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LITTLEWOOD-PALEY DECOMPOSITION ASSOCIATED WITH THE DUNKL OPERATORS AND PARAPRODUCT OPERATORS

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ABSTRACT. We define the Littlewood-Paley decomposition associated with the Dunkl operators; from this decomposition we give the characterization of the Sobolev, Hölder and Lebesgue spaces associated with the Dunkl operators. We construct the paraproduct operators associated with the Dunkl operators similar to those defined by J.M. Bony in [1]. Using the Littlewood-Paley decomposition we establish the Sobolev embedding, Gagliardo-Nirenberg inequality and we present the paraproduct algorithm.

Key words and phrases: Dunkl operators, Littlewood-Paley decomposition, Paraproduct.

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1. INTRODUCTION

The theory of function spaces appears at first to be a disconnected subject, because of the variety of spaces and the different considerations involved in their definitions. There are the Lebesgue spaces $L^p(\mathbb{R}^d)$, the Sobolev spaces $H^s(\mathbb{R}^d)$, the Besov spaces $B^s_{p,q}(\mathbb{R}^d)$, the BMO spaces (bounded mean oscillation) and others.

Nevertheless, several approaches lead to a unified viewpoint on these spaces, for example, approximation theory or interpolation theory. One of the most successful approaches is the Littlewood-Paley theory. This approach has been developed by the European school, which reached a similar unification of function space theory by a different path. Motivated by the methods of Hörmander in studying partial differential equations (see [6]), they used

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a Fourier transform approach. Pick Schwartz functions ϕ and χ on $\mathbb{R}^d$ satisfying $\operatorname{supp}\widehat{\chi} \subset B(0,2),\ \operatorname{supp}\widehat{\phi} \subset \left\{\xi \in \mathbb{R}^d,\ \frac{1}{2} \leq \|\xi\| \leq 2\right\}$, and the nondegeneracy condition $|\widehat{\chi}(\xi)|, |\widehat{\phi}(\xi)| \geq C > 0$. For $j \in \mathbb{Z}$, let $\phi_j(x) = 2^{jd}\phi(2^j x)$. In 1967 Peetre [10] proved that

$$\|f\|_{H^s(\mathbb{R}^d)} \simeq \|\chi * f\|_{L^2(\mathbb{R}^d)} + \left(\sum_{j\geq 1} 2^{2sj}\|\phi_j * f\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}}. \tag{1.1}$$

Independently, Triebel [15] in 1973 and Lizorkin [8] in 1972 introduced $F^s_{p,q}$ (the Triebel-Lizorkin spaces) defined originally for $1 \leq p < \infty$ and $1 \leq q \leq \infty$ by the norm

$$\|f\|_{F^s_{p,q}} = \|\chi * f\|_{L^p(\mathbb{R}^d)} + \left\|\left(\sum_{j\geq 1}(2^{sj}|\phi_j * f|)^q\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^d)}. \tag{1.2}$$

For the special case $q = 1$ and $s = 0$, Triebel [16] proved that

$$L^p(\mathbb{R}^d) \simeq F^0_{p,2}. \tag{1.3}$$

Thus by the Littlewood-Paley decomposition we characterize the functional spaces $L^p(\mathbb{R}^d)$, Sobolev spaces $H^s(\mathbb{R}^d)$, Hölder spaces $C^s(\mathbb{R}^d)$ and others. Using the Littlewood-Paley decomposition J.M. Bony in [1], built the paraproduct operators which have been later successfully employed in various settings.

The purpose of this paper is to generalize the Littlewood-Paley theory, to unify and extend the paraproduct operators which allow the analysis of solutions to more general partial differential equations arising in applied mathematics and other fields. More precisely, we define the Littlewood-Paley decomposition associated with the Dunkl operators. We introduce the new spaces associated with the Dunkl operators, the Sobolev spaces $H^s_k(\mathbb{R}^d)$, the Hölder spaces $C^s_k(\mathbb{R}^d)$ and the $BMO_k(\mathbb{R}^d)$ that generalizes the corresponding classical spaces. The Dunkl operators are the differential-difference operators introduced by C.F. Dunkl in [3] and which played an important role in pure Mathematics and in Physics. For example they were a main tool in the study of special functions with root systems (see [4]).

As applications of the Littlewood-Paley decomposition we establish results analogous to (1.1) and (1.3), we prove the Sobolev embedding theorems, and the Gagliardo-Nirenberg inequality. Another tool of the Littlewood-Paley decomposition associated with the Dunkl operators is to generalize the paraproduct operators defined by J.M. Bony. We prove results similar to [2].

