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ON THE APPROXIMATION OF LOCALLY BOUNDED FUNCTIONS BY OPERATORS OF BLEIMANN, BUTZER AND HAHN

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ABSTRACT. We estimate the rate of the pointwise approximation by operators of Bleimann, Butzer and Hahn of locally bounded functions, and of functions having a locally bounded derivative.

Key words and phrases: Operators of Bleimann, Butzer and Hahn, Locally bounded function, Function of bounded variation, Total variation, Rate of convergence, Binomial distribution.

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1. INTRODUCTION AND MAIN RESULTS

Bleimann, Butzer and Hahn [1] introduced the Bernstein type operator L_n over the interval $[0, \infty)$ given by

$$L_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) b_{n,k}(x), \qquad x \geq 0,\ n = 1, 2, \dots,$$

where f is a real function on $[0, \infty)$, and

$$b_{n,k}(x) := \binom{n}{k} p_x^k q_x^{n-k}, \qquad p_x := \frac{x}{1+x}, \quad q_x := 1 - p_x = \frac{1}{1+x}. \tag{1.1}$$

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The approximation of uniformly continuous functions by these operators has been considered in [1] – [4]. For other properties of L_n (preservation of global smoothness, preservation of ϕ-variation, behavior of the iterates, etc.) we refer, for instance, to [4] – [10]. In some of the mentioned works, the results are achieved by using probabilistic methods. This comes from the fact that L_n is an operator of probabilistic type. We can actually write

$$L_n(f,x) = Ef(Z_{n,x}),$$

where E denotes mathematical expectation, and $Z_{n,x}$ is the random variable given by

$$Z_{n,x} := \frac{S_{n,x}}{n - S_{n,x} + 1}, \qquad S_{n,x} := \xi_{1,x} + \cdots + \xi_{n,x}, \tag{1.2}$$

where $\xi_{1,x},\, \xi_{2,x}, \dots$ are independent random variables having the same Bernoulli distribution with parameter p_x, i.e.,

$$P(\xi_{k,x} = 1) = p_x = 1 - P(\xi_{k,x} = 0)$$

(so that $S_{n,x}$ has the binomial distribution with parameters $n,\, p_x$). This probabilistic representation also plays a significant role in the present paper (for a more refined representation useful for other purposes, see [5, 6]).

Here, we discuss the approximation of real functions f on the semi axis which are locally bounded, i.e., bounded on each finite subinterval of $[0,\infty)$. In such a case, we set, for $x > 0$ and $h \geq 0$,

$$\omega_x^+(f;h) := \sup_{x \le t \le x+h} |f(t) - f(x)|,$$

$$\omega_x^-(f;h) := \sup_{(x-h)^+ \le t \le x} |f(t) - f(x)|,$$

$$\omega_x(f;h) := \omega_x^+(f;h) + \omega_x^-(f;h),$$

where $(x-h)^+ := \max(x-h, 0)$, and we observe that these functions are (nonnegative and) nondecreasing on $[0,\infty)$. In particular, every continuous function is locally bounded. Also, if f is locally of bounded variation, i.e., such that

$$\bigvee_a^b (f) < \infty, \qquad 0 \le a < b < \infty,$$

where $\bigvee_a^b(f)$ stands for the total variation of f on the interval $[a,b]$, then f is locally bounded, and we obviously have

$$\omega_x(f;h) \le \bigvee_{x-h}^{x+h}(f), \qquad 0 \le h \le x.$$

This kind of problem has been already considered for other Bernstein-type operators (see, for instance, [11] – [14] and the references therein). Our main results are stated as follows.

Theorem 1.1. *Let g be a real locally bounded function on $[0,\infty)$ such that $g(t) = O(t^r)$ $\quad (t \to \infty)$, for some $r = 1, 2, \dots$. If g is continuous at $x > 0$, then, for n large enough, we have*

$$|L_n(g,x) - g(x)| \le \frac{7(1+x)^2}{(n+2)x} \sum_{k=1}^{n} \omega_x\left(g; \frac{x}{\sqrt{k}}\right) + O_{r,x}\left(\frac{1}{n}\right). \tag{1.3}$$

In the following statements (and throughout the paper), we use the notations:

$$f^*(x) := f(x+) - f(x-)$$

$$\tilde{f}(x) := \frac{f(x+) + f(x-)}{2},$$

$$f_x := (f - f(x-))1_{[0,x)} + (f - f(x+))1_{(x,\infty)}$$

(1_A being the indicator function of the set A), provided that the lateral limits $f(x+)$ and $f(x-)$ exist (such a condition is fulfilled when f is locally of bounded variation). We also use the symbol $\lfloor a \rfloor$ to indicate the integral part of the real number a.

