \end{vmatrix},$$

$$\begin{aligned} F_tF_zF_yF_x(B) = 2 \cdot &\begin{vmatrix} 1 & F(e) & F(g) \\ F(e) & F(e^2) & F(eg) \\ F(e^2) & F(e^3) & F(e^2g) \end{vmatrix} \cdot F(e^2f) \\ &+ 2 \cdot \begin{vmatrix} 1 & F(g) & F(e^2) \\ F(e) & F(eg) & F(e^3) \\ F(e^2) & F(e^2g) & F(e^4) \end{vmatrix} \cdot F(ef) \\ &+ 2 \cdot \begin{vmatrix} F(g) & F(e) & F(e^2) \\ F(eg) & F(e^2) & F(e^3) \\ F(e^2g) & F(e^3) & F(e^4) \end{vmatrix} \cdot F(f). \end{aligned}$$

□

Theorem 2.2. *If f, g are 3-convex (3-concave) functions on the interval $[a,b]$, then*

$$C(f,g) \geq \frac{180}{(b-a)^6} \int_a^b q(x)f(x)dx \cdot \int_a^b q(x)g(x)dx + \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) g(x)dx, \tag{2.5}$$

where

$$q(x) = \left(x - \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right)\left(x - \frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right).$$

If f is 3-convex (3-concave) and g is 3-concave (3-convex) then the reverse of the inequality in (2.5) *holds.*

The equality in (2.5) *holds when at least one of f or g is a polynomial function of degree at most two on $[a,b]$.*

Proof. We choose

$$F(f) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{2.6}$$

Then

$$F(e) = \frac{a+b}{2}, \quad F(e^2) = \frac{a^2+ab+b^2}{3}, \tag{2.7}$$
$$F(e^3) = \frac{a^3+a^2b+ab^2+b^3}{4}, \quad F(e^4) = \frac{a^4+a^3b+a^2b^2+ab^3+b^4}{5}.$$

Note that the inequality (2.2) can be written as

$$\begin{vmatrix} 1 & F(e) & F(e^2) \\ F(e) & F(e^2) & F(e^3) \\ F(e^2) & F(e^3) & F(e^4) \end{vmatrix} \cdot F(fg) - \begin{vmatrix} 1 & F(e) \\ F(e) & F(e^2) \end{vmatrix} \cdot F(e^2f)F(e^2g) \tag{2.8}$$
$$- \begin{vmatrix} 1 & F(e^2) \\ F(e^2) & F(e^4) \end{vmatrix} \cdot F(ef)F(eg) - \begin{vmatrix} F(e^2) & F(e^3) \\ F(e^3) & F(e^4) \end{vmatrix} \cdot F(f)F(g)$$
$$+ \begin{vmatrix} 1 & F(e) \\ F(e^2) & F(e^3) \end{vmatrix} \cdot [F(e^2f)F(eg) + F(ef)F(e^2g)]$$
$$- \begin{vmatrix} F(e) & F(e^2) \\ F(e^2) & F(e^3) \end{vmatrix} \cdot [F(e^2f)F(g) + F(f)F(e^2g)]$$
$$+ \begin{vmatrix} F(e) & F(e^2) \\ F(e^3) & F(e^4) \end{vmatrix} \cdot [F(ef)F(g) + F(f)F(eg)] \geq 0.$$

By calculation, we find

$$\begin{vmatrix} 1 & F(e) & F(e^2) \\ F(e) & F(e^2) & F(e^3) \\ F(e^2) & F(e^3) & F(e^4) \end{vmatrix} = \frac{(b-a)^6}{2160}, \tag{2.9}$$

$$\begin{vmatrix} 1 & F(e^2) \\ F(e^2) & F(e^4) \end{vmatrix} = \frac{(b-a)^2(4a^2+7ab+4b^2)}{45}, \tag{2.10}$$

$$\begin{vmatrix} F(e^2) & F(e^3) \\ F(e^3) & F(e^4) \end{vmatrix} = \frac{(b-a)^2(a^4+4a^3b+10a^2b^2+4ab^3+b^4)}{240}, \tag{2.11}$$

$$\begin{vmatrix} 1 & F(e) \\ F(e^2) & F(e^3) \end{vmatrix} = \frac{(b-a)^2(a+b)}{12}, \tag{2.12}$$

$$\begin{vmatrix} F(e) & F(e^2) \\ F(e^2) & F(e^3) \end{vmatrix} = \frac{(b-a)^2(a^2+4ab+b^2)}{72}, \tag{2.13}$$

$$\begin{vmatrix} F(e) & F(e^2) \\ F(e^3) & F(e^4) \end{vmatrix} = \frac{(b-a)^2(a^3+4a^2b+4ab^2+b^3)}{60}. \tag{2.14}$$

