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ON GENERALIZED PREINVEX FUNCTIONS AND MONOTONICITIES

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ABSTRACT. In this paper we consider some classes of α-preinvex and α-invex functions. We study some properties of these classes of α-preinvex (α-invex) functions. In particular, we establish the equivalence among the α-preinvex functions, α-invex functions and $\alpha\eta$-monotonicity of their differential under some suitable conditions.

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1. INTRODUCTION

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson [1]. Hanson's initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Weir and Mond [13], Jeyakumar and Mond [3] and Noor [5, 7] have studied the basic properties of the preinvex functions and their role in optimization and mathematical programming problems. It is well-known that the preinvex functions and invex sets may not be convex functions and convex sets.

In recent years, these concepts and results have been investigated extensively in [6, 7, 8, 11, 12]. It is noted that some of the results obtained in [8] are incorrect and misleading. The main purpose of this paper to suggest some appropriate and suitable modifications. We also consider some classes of preinvex and invex functions, which are called α-preinvex and α-invex functions. Several new concepts of $\alpha\eta$-monotonicity are introduced. We establish the relationship between these classes and derive some new results. As special cases, one can obtain some new and correct versions of known results. Results obtained in this paper present a refinement and improvement of previously known results.

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2. PRELIMINARIES

Let K be a nonempty closed set in H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and norm respectively. Let $F : K \to H$ and $\eta(\cdot, \cdot) : K \times K \to \mathbb{R}$ be continuous functions. Let $\alpha; K \times K \longrightarrow \mathbb{R} \backslash \{0\}$ be a bifunction.

First of all, we recall the following well known results and concepts.

Definition 2.1. Let $u \in K$. Then the set K is said to be α-invex at u with respect to $\eta(\cdot, \cdot)$ and $\alpha(\cdot, \cdot)$, if , for all $u, v \in K, t \in [0, 1]$,

$$u + t\alpha(v, u)\eta(v, u) \in K.$$

K is said to be an α-invex set with respect to η and α, if K is α-invex at each $u \in K$. The α-invex set K is also called $\alpha\eta$-connected set. Note that the convex set with $\alpha(v, u) = 1$ and $\eta(v, u) = v - u$ is an invex set, but the converse is not true. For example, the set $K = R \backslash \left(-\frac{1}{2}, \frac{1}{2}\right)$ is an invex set with respect to η and $\alpha(v, u) = 1$, where

$$\eta(v, u) = \begin{cases} v - u, & \text{for} \quad v > 0, u > 0 \quad \text{or} \quad v < 0, u < 0 \\ u - v, & \text{for} \quad v < 0, u > 0 \quad \text{or} \quad v < 0, u < 0. \end{cases}$$

It is clear that K is not a convex set.

Remark 2.1. (i) If $\alpha(v, u) = 1$, then the set K is called the invex (η-connected) set, see [6, 7, 12, 13].

(ii) If $\eta(v, u) = v - u$ and $0 < \alpha(v, u) < 1$, then the set K is called the star-shaped.

(iii) If $\alpha(v, u) = 1$ and $\eta(v, u) = v - u$, then the set K is called the convex set.

From now onward K is a nonempty closed α-invex set in H with respect to $\alpha(\cdot, \cdot)$ and $\eta(\cdot, \cdot)$, unless otherwise specified.

Definition 2.2. The function F on the α-invex set K is said to be α-preinvex with respect to η, if

$$F(u + t\alpha(v, u)\eta(v, u)) \leq (1 - t)F(u) + tF(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

The function F is said to be α-preconcave if and only if $-F$ is α-preinvex. Note that every convex function is a preinvex function, but the converse is not true. For example, the function $F(u) = -|u|$ is not a convex function, but it is a preinvex function with respect to η and $\alpha(v, u) = 1$, where

$$\eta(v, u) = \begin{cases} v - u, & \text{if} \quad v \leq 0, u \leq 0 \quad \text{and} \quad v \geq 0, u \geq 0 \\ u - v, & \text{otherwise.} \end{cases}$$

Definition 2.3. The function F on the α-invex set K is called quasi α-preinvex with respect to α and η, if

$$F(u + t\alpha(v, u)\eta(v, u)) \leq \max\{F(u), F(v)\}, \quad \forall u, v \in K, \quad t \in [0, 1].$$