Theorem 1.2. *Let f be a real locally bounded function on $[0,\infty)$ such that $f(t) = O(t^r) \quad (t \to \infty)$, for some $r = 1, 2, \dots$. If $x > 0$, and $f(x+)$ and $f(x-)$ exist, then we have for n large enough*

$$\left|L_n(f,x) - \tilde{f}(x)\right| \\ \leq \Delta_{n,x}(f_x) + \frac{1.6 + x + 2.6\,x^2}{\sqrt{nx}(1+x)} \cdot \frac{|f^*(x)|}{2} + \frac{\epsilon_{n,x}(1+x)}{\sqrt{2enx}} |f(x) - f(x-)|,$$

where $\Delta_{n,x}(f_x)$ is the right-hand side of (1.3) with g replaced by f_x, and

$$\epsilon_{n,x} := \begin{cases} 1 & \text{if } (n+1)p_x \in \{1, 2, \dots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.3. *Let g be a real function on $[0,\infty)$ such that $g(t) = O(t^r) \quad (t \to \infty)$, for some $r = 1, 2, \dots$, and having the form*

$$g(t) = c + \int_0^t f(u)\,du, \qquad t \geq 0,$$

where c is a constant and f is measurable and locally bounded on $[0,\infty)$. If $x > 0$, and $f(x+)$ and $f(x-)$ exist, then we have for n large enough

$$\left|L_n(g,x) - g(x) - \frac{\sqrt{x}(1+x)}{\sqrt{2\pi n}} f^*(x)\right| \\ \leq \frac{5(1+x)^2}{n+2} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \omega_x\left(f_x; \frac{x}{k}\right) + |f^*(x)|\, o_x\left(n^{-1/2}\right) + O_{r,x}(n^{-1}).$$

The proofs of the preceding theorems are given in Sections 3 – 5. In Section 2, we collect the necessary auxiliary results. Some remarks on moments close the paper.

2. AUXILIARY RESULTS

In the following lemma, Φ denotes the standard normal distribution function, and $F^*_{n,x}$ stands for the distribution function of $S^*_{n,x} := (S_{n,x} - np_x) \big/ \sqrt{np_xq_x}$,, where $S_{n,x}$ is the same as in (1.2). Such a lemma is nothing but the application of the well-known Berry-Esseen theorem (cf. [15]) to the situation at hand.

Lemma 2.1. *We have, for $x > 0$ and $n \geq 1$,*

$$\sup_{-\infty < t < \infty} |F^*_{n,x}(t) - \Phi(t)| \leq \frac{0.8(p_x^3 q_x + p_x q_x^3)}{\sqrt{n}(p_x q_x)^{3/2}} = \frac{0.8(1+x^2)}{\sqrt{nx}(1+x)}.$$

Lemma 2.2. *Let $x > 0$ and $n \geq 1$. Then, we have:*

(a)

$$L_n((\cdot - x)^2, x) = E(Z_{n,x} - x)^2 \leq \frac{3x(1+x)^2}{n+2}.$$

(b)

$$P(Z_{n,x} \le x-h) + P(Z_{n,x} \ge x+h) \le \frac{3x(1+x)^2}{(n+2)h^2}, \qquad h>0.$$

(c)

$$|P(Z_{n,x} > x) - P(Z_{n,x} \le x)| \le \sqrt{\frac{x}{n}} + \frac{1.6(1+x^2)}{\sqrt{nx}(1+x)}.$$

(d)

$$L_n((\cdot - x), x) = E(Z_{n,x} - x) = -xp_x^n = o_x(n^{-1}), \qquad (n \to \infty).$$