The paper is organized as follows. In Section 2 we recall the main results about the harmonic analysis associated with the Dunkl operators. We study in Section 3 the Littlewood-Paley decomposition associated with the Dunkl operators, we give the sufficient condition on u_p so that $u := \sum u_p$ belongs to Sobolev or Hölder spaces associated with the Dunkl operators. We finish this section by the Littlewood-Paley decomposition of the Lebesgue spaces $L^p_k(\mathbb{R}^d)$ associated with the Dunkl operators. In Section 4 we give some applications. More precisely we establish the Sobolev embedding theorems and the Gagliardo-Nirenberg inequality. Section 5 is devoted to defining the paraproduct operators associated with the Dunkl operators and to giving the paraproduct algorithm.

2. THE EIGENFUNCTION OF THE DUNKL OPERATORS

In this section we collect some notations and results on Dunkl operators and the Dunkl kernel (see [3], [4] and [5]).

2.1. Reflection Groups, Root System and Multiplicity Functions.

We consider $\mathbb{R}^d$ with the euclidean scalar product $\langle \cdot, \cdot \rangle$ and $\|x\| = \sqrt{\langle x, x \rangle}$. On $\mathbb{C}^d$, $\|\cdot\|$ denotes also the standard Hermitian norm, while $\langle z, w \rangle = \sum_{j=1}^d z_j \overline{w_j}$.

For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x \rangle}{\|\alpha\|^2}\alpha. \tag{2.1}$$

A finite set $R \subset \mathbb{R}^d \backslash \{0\}$ is called a root system if $R \cap \mathbb{R} \cdot \alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflections σ_α, $\alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R. All reflections in W correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \backslash \cup_{\alpha \in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R \, : \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$. We will assume that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R_+$.

A function $k : R \longrightarrow \mathbb{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W. If one regards k as a function on the corresponding reflections, this means that k is constant on the conjugacy classes of reflections in W. For brevity, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha \in R_+} k(\alpha). \tag{2.2}$$

Moreover, let ω_k denote the weight function

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \tag{2.3}$$

which is invariant and homogeneous of degree 2γ. We introduce the Mehta-type constant

$$c_k = \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \omega_k(x) dx. \tag{2.4}$$

2.2. Dunkl operators-Dunkl kernel and Dunkl intertwining operator.

Notations. We denote by

- $C(\mathbb{R}^d)$ (resp. $C_c(\mathbb{R}^d)$) the space of continuous functions on $\mathbb{R}^d$ (resp. with compact support).
- $\mathcal{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$.
- $\mathcal{S}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ which are rapidly decreasing as their derivatives.
- $D(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ which are of compact support.

We provide these spaces with the classical topology.

Consider also the following spaces

- $\mathcal{E}'(\mathbb{R}^d)$ the space of distributions on $\mathbb{R}^d$ with compact support. It is the topological dual of $\mathcal{E}(\mathbb{R}^d)$.
- $\mathcal{S}'(\mathbb{R}^d)$ the space of temperate distributions on $\mathbb{R}^d$. It is the topological dual of $\mathcal{S}(\mathbb{R}^d)$.

The Dunkl operators $T_j,\ j = 1, \dots, d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given by

(2.5) $$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^d).$$

In the case $k = 0$, the $T_j,\ j = 1, \dots, d$, reduce to the corresponding partial derivatives. In this paper, we will assume throughout that $k \geq 0$.

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x,y) = y_j u(x,y), & j = 1, \dots, d, \\ u(0,y) = 1, & \text{for all } y \in \mathbb{R}^d \end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $K(x,y)$ and called the Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. The Dunkl kernel possesses the following properties.

Proposition 2.1. *Let $z, w \in \mathbb{C}^d$, and $x, y \in \mathbb{R}^d$.*

i)

(2.6) $$K(z,w) = K(w,z), \quad K(z,0) = 1 \quad \textit{and} \quad K(\lambda z, w) = K(z, \lambda w), \textit{ for all } \lambda \in \mathbb{C}.$$

ii) *For all $\nu \in \mathbb{N}^d, x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$, we have*

(2.7) $$|D_z^\nu K(x,z)| \leq \|x\|^{|\nu|} \exp(\|x\| \|\operatorname{Re} z\|),$$

and for all $x, y \in \mathbb{R}^d$:

(2.8) $$|K(ix,y)| \leq 1,$$

with $D_z^\nu = \frac{\partial^\nu}{\partial z_1^{\nu_1} \cdots \partial z_d^{\nu_d}}$ and $|\nu| = \nu_1 + \cdots + \nu_d$.

iii) *For all $x, y \in \mathbb{R}^d$ and $w \in W$ we have*

(2.9) $$K(-ix,y) = \overline{K(ix,y)} \quad \textit{and} \quad K(wx,wy) = K(x,y).$$

The Dunkl intertwining operator V_k is defined on $C(\mathbb{R}^d)$ by

(2.10) $$V_k f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \quad \text{for all } x \in \mathbb{R}^d,$$

where $d\mu_x$ is a probability measure given on $\mathbb{R}^d$, with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$.