The relations (2.7) – (2.14) give us

$$(2.15)\quad \frac{(b-a)^5}{2160}\int_a^b f(x)g(x)dx - \frac{a^4+4a^3b+10a^2b^2+4ab^3+b^4}{240}\int_a^b f(x)dx\int_a^b g(x)dx$$
$$\geq \frac{1}{12}\int_a^b x^2f(x)dx\int_a^b x^2g(x)dx + \frac{4a^2+7ab+4b^2}{45}\int_a^b xf(x)dx\int_a^b xg(x)dx$$
$$-\frac{a+b}{12}\left[\int_a^b x^2f(x)dx\int_a^b xg(x)dx + \int_a^b xf(x)dx\int_a^b x^2g(x)dx\right]$$
$$+\frac{a^2+4ab+b^2}{72}\left[\int_a^b x^2f(x)dx\int_a^b g(x)dx + \int_a^b f(x)dx\int_a^b x^2g(x)dx\right]$$
$$-\frac{a^3+4a^2b+4ab^2+b^3}{60}\left[\int_a^b xf(x)dx\int_a^b g(x)dx + \int_a^b f(x)dx\int_a^b xg(x)dx\right],$$

or

$$(2.16)\quad C(f,g) \geq \frac{180}{(b-a)^6}\left\{\int_a^b x^2f(x)dx\int_a^b x^2g(x)dx\right.$$
$$+\frac{4(4a^2+7ab+4b^2)}{15}\int_a^b xf(x)dx\int_a^b xg(x)dx$$
$$+\frac{2a^4+10a^3b+21a^2b^2+10ab^3+2b^4}{540}\cdot\int_a^b f(x)dx\int_a^b g(x)dx$$
$$-\frac{a+b}{12}\left[\int_a^b x^2f(x)dx\int_a^b xg(x)dx + \int_a^b xf(x)dx\int_a^b x^2g(x)dx\right]$$
$$+\frac{a^2+4ab+b^2}{72}\left[\int_a^b x^2f(x)dx\int_a^b g(x)dx + \int_a^b f(x)dx\int_a^b x^2g(x)dx\right]$$
$$-\frac{a^3+4a^2b+4ab^2+b^3}{60}\left[\int_a^b xf(x)dx\int_a^b g(x)dx\right.$$
$$\left.\left.+\int_a^b f(x)dx\int_a^b xg(x)dx\right]\right\}.$$

The last inequality can be written as

$$(2.17)\quad C(f,g) \geq \frac{180}{(b-a)^6}\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)dx\cdot\int_a^b\left(x-\frac{a+b}{2}\right)^2 g(x)dx$$
$$-\frac{15}{(b-a)^4}\cdot\left[\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)dx\cdot\int_a^b g(x)dx+\right.$$
$$\left.+\int_a^b f(x)dx\cdot\int_a^b\left(x-\frac{a+b}{2}\right)^2 g(x)dx\right]$$
$$+\frac{5}{4(b-a)^2}\int_a^b f(x)dx\cdot\int_a^b g(x)dx$$
$$+\frac{12}{(b-a)^4}\int_a^b\left(x-\frac{a+b}{2}\right)f(x)dx\cdot\int_a^b\left(x-\frac{a+b}{2}\right)g(x)dx,$$

which is equivalent to (2.5). □

Corollary 2.3. *Let f and g be as in Theorem 2.2 and assume that*

$$f(x) = -f(a+b-x) \tag{2.18}$$

or

$$g(x) = -g(a+b-x) \tag{2.19}$$

for all x from $[a,b]$. Then Lupaş' inequality holds.

Proof. Note that the function denoted by q in Theorem 2.2 is symmetric about $x = \frac{a+b}{2}$, namely

$$q(x) = q(a+b-x),$$

for all x from $[a,b]$.