(e)

$$L_n(|\cdot - x|, x) = E|Z_{n,x} - x| = \frac{\sqrt{2x}(1+x)}{\sqrt{\pi n}} + o_x(n^{-1/2}), \qquad (n \to \infty).$$

Proof. Part (a) was shown in [10]. Part (b) follows from (a) and the fact that, by Markov's inequality,

$$P(Z_{n,x} \le x-h) + P(Z_{n,x} \ge x+h) = P(|Z_{n,x} - x| \ge h) \le \frac{E(Z_{n,x} - x)^2}{h^2}.$$

To show (c), observe that

$$\begin{aligned} |P(Z_{n,x} > x) - P(Z_{n,x} \le x)| &= |1 - 2P(Z_{n,x} \le x)| \\ &= |1 - 2P(S_{n,x} \le (n+1)p_x)| \\ &= \left|1 - 2F^*_{n,x}\left(\sqrt{\frac{x}{n}}\right)\right| \\ &\le 2\left|\Phi\left(\sqrt{\frac{x}{n}}\right) - F^*_{n,x}\left(\sqrt{\frac{x}{n}}\right)\right| + \left|1 - 2\Phi\left(\sqrt{\frac{x}{n}}\right)\right|. \end{aligned}$$

Thus, the conclusion in part (c) follows from Lemma 2.1 and the fact that (cf. [16])

$$0 < 2\Phi(t) - 1 \le \left(1 - e^{-t^2}\right)^{1/2} \le t, \qquad (t > 0).$$

Part (d) is immediate. Finally, to show (e), let $m := \lfloor (n+1)p_x \rfloor$. We have

$$\begin{aligned} L_n(|\cdot - x|, x) - L_n((\cdot - x), x) &= 2\sum_{k=0}^{m}\left(x - \frac{k}{n-k+1}\right) b_{n,k}(x) \\ &= 2x\sum_{k=0}^{m} b_{n,k}(x) - 2\sum_{k=1}^{m}\frac{n!}{(k-1)!(n-k+1)!}\, p_x^k q_x^{n-k} \\ &= 2x\sum_{k=0}^{m} b_{n,k}(x) - 2x\sum_{k=0}^{m-1} b_{n,k}(x) \\ &= 2x\, b_{n,m}(x) \\ &= \frac{\sqrt{2x}(1+x)}{\sqrt{\pi n}} + o_x(n^{-1/2}), \qquad (n \to \infty), \end{aligned}$$

the last equality by [13, Lemma 1], and the conclusion follows from (d). □

Lemma 2.3. *Let $x > 0$ and $r = 1, 2, \ldots$. Then, we have for all integers n such that $(n + 1)(p_{2x} - p_{3x/2}) \geq r$,*

$$\sum_{k\in K} \frac{k^r}{(n-k+1)^r}\, b_{n,k}(x) \leq 12\, r! \sum_{s=1}^{r} \left\{ {r \atop s} \right\} \frac{x^{s-1}(1+x)^{r-s+2}}{n+r-s+2} \cdot \frac{n!}{(n+r-s)!}$$
$$= O_{r,x}(n^{-1}), \qquad (n \to \infty),$$

where the $\left\{ {r \atop s} \right\}$ are the Stirling numbers of the second kind, and K is the set of all integers k such that $n \geq k > (n-k+1)2x$ (i.e., $n \geq k > (n+1)p_{2x}$).

Proof. Using the well known identity

$$a^r = \sum_{s=1}^{r} \left\{ {r \atop s} \right\} a(a-1)\cdots(a-s+1),$$

we can write

$$\sum_{k\in K} \frac{k^r}{(n-k+1)^r}\, b_{n,k}(x) = \sum_{s=1}^{r} \left\{ {r \atop s} \right\} A_s, \tag{2.1}$$

where

$$A_s := \sum_{k\in K} \frac{k(k-1)\cdots(k-s+1)}{(n-k+1)^r}\, b_{n,k}(x)$$
$$= \sum_{k\in K} \frac{1}{(n-k+1)^r} \cdot \frac{n!}{(k-s)!(n-k)!}\, p_x^k q_x^{n-k}.$$