Definition 2.4. The function F on the α-invex set K is said to be logarithmic α-preinvex with respect to α and η, if

$$F(u + t\alpha(v, u)\eta(v, u)) \leq (F(u))^{1-t}(F(v))^t, \quad u, v \in K, \quad t \in [0, 1],$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} F(u+t\alpha(v,u)\eta(v,u)) &\leq (F(u))^{1-t}(F(v))^t \\ &\leq (1-t)F(u)+tF(v) \\ &\leq \max\{F(u),F(v)\} \\ &< \max\{F(u),F(v)\}. \end{aligned}$$

For $t=1$, Definitions 2.2 and 2.4 reduce to:

Condition A.

$$F(u+\alpha(v,u)\eta(v,u)) \leq F(v), \quad \forall u,v \in K,$$

which plays an important part in studying the properties of the α-preinvex (α-invex) functions. Some properties of the α-preinvex functions have been studied in [7, 11].

For $\alpha(v,u)=1$, Condition A reduces to the following for preinvex functions.

Condition B.

$$F(u+\eta(v,u)) \leq F(v), \quad \forall u,v \in K.$$

For the applications of Condition B, see [7, 11, 12].

Definition 2.5. The function F on the α-invex set K is said to be pseudo α-preinvex with respect to α and η, if there exists a strictly positive function $b(\cdot,\cdot)$ such that

$$F(v) \leq F(u) \Longrightarrow \quad F(u+t\alpha(v,u)\eta(v,u)) \leq F(u)+t(t-1)b(u,v), \quad u,v \in K \quad t \in [0,1].$$

Lemma 2.2. *If the function F is α-preinvex function with respect to α and η, then F is pseudo α-preinvex function with respect to α and η.*

Proof. Without loss of generality, we assume that $F(v) < F(u), \quad \forall u,v \in K$. For every $t \in [0,1]$, we have

$$\begin{aligned} F(u+t\alpha(v,u)\eta(v,u)) &\leq (1-t)F(u)+tF(v) \\ &< F(u)+t(t-1)\{F(u)-F(v)\} \\ &= F(u)+t(t-1)b(v,u), \end{aligned}$$

where $b(v,u)=F(v)-F(u)>0$.

Thus it follows that the function F is pseudo α-preinvex function with respect to α and η, the required result. □

Definition 2.6. The differentiable function F on the α-invex set K is said to be an α-invex function with respect to $\alpha(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$, if

$$F(v)-F(u) \geq \langle \alpha(v,u)F'(u), \eta(v,u)\rangle, \quad \forall u,v \in K,$$

where $F'(u)$ is the differential of F at u. The concepts of the α-invex and α-preinvex functions have played a very important role in the development of convex programming, see [2, 3].

Definition 2.7. An operator $T : K \longrightarrow H$ is said to be:

(i). *strongly $\alpha\eta$-monotone,* iff, there exists a constant $\alpha>0$ such that

$$\begin{aligned} \langle \alpha(v,u)Tu, \eta(v,u)\rangle + \langle \alpha(u,v)Tv, \eta(u,v)\rangle \leq -\alpha\{&\|\eta(v,u)\|^2 \\ &+\|\eta(u,v)\|^2\}, \quad \forall u,v \in K. \end{aligned}$$

(ii). *$\alpha\eta$-monotone,* iff,

$$\langle \alpha(v,u)Tu, \eta(v,u)\rangle + \langle \alpha(u,v)Tv, \eta(u,v)\rangle \leq 0, \quad \forall u,v \in K.$$

(iii). *strongly $\alpha\eta$-pseudomonotone,* iff, there exists a constant $\nu > 0$ such that

$$\langle \alpha(v,u)Tu, \eta(v,u)\rangle + \nu\|\eta(v,u)\|^2 \geq 0 \quad \Longrightarrow \quad -\langle \alpha(u,v)Tv, \eta(u,v)\rangle \geq 0, \quad \forall u,v \in K.$$

(iv). *strictly η-monotone,* iff,

$$\langle \alpha(v,u)Tu, \eta(v,u)\rangle + \langle \alpha(u,v)Tv, \eta(u,v)\rangle < 0, \quad \forall u,v \in K.$$