Assume that (2.18) is satisfied. Then we have

$$\begin{aligned} \int_a^b q(x)f(x)dx &= \frac{1}{2}\int_a^b q(x)[f(x) - f(a+b-x)]dx \\ &= \frac{1}{2}\int_a^b q(x)f(x)dx - \frac{1}{2}\int_a^b q(a+b-x)f(a+b-x)dx \\ &= \frac{1}{2}\int_a^b q(x)f(x)dx - \frac{1}{2}\int_a^b q(t)f(t)dt = 0. \end{aligned} \tag{2.20}$$

From (2.5) and (2.20), we deduce (1.1). □

Note that the condition (2.3) is important. The same results are valid if we suppose that this (or its reverse) is satisfied. Thus, we obtain a more general result:

Theorem 2.4. *If the functions f and g are integrable on the interval $[a,b]$ and satisfy* (2.3) *(or its reverse), then we have* (2.5) *(or its reverse).*

The equality in (2.5) *holds when at least one of f or g is a polynomial function of degree at most two on $[a,b]$.*

Corollary 2.5. *If the function f is integrable on $[a,b]$, then we have*

$$\begin{aligned} &(b-a)\int_a^b f^2(x)dx - \left(\int_a^b f(x)dx\right)^2 \\ &\qquad \geq \frac{12}{(b-a)^2}\left(\int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx\right)^2 + \frac{180}{(b-a)^4}\left(\int_a^b q(x)f(x)dx\right)^2, \end{aligned} \tag{2.21}$$

where $q(x)$ is defined in Theorem 2.2.

Proof. Considering $g(x) = f(x)$ in (2.5), we find the inequality (2.21). □

Remark 1. The inequality (2.21) is better than the well-known inequality

$$(b-a)\int_a^b f^2(x)dx \geq \left(\int_a^b f(x)dx\right)^2, \tag{2.22}$$

valid for all integrable functions f on $[a,b]$.

Corollary 2.6. *If the functions f, g satisfy the following conditions:*

(i) *f, g are 3-convex (3-concave) functions on $[a,b]$;*
(ii) *f, g are differentiable functions on $[a,b]$,*

then we have

$$\int_a^b f'(x)g'(x)dx \ge \frac{[f(b)-f(a)][g(b)-g(a)]}{b-a} + \frac{12}{b-a}\left(\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right) \times \left(\frac{1}{b-a}\int_a^b g(x)dx - \frac{g(a)+g(b)}{2}\right). \tag{2.23}$$

Proof. In (1.1), we use the fact that if f, g are 3-convex functions on $[a, b]$, then f', g' are convex functions on $[a, b]$. We have

$$\frac{1}{b-a}\int_a^b f'(x)g'(x)dx - \frac{1}{(b-a)^2}\int_a^b f'(x)dx \cdot \int_a^b g'(x)dx \ge \frac{12}{(b-a)^4}\int_a^b \left(x-\frac{a+b}{2}\right) f'(x)dx \cdot \int_a^b \left(x-\frac{a+b}{2}\right) g'(x)dx,$$

which is equivalent to (2.23). □

Remark 2. If, in addition, f and g are convex (concave) on $[a, b]$, then the inequality (2.23) is better than the inequality

$$\int_a^b f'(x)\cdot g'(x)dx \ge \frac{[f(b)-f(a)][g(b)-g(a)]}{b-a}, \tag{2.24}$$

which is valid for all convex (concave) functions f, g on $[a, b]$.

Remark 3. Lemma 2.1 can be generalized for n-convex functions, obtaining a result similar to (2.2), from where the inequality

$$\begin{vmatrix} b-a & \frac{b^2-a^2}{2} & \dots & \frac{b^n-a^n}{n} & \int_a^b g(x)dx \\ \frac{b^2-a^2}{2} & \frac{b^3-a^3}{3} & \dots & \frac{b^{n+1}-a^{n+1}}{n+1} & \int_a^b xg(x)dx \\ \dots & \dots & \dots & \dots & \dots \\ \frac{b^n-a^n}{n} & \frac{b^{n+1}-a^{n+1}}{n+1} & \dots & \frac{b^{2n-1}-a^{2n-1}}{2n-1} & \int_a^b x^{n-1}g(x)dx \\ \int_a^b f(x)dx & \int_a^b xf(x)dx & \dots & \int_a^b x^{n-1}f(x)dx & \int_a^b f(x)g(x)dx \end{vmatrix} \ge 0, \tag{2.25}$$

holds for all integer numbers $n \ge 3$.

Some similar results related to the Chebyshev functional are given in [1] – [6].