Since

$$\frac{1}{(n-k+1)^r} = \prod_{i=1}^{r} \left[\frac{1}{n-k+i} \frac{n-k+i}{n-k+1} \right]$$
$$= \prod_{i=1}^{r} \left[\frac{1}{n-k+i} \left(1 + \frac{i-1}{n-k+1} \right) \right]$$
$$\leq \prod_{i=1}^{r} \frac{i}{n-k+i} = \frac{r!(n-k)!}{(n-k+r)!},$$

we have

$$A_s \leq r! \sum_{k\in K} \frac{n!}{(k-s)!(n-k+r)!}\, p_x^k q_x^{n-k}$$
$$= r! \sum_{l\in K_s} \frac{n!}{l!(n+r-s-l)!}\, p_x^{l+s} q_x^{n-l-s}$$
$$= \frac{r!n!p_x^s q_x^{-r}}{(n+r-s)!} \sum_{l\in K_s} \binom{n+r-s}{l} \frac{x^l}{(1+x)^{n+r-s}}$$
$$\leq \frac{r!n!p_x^s q_x^{-r}}{(n+r-s)!} \sum_{l\in K'} \binom{n+r-s}{l} \frac{x^l}{(1+x)^{n+r-s}},$$

where $K_s := \{k - s : k \in K\}$, and K' stands for the set of all integers l such that $n \geq l > (n-l+1)(3x/2)$ (observe that, by the assumption on n, we have $K_s \subset K'$). The probabilistic interpretation of the last sum together with Lemma 2.2(b) yield

$$\begin{aligned} A_s &\leq \frac{r!n!x^s(1+x)^{r-s}}{(n+r-s)!} P\left(Z_{n+r-s,x} > \frac{3x}{2}\right) \\ &\leq \frac{12\, r!n!x^{s-1}(1+x)^{r-s+2}}{(n+r-s)!(n+r-s+2)}, \end{aligned} \tag{2.2}$$

and the conclusion follows from (2.1) and (2.2). □

Remark 2.4. The same procedure as in the preceding proof leads to the following upper bound for the integral moments of L_n (or $Z_{n,x}$):

$$\begin{aligned} L_n(t^r,x) &= E(Z_{n,x})^r \\ &= \sum_{k=0}^{n} \frac{k^r}{(n-k+1)^r}\, b_{n,k}(x) \\ &\leq r! \sum_{s=1}^{r} \left\{ {r \atop s} \right\} \frac{n!x^s(1+x)^{r-s}}{(n+r-s)!}. \end{aligned}$$

3. PROOF OF THEOREM 1.1

Without loss of generality, we assume that $g(x)=0$. Denote by $K_{n,x}$ the distribution function of $Z_{n,x}$, i.e.,

$$K_{n,x}(t) := P(Z_{n,x} \leq t) = \sum_{k \leq (n-k+1)t} b_{n,k}(x) \qquad t \geq 0.$$

We can write $L_n(g,x)$ as the Lebesgue-Stieltjes integral

$$L_n(g,x) = Eg(Z_{n,x}) = \int_{[0,\infty)} g(t)\, dK_{n,x}(t) = \sum_{j=1}^{4} \int_{I_j} g(t)\, dK_{n,x}(t),$$

where

$$I_1 := \left[0, x - \frac{x}{\sqrt{n}}\right], \qquad I_2 := \left(x - \frac{x}{\sqrt{n}}, x + \frac{x}{\sqrt{n}}\right],$$

$$I_3 := \left(x + \frac{x}{\sqrt{n}}, 2x\right] \qquad \text{and} \qquad I_4 := (2x, \infty).$$

We obviously have

$$\begin{aligned} \int_{I_2} |g(t)|\, dK_{n,x}(t) &\leq \omega_x\left(g; \frac{x}{\sqrt{n}}\right) \int_{I_2} dK_{n,x}(t) \\ &\leq \omega_x\left(g; \frac{x}{\sqrt{n}}\right) \\ &\leq \frac{1}{n} \sum_{k=1}^{n} \omega_x\left(g; \frac{x}{\sqrt{k}}\right). \end{aligned} \tag{3.1}$$