2.3. **The Dunkl Transform.** The results of this subsection are given in [7] and [18].

Notations. We denote by

– $L_k^p(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{L_k^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(x)|^p \omega_k(x)dx\right)^{\frac{1}{p}} < \infty, \quad \text{if } 1 \leq p < \infty,$$
$$\|f\|_{L_k^\infty(\mathbb{R}^d)} = \text{ess} \sup_{x\in\mathbb{R}^d} |f(x)| < \infty.$$

– $H(\mathbb{C}^d)$ the space of entire functions on $\mathbb{C}^d$, rapidly decreasing of exponential type.

– $\mathcal{H}(\mathbb{C}^d)$ the space of entire functions on $\mathbb{C}^d$, slowly increasing of exponential type.

We provide these spaces with the classical topology.

The Dunkl transform of a function f in $D(\mathbb{R}^d)$ is given by

$$\mathcal{F}_D(f)(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)K(-iy,x)\omega_k(x)dx, \quad \text{for all } y \in \mathbb{R}^d. \tag{2.11}$$

It satisfies the following properties:

i) For f in $L_k^1(\mathbb{R}^d)$ we have

$$\|\mathcal{F}_D(f)\|_{L_k^\infty(\mathbb{R}^d)} \leq \frac{1}{c_k}\|f\|_{L_k^1(\mathbb{R}^d)}. \tag{2.12}$$

ii) For f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\forall y \in \mathbb{R}^d, \quad \mathcal{F}_D(T_jf)(y) = iy_j\mathcal{F}_D(f)y), \quad j = 1,\dots,d. \tag{2.13}$$

iii) For all f in $L_k^1(\mathbb{R}^d)$ such that $\mathcal{F}_D(f)$ is in $L_k^1(\mathbb{R}^d)$, we have the inversion formula

$$f(y) = \int_{\mathbb{R}^d} \mathcal{F}_D(f)(x)K(ix,y)\omega_k(x)\,dx, \quad a.e. \tag{2.14}$$

Theorem 2.2. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism.*

i) *From $\mathcal{S}(\mathbb{R}^d)$ onto itself.*

ii) *From $D(\mathbb{R}^d)$ onto $H(\mathbb{C}^d)$.*

The inverse transform $\mathcal{F}_D^{-1}$ is given by

$$\forall y \in \mathbb{R}^d, \quad \mathcal{F}_D^{-1}(f)(y) = \mathcal{F}_D(f)(-y), \quad f \in S(\mathbb{R}^d). \tag{2.15}$$

Theorem 2.3. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism.*

i) *From $\mathcal{S}'(\mathbb{R}^d)$ onto itself.*

ii) *From $\mathcal{E}'(\mathbb{R}^d)$ onto $\mathcal{H}(\mathbb{C}^d)$.*

Theorem 2.4.

i) *Plancherel formula for $\mathcal{F}_D$. For all f in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} |f(x)|^2\omega_k(x)dx = \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2\omega_k(\xi)d\xi. \tag{2.16}$$

ii) *Plancherel theorem for $\mathcal{F}_D$. The Dunkl transform $f \to \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L_k^2(\mathbb{R}^d)$.*

2.4. **The Dunkl Convolution Operator.**

Definition 2.1. Let y be in $\mathbb{R}^d$. The Dunkl translation operator $f \mapsto \tau_y f$ is defined on $\mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}_D(\tau_y f)(x) = K(ix, y)\mathcal{F}_D(f)(x), \quad \text{for all } x \in \mathbb{R}^d. \tag{2.17}$$

Example 2.1. Let $t > 0$, we have

$$\tau_x(e^{-t\|\xi\|^2})(y) = e^{-t(\|x\|^2+\|y\|^2)}K(2tx, y), \quad \text{for all } x \in \mathbb{R}^d.$$

Remark 1. The operator τ_y, $y \in \mathbb{R}^d$, can also be defined on $\mathcal{E}(\mathbb{R}^d)$ by

$$\tau_y f(x) = (V_k)_x(V_k)_y[(V_k)^{-1}(f)(x+y)], \quad \text{for all } x \in \mathbb{R}^d \tag{2.18}$$

(see [18]).