(v). *$\alpha\eta$-pseudomonotone,* iff,

$$\langle \alpha(v,u)Tu, \eta(v,u)\rangle \geq 0 \quad \Longrightarrow \quad \langle \alpha(u,v)Tv, \eta(u,v)\rangle \leq 0, \quad \forall u,v \in K.$$

(vi). *quasi $\alpha\eta$-monotone,* iff,

$$\langle \alpha(v,u)Tu, \eta(v,u)\rangle > 0 \quad \Longrightarrow \quad \langle \alpha(u,v)Tv, \eta(u,v)\rangle \leq 0, \quad \forall u,v \in K.$$

(vii). *strictly η-pseudomonotone,* iff,

$$\langle \alpha(v,u)Tu, \eta(v,u)\rangle \geq 0 \quad \Longrightarrow \quad \langle \alpha(u,v)Tv, \eta(u,v)\rangle < 0, \quad \forall u,v \in K.$$

Note for $\alpha(v,u) = 1, \quad \forall u,v \in K$, the α-invex set K becomes an invex set. In this case, Definition 2.7 is exactly the same as in [7, 11]. In addition, if $\alpha(v,u) = 1$ and $\eta(v,u) = v - u$, then the α-invex set K is the convex set K and consequently Definition 2.7 reduces to the one in [9] for the convex set K. This clearly shows that Definition 2.7 is more general than and includes the ones in [7, 9, 12] as special cases.

Definition 2.8. A differentiable function F on an α-invex set K is said to be strongly pseudo $\alpha\eta$-invex function, iff, there exists a constant $\mu > 0$ such that

$$\langle \alpha(v,u)F'(u), \eta(v,u)\rangle + \mu\|\eta(u,v)\|^2 \geq 0 \Longrightarrow F(v) - F(u) \geq 0, \quad \forall u,v \in K.$$

Definition 2.9. A differentiable function F on the α-invex set K is said to be strongly quasi α-invex, if there exists a constant $\mu > 0$ such that

$$F(v) \leq F(u) \quad \Longrightarrow \quad \langle \alpha(v,u)F'(u), \eta(v,u) + \mu\|\eta(v,u)\|^2 \leq 0, \quad \forall u,v \in K.$$

Definition 2.10. The function F on the α-invex set K is said to be pseudo α-invex, if

$$\langle \alpha(v,u)F'(u), \eta(v,u)\rangle \geq 0, \quad \Longrightarrow \quad F(v) \geq F(u), \quad \forall u,v \in K.$$

Note that if $\alpha(v,u) = 1$, then the α-invex set K is exactly the invex set K and consequently Definitions 2.8 – 2.10 are exactly the same same as in [7]. In particular, if $\eta(v,u) = -\eta(v,u), \forall u,v \in K$, that is, the function $\eta(\cdot,\cdot)$ is skew-symmetric, then Definitions 2.7 – 2.10 reduce to the ones in [8, 11]. This shows that the concepts introduced in this paper represent an improvement of the previously known ones. All the concepts defined above play an important and fundamental part in mathematical programming and optimization problems.

We also need the following assumption regarding the functions $\eta(\cdot,\cdot)$, and $\alpha(\cdot,\cdot)$.

Condition C. *Let $\eta(\cdot,\cdot) : K \times K \longrightarrow H$ and $\alpha(\cdot,\cdot) : K \times K \longrightarrow R\backslash\{0\}$ satisfy the assumptions*

$$\eta(u, u + t\alpha(v,u)\eta(v,u)) = -t\eta(v,u)$$
$$\eta(v, u + t\alpha(v,u)\eta(v,u)) = (1-t)\eta(v,u), \quad \forall u,v \in K, \quad t \in [0,1].$$

Clearly for $t = 0$, we have $\eta(u,v) = 0$, if and only if $u = v, \forall u,v \in K$. One can easily show [11] *that*

$$\eta(u + t\alpha(v,u)\eta(v,u), u) = t\eta(v,u), \quad \forall u,v \in K.$$

Note that for $\alpha(v,u) = 1$, Condition C collapses to the following condition, which is due to Mohan and Neogy [4].