On the other hand, from the asymptotic assumption on g, we have

$$|g(t)| \leq M\, t^r, \qquad t \geq \alpha,$$

for some constants $M > 0$ and $\alpha \geq 2x$. Therefore,

$$\int_{I_4} |g(t)|\, dK_{n,x}(t) = \left(\int_{(2x,\alpha]} + \int_{(\alpha,\infty)}\right) |g(t)|\, dK_{n,x}(t)$$
$$\leq \omega_x^+(g;\alpha - x) P(Z_{n,x} > 2x) + M \sum_{k>(n-k+1)\alpha} \frac{k^r}{(n-k+1)^r}\, b_{n,k}(x).$$

By Lemma 2.2(b) and Lemma 2.3, this shows that

$$\int_{I_4} |g(t)|\, dK_{n,x}(t) = O_{r,x}(n^{-1}) \qquad (n \to \infty). \tag{3.2}$$

Finally, using Lemma 2.2(b) and integration by parts (follow the same procedure as in the proof of Theorem 1 in [13]), we obtain

$$\int_{I_1} |g(t)|\, dK_{n,x}(t) \leq \int_{I_1} \omega_x^-(g; x - t)\, dK_{n,x}(t)$$
$$\leq \frac{3x(1+x)^2}{(n+2)} \left[\frac{\omega_x^-(g;x)}{x^2} + 2\int_0^{x - x/\sqrt{n}} \frac{\omega_x^-(g; x-t)}{(x-t)^3}\, dt\right]$$
$$\leq \frac{6(1+x)^2}{(n+2)x} \sum_{k=1}^{n} \omega_x^- \left(g; \frac{x}{\sqrt{k}}\right), \tag{3.3}$$

and, analogously,

$$\int_{I_3} |g(t)|\, dK_{n,x}(t) \leq \frac{6(1+x)^2}{(n+2)x} \sum_{k=1}^{n} \omega_x^+ \left(g; \frac{x}{\sqrt{k}}\right). \tag{3.4}$$

The conclusion follows from (3.1) – (3.4).

4. PROOF OF THEOREM 1.2

We can write, for $t \geq 0$,

$$f(t) - \tilde{f}(x) = f_x(t) + \frac{f^*(x)}{2}\sigma_x(t) + (f(x) - \tilde{f}(x))\delta_x(t), \tag{4.1}$$

where $\sigma_x := -1_{[0,x)} + 1_{(x,\infty)}$, and $\delta_x := 1_{\{x\}}$ is Dirac's delta at x (this is the so called Bojanic-Vuilleumier-Cheng decomposition).

By Theorem 1.1, we have

$$|L_n(f_x, x)| \leq \Delta_{n,x}(f_x), \tag{4.2}$$

where $\Delta_{n,x}(f_x)$ is the right-hand side of (1.2) with g replaced by f_x. Moreover,

$$L_n(\sigma_x, x) = P(Z_{n,x} > x) - P(Z_{n,x} < x)$$
$$= (P(Z_{n,x} > x) - P(Z_{n,x} \leq x)) + P(Z_{n,x} = x), \tag{4.3}$$

and

$$L_n(\delta_x, x) = P(Z_{n,x} = x). \tag{4.4}$$

Using Lemma 2.2(c) and the fact that (cf. [17, Theorem 1])

$$P(Z_{n,x} = x) = \begin{cases} \binom{n}{k} p_x^k q_x^{n-k} \leq \frac{(1+x)}{\sqrt{2enx}} & \text{if } (n+1)p_x = k \in \{1, 2, \ldots, n\} \\ 0 & \text{otherwise,} \end{cases}$$

the conclusion readily follows from (4.1) – (4.4).

5. PROOF OF THEOREM 1.3

Using the decomposition (4.1), it is easily checked that

$$L_n(g,x) - g(x) = \sum_{i=1}^{4} A_i(n,x), \tag{5.1}$$

where

$$A_1(n,x) := \tilde{f}(x) L_n((\cdot - x), x) + \frac{f^*(x)}{2} L_n(|\cdot - x|, x),$$
$$A_2(n,x) := \int_{[0,x]} \left(\int_t^x f_x(u)\,du \right) dK_{n,x}(t),$$
$$A_3(n,x) := \int_{(x,2x]} \left(\int_x^t f_x(u)\,du \right) dK_{n,x}(t),$$
$$A_4(n,x) := \int_{(2x,\infty)} \left(\int_x^t f_x(u)\,du \right) dK_{n,x}(t),$$

and $K_{n,x}(t)$ is the same as in the preceding proofs.