At the moment an explicit formula for the Dunkl translation operators is known only in the following two cases. (See [11] and [13]).

- 1^{st} case: $d = 1$ and $W = \mathbb{Z}_2$.
- 2^{nd} case: For all f in $\mathcal{E}(\mathbb{R}^d)$ radial we have

$$\tau_y f(x) = V_k\left[f_0\left(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle x, \cdot\rangle}\right)\right](x), \quad \text{for all } x \in \mathbb{R}^d, \tag{2.19}$$

  with f_0 the function on $[0, \infty[$ given by

$$f(x) = f_0(\|x\|).$$

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (see [11] and [18]).

Definition 2.2. The Dunkl convolution product of f and g in $D(\mathbb{R}^d)$ is the function $f *_D g$ defined by

$$f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y)g(y)\omega_k(y)dy, \quad \text{for all } x \in \mathbb{R}^d. \tag{2.20}$$

This convolution is commutative, associative and satisfies the following properties. (See [13]).

Proposition 2.5.

i) *For f and g in $D(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$) the function $f *_D g$ belongs to $D(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$) and we have*

$$\mathcal{F}_D(f *_D g)(y) = \mathcal{F}_D(f)(y)\mathcal{F}_D(g)(y), \quad \textit{for all } y \in \mathbb{R}^d.$$

ii) *Let $1 \leq p, q, r \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If f is in $L^p_k(\mathbb{R}^d)$ and g is a radial element of $L^q_k(\mathbb{R}^d)$, then $f *_D g \in L^r_k(\mathbb{R}^d)$ and we have*

$$\|f *_D g\|_{L^r_k(\mathbb{R}^d)} \leq \|f\|_{L^p_k(\mathbb{R}^d)} \|g\|_{L^q_k(\mathbb{R}^d)}. \tag{2.21}$$

iii) *Let $W = \mathbb{Z}_2^d$. We have the same result for all $f \in L^p_k(\mathbb{R}^d)$ and $g \in L^q_k(\mathbb{R}^d)$.*

3. LITTLEWOOD-PALEY THEORY ASSOCIATED WITH DUNKL OPERATORS

We consider now a dyadic decomposition of $\mathbb{R}^d$.

3.1. Dyadic Decomposition.

For $p \geq 0$ be a natural integer, we set

$$C_p = \{\xi \in \mathbb{R}^d;\ 2^{p-1} \leq \|\xi\| \leq 2^{p+1}\} = 2^p C_0 \tag{3.1}$$

and

$$C_{-1} = B(0,1) = \{\xi \in \mathbb{R}^d;\ \|\xi\| \leq 1\}. \tag{3.2}$$

Clearly $\mathbb{R}^d = \bigcup_{p=-1}^{\infty} C_p$.

Remark 2. We remark that

$$\operatorname{card}\left\{q;\ C_p \bigcap C_q \neq \varnothing\right\} \leq 2. \tag{3.3}$$

Now, let us define a dyadic partition of unity that we shall use throughout this paper.

Lemma 3.1. *There exist positive functions φ and ψ in $D(\mathbb{R}^d)$, radial with supp $\psi \subset C_{-1}$, and supp $\varphi \subset C_0$, such that for any $\xi \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we have*

$$\psi(\xi) + \sum_{p=0}^{\infty} \varphi(2^{-p}\xi) = 1$$

and

$$\psi(\xi) + \sum_{p=0}^{n} \varphi(2^{-p}\xi) = \psi(2^{-n}\xi).$$

Remark 3. It is not hard to see that for any $\xi \in \mathbb{R}^d$

$$\frac{1}{2} \leq \psi^2(\xi) + \sum_{p=0}^{\infty} \varphi^2(2^{-p}\xi) \leq 2. \tag{3.4}$$

Definition 3.1. Let $\lambda \in \mathbb{R}$. For χ in $\mathcal{S}(\mathbb{R}^d)$, we define the pseudo-differential-difference operator $\chi(\lambda T)$ by

$$\mathcal{F}_D(\chi(\lambda T)u) = \chi(\lambda\xi)\mathcal{F}_D(u), \qquad u \in \mathcal{S}'(\mathbb{R}^d).$$

Definition 3.2. For u in $\mathcal{S}'(\mathbb{R}^d)$, we define its Littlewood-Paley decomposition associated with the Dunkl operators (or dyadic decomposition) $\{\Delta_p u\}_{p=-1}^{\infty}$ as $\Delta_{-1}u = \psi(T)u$ and for $q \geq 0$, $\Delta_q u = \varphi(2^{-q}T)u$.