Condition D. *Let* $\eta(\cdot,\cdot) : K \times K \longrightarrow H$ *satisfy the assumptions*

$$\eta(u, u + t\eta(v,u)) = -t\eta(v,u)$$
$$\eta(v, u + t\eta(v,u)) = (1-t)\eta(v,u), \quad \forall u, v \in K, \quad t \in [0,1]$$

For the applications of Condition D, see [7, 11, 12] *and the references therein.*

3. MAIN RESULTS

In this section, we study some basic properties of α-preinvex functions on the α-invex set K.

Theorem 3.1. *Let* F *be a differentiable function on the* α*-invex set* K *and let Condition C hold. Then the function* F *is a* α*-preinvex function if and only if* F *is a* α*-invex function.*

Proof. Let F be a α-preinvex function on the α-invex set K. Then, $\forall u, v \in K, t \in [0,1], \quad u + t\alpha(v,u)\eta(v,u) \in K$ and

$$F(u + t\alpha(v,u)\eta(v,u)) \le (1-t)F(u) + tF(v), \quad \forall u, v \in K,$$

which can be written as

$$F(v) - F(u) \ge \frac{F(u + t\alpha(v,u)\eta(v,u)) - F(u)}{t}.$$

Letting $t \longrightarrow 0$ in the above inequality, we have

$$F(v) - F(u) \ge \langle \alpha(v,u)F'(u), \eta(v,u)\rangle,$$

which implies that F is a α-invex function.

Conversely, let F be a α-invex function on the α-invex function K. Then $\quad \forall u, v \in K, t \in [0,1], \quad v_t = u + t\alpha(v,u)\eta(v,u) \in K$ and using Condition C, we have

$$\begin{aligned} &F(v) - F(u + t\alpha(v,u)\eta(v,u)) \\ &\qquad \ge \langle \alpha(v,u)F'(u + t\alpha(v,u)\eta(v,u)), \eta(v, u + t\alpha(v,u)\eta(v,u))\rangle \\ &\qquad = (1-t)\langle \alpha(v,u)F'(u + t\alpha(v,u)\eta(v,u)), \eta(v,u)\rangle. \end{aligned} \tag{3.1}$$

In a similar way, we have

$$\begin{aligned} &F(u) - F(u + t\alpha(v,u)\eta(v,u)) \\ &\qquad \ge \langle \alpha(v,u)F'(u + t\alpha(v,u)\eta(v,u)), \eta(u, u + t\alpha(v,u)\eta(v,u))\rangle \\ &\qquad = -t\langle \alpha(v,u)F'(u + t\alpha(v,u)\eta(v,u)), \eta(v,u))\rangle. \end{aligned} \tag{3.2}$$

Multiplying (3.1) by t and (3.2) by $(1-t)$ and adding the resultant, we have

$$F(u + t\alpha(v,u)\eta(v,u)) \le (1-t)F(u) + tF(v).$$

showing that F is a α-preinvex function. □

If $\alpha(v,u) = 1$, then Theorem 3.1 reduces to the following result, which is mainly due to Mohan and Neogy [4] for the preinvex and invex functions on the invex set.

Theorem 3.2. *Let* F *be a differentiable function on the invex set* K *and let Condition D hold. Then the function* F *is a preinvex function if and only if* F *is an invex function.*

Theorem 3.3. *Let* F *be differntiable function on the invex set* K*. If* F *is* α*-invex (*α*-preinvex) function, then its differential* $F'(u)$ *is* $\alpha\eta$*-monotone.*

Proof. Let F be a α-invex function on the α-invex set K. Then

$$F(v) - F(u) \geq \langle \alpha(v,u)F'(u), \eta(v,u)\rangle, \quad \forall u, v \in K. \tag{3.3}$$

Changing the role of u and v in (3.3), we have

$$F(u) - F(v) \geq \langle \alpha(u,v)F'(v), \eta(u,v)\rangle, \quad \forall u, v \in K. \tag{3.4}$$

Adding (3.3) and (3.4), we have

$$\langle \alpha(v,u)F'(u), \eta(v,u)\rangle + \langle \alpha(u,v)F'(v), \eta(u,v)\rangle \leq 0,$$

which shows that F' is $\alpha\eta$-monotone. □

We now prove the converse of Theorem 3.3 for the case $\alpha(v,u) = \alpha(u,v)$, that is, the function $\alpha(v,u)$ is a symmetric function. However, in general, the converse of Theorem 3.3 is an open problem.