3. AN APPLICATION

Let f, g be two 3-time differentiable functions defined on a nonempty interval $[a, b]$. Denote

$$m_1 = \inf_{x\in[a,b]} f^{(3)}(x), \qquad M_1 = \sup_{x\in[a,b]} f^{(3)}(x),$$
$$m_2 = \inf_{x\in[a,b]} g^{(3)}(x), \qquad M_2 = \sup_{x\in[a,b]} g^{(3)}(x).$$

Considering the functions $F_1, G_1, F_2, G_2 : [a,b] \to \mathbb{R}$, defined by

$$F_1(x) = \frac{m_1 x^3}{6} - f(x), \quad G_1(x) = \frac{m_2 x^3}{6} - g(x),$$
$$F_2(x) = \frac{M_1 x^3}{6} - f(x), \quad G_2(x) = \frac{M_2 x^3}{6} - g(x),$$

we note that these are 3-differentiable on $[a,b]$ and $F_1^{(3)}(x) \le 0$, $G_1^{(3)}(x) \le 0$, $F_2^{(3)}(x) \ge 0$, $G_2^{(3)}(x) \ge 0$ for all $x \in [a,b]$. Therefore F_1, G_1 are 3-concave on $[a,b]$ and F_2, G_2 are 3-convex on $[a,b]$.

Applying Theorem 2.2 we shall prove the following result:

Theorem 3.1. *Let f, g be two 3-differentiable functions on the nonempty interval $[a,b]$. Then, we have*

$$\left| L(f,g) - \frac{1}{6(b-a)} \int_a^b \left(x - \frac{a+b}{2}\right) r(x)h(x)dx + \frac{(m_1+M_1)(m_2+M_2)}{403200} \cdot (b-a)^6 \right| \le \frac{(M_1-m_1)(M_2-m_2)}{403200} \cdot (b-a)^6, \tag{3.1}$$

where

$$L(f,g) = C(f,g) - \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) g(x)dx - \frac{180}{(b-a)^6} \int_a^b q(x)f(x)dx \cdot \int_a^b q(x)g(x)dx, \tag{3.2}$$

$$h(x) = \frac{m_1+M_1}{2} \cdot g(x) + \frac{m_2+M_2}{2} \cdot f(x), \tag{3.3}$$

$$r(x) = \left(x - \frac{a+b}{2} - \frac{(b-a)\sqrt{15}}{10}\right)\left(x - \frac{a+b}{2} + \frac{(b-a)\sqrt{15}}{10}\right). \tag{3.4}$$

Proof. Applying Theorem 2.2, we have

$$C(F_1,G_1) \ge \frac{180}{(b-a)^6} \int_a^b q(x)F_1(x)dx \cdot \int_a^b q(x)G_1(x)dx + \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) F_1(x)dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) G_1(x)dx, \tag{3.5}$$

$$C(F_2,G_2) \ge \frac{180}{(b-a)^6} \int_a^b q(x)F_2(x)dx \cdot \int_a^b q(x)G_2(x)dx + \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) F_2(x)dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) G_2(x)dx, \tag{3.6}$$

$$C(F_1,G_2) \le \frac{180}{(b-a)^6}\int_a^b q(x)F_1(x)dx \cdot \int_a^b q(x)G_2(x)dx + \frac{12}{(b-a)^4}\int_a^b \left(x-\frac{a+b}{2}\right)F_1(x)dx \cdot \int_a^b \left(x-\frac{a+b}{2}\right)G_2(x)dx, \tag{3.7}$$

$$C(F_2,G_1) \le \frac{180}{(b-a)^6}\int_a^b q(x)F_2(x)dx \cdot \int_a^b q(x)G_1(x)dx + \frac{12}{(b-a)^4}\int_a^b \left(x-\frac{a+b}{2}\right)F_2(x)dx \cdot \int_a^b \left(x-\frac{a+b}{2}\right)G_1(x)dx, \tag{3.8}$$

where $q(x)$ is defined in Theorem 2.2.

By calculation, we find

$$C(F_1,G_1) = C(f,g) + \frac{m_1m_2}{4032}(b-a)^2(9a^4+20a^3b+26a^2b^2+20ab^3+9b^4) - \frac{1}{6(b-a)}\int_a^b x^3[m_1g(x)+m_2f(x)]dx + \frac{a^3+a^2b+ab^2+b^3}{24(b-a)}\int_a^b [m_1g(x)+m_2f(x)]dx, \tag{3.9}$$