Now we go to see in which case we can have the identity

$$Id = \sum_{p \geq -1} \Delta_p.$$

This is described by the following proposition.

Proposition 3.2. *For u in $\mathcal{S}'(\mathbb{R}^d)$, we have $u = \sum_{p=-1}^{\infty} \Delta_p u$, in the sense of $\mathcal{S}'(\mathbb{R}^d)$.*

Proof. For any f in $\mathcal{S}(\mathbb{R}^d)$, it is easy to see that $\mathcal{F}_D(f) = \sum_{p=-1}^{\infty} \mathcal{F}_D(\Delta_p f)$ in the sense of $\mathcal{S}(\mathbb{R}^d)$. Then for any u in $\mathcal{S}'(\mathbb{R}^d)$, we have

$$\begin{aligned}\langle u, f\rangle &= \langle \mathcal{F}_D(u), \mathcal{F}_D(f)\rangle \\ &= \sum_{p=-1}^{\infty} \langle \mathcal{F}_D(u), \mathcal{F}_D(\Delta_p f)\rangle \\ &= \sum_{p=-1}^{\infty} \langle \mathcal{F}_D(\Delta_p u), \mathcal{F}_D(f)\rangle \\ &= \left\langle \sum_{p=-1}^{\infty} \mathcal{F}_D(\Delta_p u), \mathcal{F}_D(f)\right\rangle = \left\langle \sum_{p=-1}^{\infty} \Delta_p u, f\right\rangle.\end{aligned}$$

The proof is finished. □

3.2. **The Generalized Sobolev Spaces.** In this subsection we will give a characterization of Sobolev spaces associated with the Dunkl operators by a Littlewood-Paley decomposition. First, we recall the definition of these spaces (see [9]).

Definition 3.3. Let s be in $\mathbb{R}$, we define the space $H_k^s(\mathbb{R}^d)$ by

$$\left\{u \in \mathcal{S}'(\mathbb{R}^d) : (1 + \|\xi\|^2)^{\frac{s}{2}} \mathcal{F}_D(u) \in L_k^2(\mathbb{R}^d)\right\}.$$

We provide this space by the scalar product

$$\langle u, v\rangle_{H_k^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^s \mathcal{F}_D(u)(\xi)\, \overline{\mathcal{F}_D(v)(\xi)} \omega_k(\xi) d\xi, \tag{3.5}$$

and the norm

$$\|u\|^2_{H_k^s(\mathbb{R}^d)} = \langle u, u\rangle_{H_k^s(\mathbb{R}^d)}. \tag{3.6}$$

Another proposition will be useful. Let $S_q u = \sum_{p \le q-1} \Delta_p u$.

Proposition 3.3. *For all s in $\mathbb{R}$ and for all distributions u in $H_k^s(\mathbb{R}^d)$, we have*

$$\lim_{n\to\infty} S_n u = u.$$

Proof. For all ξ in $\mathbb{R}^d$, we have

$$\mathcal{F}_D(S_n u - u)(\xi) = (\psi(2^{-n}\xi) - 1)\mathcal{F}_D(u)(\xi).$$

Hence

$$\lim_{n\to\infty} \mathcal{F}_D(S_n u - u)(\xi) = 0.$$

On the other hand

$$(1 + \|\xi\|^2)^s |\mathcal{F}_D(S_n u - u)(\xi)|^2 \le 2(1 + \|\xi\|^2)^s |\mathcal{F}_D(u)(\xi)|^2.$$

Thus the result follows from the dominated convergence theorem. □

The first application of the Littlewood-Paley decomposition associated with the Dunkl operators is the characterization of the Sobolev spaces associated with these operators through the behavior on q of $\|\Delta_q u\|_{L^2_k(\mathbb{R}^d)}$. More precisely, we now define a norm equivalent to the norm $\|\cdot\|_{H^s_k(\mathbb{R}^d)}$ in terms of the dyadic decomposition.