Theorem 3.4. *Let Conditions A and C hold and the function $\alpha(v,u)$ be symmetric. If the differential $F'(u)$ of a function $F(u)$ is $\alpha\eta$-monotone, then the function $F(u)$ is α-invex (α-preinvex) function.*

Proof. Let $F'(u)$ be $\alpha\eta$-monotone, that is,

$$\langle \alpha(u,v)F'(v), \eta(u,v)\rangle + \langle \alpha(v,u)F'(u), \eta(v,u)\rangle \leq 0, \quad \forall u, v \in K,$$

which implies that

$$\langle F'(v), \eta(u,v)\rangle \leq -\langle F'(u), \eta(v,u)\rangle, \tag{3.5}$$

since $\alpha(v,u)$ is a positive symmetric function.

Since K is a α-invex set, $\forall u, v \in K, t \in [0,1], v_t = u + t\alpha(v,u)\eta(v,u) \in K$. Taking $v \equiv v_t$, in (3.5) and using Condition C, we have

$$-t\langle F'(u + t\alpha(v,u)\eta(v,u)), \eta(v,u)\rangle \leq -t\langle F'(u), \eta(v,u)\rangle,$$

which implies that

$$\langle F'(u + t\alpha(v,u)\eta(v,u)), \eta(v,u)\rangle \geq \langle F'(u), \eta(v,u)\rangle. \tag{3.6}$$

Let

$$g(t) = F(u + t\alpha(v,u)\eta(v,u)), \quad \forall u, v \in K, t \in [0,1].$$

Then, using (3.6), we have

$$\begin{aligned} g'(t) &= \langle \alpha(v,u)F'(u + t\alpha(v,u)\eta(v,u)), \eta(v,u)\rangle \\ &\geq \langle \alpha(v,u)F'(u), \eta(v,u)\rangle. \end{aligned}$$

Integrating the above relation between 0 and 1, we have

$$g(1) - g(0) \geq \langle \alpha(v,u)F'(u), \eta(v,u)\rangle,$$

that is,

$$F(u + \alpha(v,u)\eta(v,u)) - F(u) \geq \langle \alpha(v,u)F'(u), \eta(v,u)\rangle,$$

which implies, using Condition A,

$$F(v) - F(u) \geq \langle \alpha(v,u)F'(u), \eta(v,u)\rangle,$$

which shows that the function $F(u)$ is a α-invex (α-preinvex) function, the required result. □

For $\alpha(v,u) = 1$, the α-invex set K becomes the invex set and consequently from Theorem 3.3 and Theorem 3.4, we have the following result for preinvex and invex functions.

Theorem 3.5. *Let Conditions B and D hold and let K be an invex set. Then the differential $F'(u)$ of a function $F(u)$ is η-monotone if and only if $F(u)$ is a preinvex(invex) function on the invex set K.*

We now give a necessary condition for strongly $\alpha\eta$-pseudo-invex functions, which is also a generalization and refinement of a result proved in [8, 11].

Theorem 3.6. *Let the differential $F'(u)$ of a function $F(u)$ on the α-invex set K be strongly $\alpha\eta$-pseudomonotone. If Conditions A and C hold, then F is strongly pseudo $\alpha\eta$-invex function.*

Proof. Let $F'(u)$ be strongly $\alpha\eta$-pseudomonotone. Then

$$\langle \alpha(v,u)F'(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2 \geq 0, \quad \forall u, v \in K,$$

implies that

$$-\langle \alpha(u,v)F'(v), \eta(u,v)\rangle \geq 0, \quad \forall u, v \in K. \tag{3.7}$$

Since K is an α-invex set, $\forall u, v \in K$, $t \in [0,1]$, $v_t = u + t\alpha(v,u)\eta(v,u) \in K$. Taking $v = v_t$ in (3.7) and using Condition C, we have

$$\langle \alpha(v_t,u)F'(u + t\alpha(v,u)\eta(v,u)), \eta(v,u)\rangle \geq 0, \quad \forall u, v \in K,$$

which implies that

$$\langle F'(u + t\alpha(v,u)\eta(v,u)), \eta(v,u)\rangle \geq 0, \quad \forall u, v \in K. \tag{3.8}$$