From Lemma 2.2(d,e), we have

$$A_1(n,x) = \frac{\sqrt{x}(1+x)}{\sqrt{2\pi n}} f^*(x) + f^*(x)\, o_x(n^{-1/2}) + o_x(n^{-1}), \qquad (n \to \infty). \tag{5.2}$$

Next, we estimate $A_2(n,x)$. By Fubini's theorem,

$$A_2(n,x) = \int_0^x K_{n,x}(u)\, f_x(u)\,du = \left(\int_0^{x - x/\sqrt{n}} + \int_{x - x/\sqrt{n}}^x \right) K_{n,x}(u)\, f_x(u)\,du.$$

It is clear that

$$\begin{aligned} \left| \int_{x-x/\sqrt{n}}^x K_{n,x}(u)\, f_x(u)\,du \right| &\le \int_{x-x/\sqrt{n}}^x |f_x(u)|\,du \\ &\le \int_{x-x/\sqrt{n}}^x \omega_x^-(f_x; x-u)\,du \\ &\le \frac{x}{\sqrt{n}}\, \omega_x^- \left(f_x; \frac{x}{\sqrt{n}} \right) \\ &\le \frac{2x}{n} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \omega_x^- \left(f_x; \frac{x}{k} \right), \end{aligned}$$

and, using Lemma 2.2(b),

$$\begin{aligned} \left| \int_0^{x-x/\sqrt{n}} K_{n,x}(u)\, f_x(u)\,du \right| &\le \frac{3x(1+x)^2}{(n+2)} \int_0^{x-x/\sqrt{n}} \frac{|f_x(u)|}{(x-u)^2}\,du \\ &\le \frac{3x(1+x)^2}{(n+2)} \int_0^{x-x/\sqrt{n}} \frac{\omega_x^-(f_x; x-u)}{(x-u)^2}\,du \\ &\le \frac{3(1+x)^2}{(n+2)} \int_1^{\sqrt{n}} \omega_x^- \left(f_x; \frac{x}{t} \right) dt \\ &\le \frac{3(1+x)^2}{n+2} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \omega_x^- \left(f_x; \frac{x}{k} \right). \end{aligned}$$

We therefore conclude that

$$|A_2(n,x)| \le \frac{5(1+x)^2}{n+2} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \omega_x^- \left(f_x; \frac{x}{k}\right). \tag{5.3}$$

Similarly,

$$|A_3(n,x)| \le \frac{5(1+x)^2}{n+2} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \omega_x^+ \left(f_x; \frac{x}{k}\right). \tag{5.4}$$

Finally,

$$A_4(n,x) = \int_{(2x,\infty)} g(t)\, dK_{n,x}(t) - \int_{(2x,\infty)} [g(x) + f(x+)(t-x)]\, dK_{n,x}(t),$$

and, by the asymptotic assumption on g, Lemma 2.2(b) and Lemma 2.3, we obtain

$$|A_4(n,x)| = O_{r,x}(n^{-1}), \qquad (n \to \infty). \tag{5.5}$$

The conclusion follows from (5.1) – (5.5).

6. REMARKS ON MOMENTS

Fix $x > 0$, and let $g(\cdot) := |\cdot - x|^\beta$, with $\beta > 2$. Since

$$\omega_x(g,h) = 2\,h^\beta, \qquad 0 \le h \le x,$$

and

$$\sum_{k=1}^{n} k^{-\beta/2} = O(1), \qquad (n \to \infty),$$

we conclude from Theorem 1.1 that

$$L_n(|\cdot - x|^\beta, x) = O_{r,x}(n^{-1}), \qquad (n \to \infty).$$

In the case that $0 < \beta \le 2$, we have, by Jensen's inequality (or Hölder's inequality) and Lemma 2.2(a),

$$L_n(|\cdot - x|^\beta, x) = E|Z_{n,x} - x|^\beta \le \left(E(Z_{n,x} - x)^2\right)^{\beta/2} \le \left(\frac{3x(1+x)^2}{n+2}\right)^{\frac{\beta}{2}},$$

for all $n \ge 1$.

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