Proposition 3.4. *There exists a positive constant C such that for all s in $\mathbb{R}$, we have*

$$\frac{1}{C^{|s|+1}}\|u\|^2_{H^s_k(\mathbb{R}^d)} \le \sum_{q\ge -1} 2^{2qs}\|\Delta_q u\|^2_{L^2_k(\mathbb{R}^d)} \le C^{|s|+1}\|u\|^2_{H^s_k(\mathbb{R}^d)}.$$

Proof. Since $\operatorname{supp}\mathcal{F}_{\mathrm{D}}(\Delta_{\mathrm{q}}\mathrm{u}) \subset \mathrm{C_q}$, from the definition of the norm $\|\cdot\|_{H^s_k(\mathbb{R}^d)}$, there exists a positive constant C such that we have

$$\frac{1}{C^{|s|+1}}2^{qs}\|\Delta_q u\|_{L^2_k(\mathbb{R}^d)} \le \|\Delta_q u\|_{H^s_k(\mathbb{R}^d)} \le C^{|s|+1}2^{qs}\|\Delta_q u\|_{L^2_k(\mathbb{R}^d)}. \tag{3.7}$$

From (3.4) we deduce that

$$\begin{aligned}\frac{1}{2}\|u\|^2_{H^s_k(\mathbb{R}^d)} &\le \int_{\mathbb{R}^d}\left[\psi^2(\xi)+\sum_{q=0}^{\infty}\varphi^2(2^{-q}\xi)\right](1+\|\xi\|^2)^s|\mathcal{F}_D(u)(\xi)|^2\omega_k(\xi)d\xi\\ &\le 2\|u\|^2_{H^s_k(\mathbb{R}^d)}.\end{aligned}$$

Hence

$$\frac{1}{2}\|u\|^2_{H^s_k(\mathbb{R}^d)} \le \sum_{q\ge -1}\|\Delta_q u\|^2_{H^s_k(\mathbb{R}^d)} \le 2\|u\|^2_{H^s_k(\mathbb{R}^d)}.$$

Thus from this and (3.7) we deduce the result. □

The following theorem is a consequence of Proposition 3.4.

Theorem 3.5. *Let u be in $\mathcal{S}'(\mathbb{R}^d)$ and $u = \sum_{q\ge -1}\Delta_q u$ its Littlewood-Paley decomposition. The following are equivalent:*

i) $u \in H^s_k(\mathbb{R}^d)$.
ii) $\sum_{q\ge -1} 2^{2qs}\|\Delta_q u\|^2_{L^2_k(\mathbb{R}^d)} < \infty$.
iii) $\|\Delta_q u\|_{L^2_k(\mathbb{R}^d)} \le c_q 2^{-qs}$, *with* $\{c_q\} \in l^2$.

Remark 4. Since for u in $\mathcal{S}'(\mathbb{R}^d)$ we have $\Delta_p u$ in $\mathcal{S}'(\mathbb{R}^d)$ and $\operatorname{supp}\mathcal{F}_D(\Delta_p u) \subset C_p$, from Theorem 2.3 ii) we deduce that $\Delta_p u$ is in $\mathcal{E}(\mathbb{R}^d)$.

The following propositions will be very useful.

Proposition 3.6. *Let $\widetilde{C}$ be an annulus in $\mathbb{R}^d$ and s in $\mathbb{R}$. Let $(u_p)_{p\in\mathbb{N}}$ be a sequence of smooth functions. If the sequence $(u_p)_{p\in\mathbb{N}}$ satisfies*

$$\operatorname{supp}\mathcal{F}_D(u_p) \subset 2^p\widetilde{C} \quad \text{and} \quad \|u_p\|_{L^2_k(\mathbb{R}^d)} \le Cc_p 2^{-ps}, \{c_p\} \in l^2,$$

then we have

$$u = \sum_{p\ge 0} u_p \in H^s_k(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{H^s_k(\mathbb{R}^d)} \le C(s)\left(\sum_{p\ge 0} 2^{2ps}\|u_p\|^2_{L^2_k(\mathbb{R}^d)}\right)^{\frac{1}{2}}.$$

Proof. Since $\widetilde{C}$ and C_0 are two annuli, there exists an integer N_0 so that

$$|p-q| > N_0 \implies 2^p C_0 \bigcap 2^q \widetilde{C} = \varnothing.$$

It is clear that

$$|p-q| > N_0 \Longrightarrow \mathcal{F}_D(\Delta_q u_p) = 0.$$

Then

$$\Delta_q u = \sum_{|p-q| \leq N_0} \Delta_q u_p.$$

By the triangle inequality and definition of $\Delta_q u_p$ we deduce that

$$\|\Delta_q u\|_{L^2_k(\mathbb{R}^d)} \leq \sum_{|p-q| \leq N_0} \|u_p\|_{L^2_k(\mathbb{R}^d)}.$$

Thus the Cauchy-Schwartz inequality implies that

$$\sum_{q \geq 0} 2^{2qs} \|\Delta_q u\|^2_{L^2_k(\mathbb{R}^d)} \leq C \left(\sum_{q/|p-q| \leq N_0} 2^{2(q-p)s} \right) \left(\sum_{p \geq 0} 2^{2ps} \|u_p\|^2_{L^2_k(\mathbb{R}^d)} \right).$$