$$\frac{12}{(b-a)^4}\int_a^b \left(x-\frac{a+b}{2}\right)F_1(x)dx \cdot \int_a^b \left(x-\frac{a+b}{2}\right)G_1(x)dx = \frac{12}{(b-a)^4}\int_a^b \left(x-\frac{a+b}{2}\right)f(x)dx \cdot \int_a^b \left(x-\frac{a+b}{2}\right)g(x)dx + \frac{m_1m_2}{4800}(b-a)^2(3a^2+4ab+3b^2)^2 - \frac{3a^2+4ab+3b^2}{20(b-a)}\int_a^b \left(x-\frac{a+b}{2}\right)[m_1g(x)+m_2f(x)]dx, \tag{3.10}$$

$$\frac{180}{(b-a)^6}\int_a^b q(x)F_1(x)dx \cdot \int_a^b q(x)G_1(x)dx = \frac{180}{(b-a)^6}\int_a^b q(x)f(x)dx \cdot \int_a^b q(x)g(x)dx + \frac{m_1m_2}{2880}(b-a)^4(a+b)^2 - \frac{a+b}{4(b-a)} \cdot \int_a^b q(x)[m_1g(x)+m_2f(x)]dx. \tag{3.11}$$

From (3.5) and (3.9) – (3.11), we obtain

$$L(f,g) + \frac{m_1m_2}{100800}(b-a)^6 \ge \frac{1}{6(b-a)}\int_a^b \left(x-\frac{a+b}{2}\right)r(x)[m_1g(x)+m_2f(x)]dx. \tag{3.12}$$

In a similar way we can prove that the inequality (3.6) is equivalent to

$$L(f,g) + \frac{M_1M_2}{100800}(b-a)^6 \ge \frac{1}{6(b-a)}\int_a^b \left(x-\frac{a+b}{2}\right)r(x)[M_1g(x)+M_2f(x)]dx, \tag{3.13}$$

the inequality (3.7) is equivalent to

$$L(f,g) + \frac{m_1M_2}{100800}(b-a)^6 \le \frac{1}{6(b-a)}\int_a^b \left(x-\frac{a+b}{2}\right)r(x)[m_1g(x)+M_2f(x)]dx, \tag{3.14}$$

and the inequality (3.8) is equivalent to

$$L(f,g)+\frac{M_1m_2}{100800}(b-a)^6\le\frac{1}{6(b-a)}\int_a^b\left(x-\frac{a+b}{2}\right)r(x)[M_1g(x)+m_2f(x)]dx. \tag{3.15}$$

From (3.12) and (3.13) we deduce

$$L(f,g)-\frac{1}{6(b-a)}\int_a^b\left(x-\frac{a+b}{2}\right)r(x)h(x)dx\ge-\frac{m_1m_2+M_1M_2}{201600}(b-a)^6. \tag{3.16}$$

From (3.14) and (3.15) we find

$$L(f,g)-\frac{1}{6(b-a)}\int_a^b\left(x-\frac{a+b}{2}\right)r(x)h(x)dx\le-\frac{m_1M_2+M_1m_2}{201600}(b-a)^6. \tag{3.17}$$

The inequalities (3.16) and (3.17) prove (3.1). □

Corollary 3.2. *If f,g are 3-time differentiable on $[a,b]$ and symmetric about $x=\frac{a+b}{2}$, then we have*

$$\left|L(f,g)+\frac{(m_1+M_1)(m_2+M_2)}{403200}(b-a)^6\right|\le\frac{(M_1-m_1)(M_2-m_2)}{403200}(b-a)^6. \tag{3.18}$$

Proof. Note that the functions h and r defined on $[a,b]$ by (3.3) and (3.4) are symmetric about $x=\frac{a+b}{2}$. Hence, their product $h\cdot r$ is symmetric about $x=\frac{a+b}{2}$ and

$$\int_a^b\left(x-\frac{a+b}{2}\right)r(x)h(x)dx=0. \tag{3.19}$$

From (3.1) and (3.19), we obtain (3.18). □

REFERENCES

[1] E.V. ATKINSON, An inequality, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, (357-380) (1971), 5–6.

[2] A. LUPAŞ, An integral inequality for convex functions, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, (381-409) (1972), 17–19.

[3] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalitites in Analysis*, Kluwer Academic Publishers, Dordrecht, 1992.

[4] D.S. MITRINOVIĆ AND P.M. VASIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin and New-York, 1970.

[5] J.E. PEČARIĆ, F. PROSCHAN AND Y.I. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, San Diego, 1992.

[6] P.M. VASIĆ AND I.B. LACKOVIĆ, Notes on convex functions VI: On an inequality for convex functions proved by A. Lupaş, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, (634-677) (1979), 36–41.