Let

$$g(t) = F(u + t\alpha(v,u)\eta(v,u)), \quad \forall u, v \in K, t \in [0,1].$$

Then, using (3.8), we have

$$g'(u) = \langle \alpha(v,u)F'(u + t\alpha(v,u)\eta(v,u)), \eta(v,u)\rangle \geq 0.$$

Integrating the above relation between 0 and 1, we have

$$g(1) - g(0) \geq 0,$$

that is,

$$F(u + \alpha(v,u)\eta(v,u)) - F(u) \geq 0,$$

which implies, using Condition A, that

$$F(v) - F(u) \geq 0,$$

showing that the function $F(u)$ is strongly pseudo $\alpha\eta$-invex function. □

As special cases of Theorem 3.6, we have the following:

Theorem 3.7. *Let the differential $F'(u)$ of a function $F(u)$ on the α-invex set K be $\alpha\eta$-pseudomonotone. If Conditions A and C hold, then F is pseudo $\alpha\eta$-invex function.*

Theorem 3.8. *Let the differential $F'(u)$ of a function $F(u)$ on the α-invex set K be strongly η-pseudomonotone. If Conditions A and C hold, then F is a strongly pseudo η-invex function.*

Theorem 3.9. *Let the differential $F'(u)$ of a function $F(u)$ on the invex set K be strongly η-pseudomonotone. If Conditions B and D hold, then F is a strongly pseudo η-invex function.*

Theorem 3.10. *Let the differential $F'(u)$ of a function $F(u)$ on the invex set K be η-pseudomonotone. If Conditions B and D hold, then F is a pseudo invex function.*

Theorem 3.11. *Let the differential $F'(u)$ of a differentiable α-preinvex function $F(u)$ be Lipschitz continuous on the α-invex set K with a constant $\beta > 0$. If Condition A holds, then*

$$F(v) - F(u) \leq \langle \alpha(v,u)F'(u), \eta(v,u)\rangle + \frac{\beta}{2}\|\alpha(v,u)\eta(v,u)\|^2, \quad \forall u, v \in K.$$

Proof. $\forall u, v \in K, t \in [0,1], \quad u + t\alpha(v,u)\eta(v,u) \in K$, since K is an α-invex set. Now we consider the function

$$\varphi(t) = F(u + t\alpha(v,u)\eta(v,u)) - F(u) - t\langle \alpha(v,u)F'(u), \eta(v,u)\rangle.$$

from which it follows that $\varphi(0) = 0$ and

$$\varphi'(t) = \langle \alpha(v,u)F'(u + t\alpha(v,u)\eta(v,u)), \eta(v,u)\rangle - \langle \alpha(v,u)F'(u), \eta(v,u)\rangle. \tag{3.9}$$

Integrating (3.9) between 0 and 1, we have

$$\begin{aligned}
\varphi(1) &= F(u + \alpha(v,u)\eta(v,u)) - F(u) - \langle \alpha(v,u)F'(u), \eta(v,u)\rangle \\
&\leq \int_0^1 |\varphi'(t)|dt \\
&= \int_0^1 |\langle \alpha(v,u)F'(u + t\alpha(v,u)\eta(v,u)), \eta(v,u)\rangle - \langle \alpha(v,u)F'(u), \eta(v,u)\rangle|\, dt \\
&\leq \beta \int_0^1 t\|\alpha(v,u)\eta(v,u)\|^2 dt \\
&= \frac{\beta}{2}\|\alpha(v,u)\eta(v,u)\|^2,
\end{aligned}$$

which implies that

$$F(u + \alpha(v,u)\eta(v,u)) - F(u) \leq \langle \alpha(v,u)F'(u), \eta(v,u)\rangle + \frac{\beta}{2}\|\alpha(v,u)\eta(v,u)\|^2. \tag{3.10}$$

from which, using Condition A, we obtain

$$F(v) - F(u) \leq \langle \alpha(v,u)F'(u), \eta(v,u)\rangle + \frac{\beta}{2}\|\alpha(v,u)\eta(v,u)\|^2.$$

□

Remark 3.12. For $\eta(v,u) = v - u$ and $\alpha(v,u) = 1$, the α-invex set K becomes a convex set and consequently Theorem 3.11 reduces to the well known result in convexity, see [14].

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