From Theorem 3.5 we deduce that if $\|u_p\|_{L^2_k(\mathbb{R}^d)} \leq C c_p 2^{-ps}$ then $u \in H^s_k(\mathbb{R}^d)$. □

Proposition 3.7. *Let $K > 0$ and $s > 0$. Let $(u_p)_{p \in \mathbb{N}}$ be a sequence of smooth functions. If the sequence $(u_p)_{p \in \mathbb{N}}$ satisfies*

$$\operatorname{supp} \mathcal{F}_D(u_p) \subset B(0, K2^p) \quad \text{and} \quad \|u_p\|_{L^2_k(\mathbb{R}^d)} \leq C c_p 2^{-ps}, \{c_p\} \in l^2,$$

then we have

$$u = \sum_{p \geq 0} u_p \in H^s_k(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{H^s_k(\mathbb{R}^d)} \leq C(s) \left(\sum_{q \geq 0} 2^{2ps} \|u_p\|^2_{L^2_k(\mathbb{R}^d)} \right)^{\frac{1}{2}}.$$

Proof. Since $\operatorname{supp} \mathcal{F}_D(u_p) \subset B(0, K2^p)$, there exists N_1 such that

$$\Delta_q u = \sum_{p \geq q - N_1} \Delta_q u_p.$$

So, we get that

$$\begin{aligned} 2^{qs} \|\Delta_q u\|_{L^2_k(\mathbb{R}^d)} &\leq \sum_{p \geq q - N_1} 2^{qs} \|u_p\|_{L^2_k(\mathbb{R}^d)} \\ &= \sum_{p \geq q - N_1} 2^{(q-p)s} 2^{ps} \|u_p\|_{L^2_k(\mathbb{R}^d)}. \end{aligned}$$

Since $s > 0$, the Cauchy-Schwartz inequality implies

$$\sum_q 2^{2qs} \|\Delta_q u\|^2_{L^2_k(\mathbb{R}^d)} \leq \frac{2^{2N_1 s}}{1 - 2^{-s}} \sum_p 2^{2ps} \|u_p\|^2_{L^2_k(\mathbb{R}^d)}.$$

From Theorem 3.5 we deduce the result. □

Proposition 3.8. *Let $s > 0$ and $(u_p)_{p\in\mathbb{N}}$ be a sequence of smooth functions. If the sequence $(u_p)_{p\in\mathbb{N}}$ satisfies*

$$u_p \in \mathcal{E}(\mathbb{R}^d) \quad \text{and} \quad \textit{for all } \mu \in \mathbb{N}^d,\ \|T^\mu u_p\|_{L^2_k(\mathbb{R}^d)} \le C c_{p,\mu} 2^{-p(s-|\mu|)},\ \{c_{p,\mu}\} \in l^2,$$

then we have

$$u = \sum_{p\ge 0} u_p \in H^s_k(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{H^s_k(\mathbb{R}^d)} \le C(s) \left(\sum_{p\ge 0} 2^{2ps}\|u_p\|^2_{L^2_k(\mathbb{R}^d)}\right)^{\frac{1}{2}}.$$

Proof. By the assumption we first have $u = \sum u_p \in L^2_k(\mathbb{R}^d)$. Take $\mu \in \mathbb{N}^d$ with $|\mu| = s_0 > s > 0$, and $\chi_p(\xi) = \chi(2^{-p}\xi) \in D(\mathbb{R}^d)$ with $\operatorname{supp}\chi \subset \mathrm{B}(0,2)$, $\chi(\xi) = 1$, $\|\xi\| \le 1$ and $0 \le \chi \le 1$, then

$$\operatorname{supp}\chi_{\mathrm{p}}(1-\chi_{\mathrm{p}}) \subset \left\{\xi \in \mathbb{R}^{\mathrm{d}}; 2^{\mathrm{p}} \le \|\xi\| \le 2^{\mathrm{p}+2}\right\}.$$

Set

$$\begin{aligned}\mathcal{F}_D(u_p)(\xi) &= \chi_p(\xi)\mathcal{F}_D(u_p)(\xi) + (1-\chi_p(\xi))\mathcal{F}_D(u_p)(\xi)\\ &= \mathcal{F}_D(u_p^{(1)})(\xi) + \mathcal{F}_D(u_p^{(2)})(\xi),\end{aligned}$$

and we have

$$\begin{aligned}\|u_p\|^2_{L^2_k(\mathbb{R}^d)} &= \|\mathcal{F}_D(u_p)\|^2_{L^2_k(\mathbb{R}^d)}\\ &= \left[\int_{\mathbb{R}^d} |\mathcal{F}_D(u_p^{(1)})(\xi)|^2\omega_k(\xi)d\xi + \int_{\mathbb{R}^d} |\mathcal{F}_D(u_p^{(2)})(\xi)|^2\omega_k(\xi)d\xi\right.\\ &\qquad \left. +2\int_{\mathbb{R}^d} |\mathcal{F}_D(u_p)(\xi)|^2\chi_p(\xi)(1-\chi_p(\xi))\omega_k(\xi)d\xi\right].\end{aligned}$$

Since $0 \le \chi_p(\xi)(1-\chi_p(\xi)) \le 1$, we deduce that

$$\left\|u_p^{(1)}\right\|^2_{L^2_k(\mathbb{R}^d)} + \left\|u_p^{(2)}\right\|^2_{L^2_k(\mathbb{R}^d)} \le \|u_p\|^2_{L^2_k(\mathbb{R}^d)} \le c_p^2 2^{-2ps}.$$

Similarly, using Theorem 3.1 of [9], we obtain

$$\left\|u_p^{(1)}\right\|^2_{H^{s_0}_k(\mathbb{R}^d)} + \left\|u_p^{(2)}\right\|^2_{H^{s_0}_k(\mathbb{R}^d)} \le \|u_p\|^2_{H^{s_0}_k(\mathbb{R}^d)} \le c_p^2 2^{-2p(s-s_0)}.$$

Set $u^{(1)} = \sum_p u_p^{(1)}$, $u^{(2)} = \sum_p u_p^{(2)}$, then $u = u^{(1)} + u^{(2)}$, and from Proposition 3.7 we deduce that $u^{(1)}$ belongs to $H^s_k(\mathbb{R}^d)$. For $u^{(2)}$ the definition of $u_p^{(2)}$ gives that

$$\|\Delta_q(u^{(2)})\|^2_{L^2_k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left|\sum_{p\le q+1} \varphi(2^{-q}\xi)\mathcal{F}_D(u_p^{(2)})(\xi)\right|^2 \omega_k(\xi)d\xi.$$

Thus by the Cauchy-Schwartz inequality we have

$$\begin{aligned}&\|\Delta_q(u^{(2)})\|^2_{L^2_k(\mathbb{R}^d)}\\ &\le \left(\sum_{p\le q+1} 2^{-2p(s-s_0)}\right)\left(\int_{\mathbb{R}^d}\sum_{p\le q+1} 2^{2p(s-s_0)}|\varphi(2^{-q}\xi)\mathcal{F}_D(u_p^{(2)})(\xi)|^2\omega_k(\xi)d\xi\right)\\ &\le \frac{1-2^{-2(q+2)(s-s_0)}}{1-2^{-(s-s_0)}} 2^{-2qs_0} \sum_{p\le q+1} 2^{2p(s-s_0)}\left\|\Delta_q\left(u_p^{(2)}\right)\right\|^2_{H^{s_0}_k(\mathbb{R}^d)}.\end{aligned}$$

Moreover, since $s_0 > s > 0$,

$$\frac{1-2^{-2(q+2)(s-s_0)}}{1-2^{-(s-s_0)}}2^{-2qs_0} \leq C2^{-2qs},$$

and C is independent of q. Now set

$$c_q^2 = \sum_{p\leq q+1} 2^{2p(s-s_0)} \left\|\Delta_q\left(u_p^{(2)}\right)\right\|^2_{H_k^{s_0}(\mathbb{R}^d)},$$

then

$$\sum_{q\geq -1} 2^{2qs}\|\Delta_q(u^{(2)})\|^2_{L_k^2(\mathbb{R}^d)} \leq \sum_{q\geq -1} c_q^2 \leq \sum_p 2^{2p(s-s_0)}\left\|u_p^{(2)}\right\|^2_{H_k^{s_0}(\mathbb{R}^d)} < \infty.$$

Thus by Theorem 3.5 we deduce that $u^{(2)} = \sum_q \Delta_q(u^{(2)})$ belongs to $H_k^s(\mathbb{R}^d)$. □

Corollary 3.9. *The spaces $H_k^s(\mathbb{R}^d)$ do not depend on the choice of the function φ and ψ used in the Definition 3.2.*

3.3. **The Generalized Hölder Spaces.**

Definition 3.4. For α in $\mathbb{R}$, we define the Hölder space $C_k^\alpha(\mathbb{R}^d)$ associated with the Dunkl operators as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ satisfying

$$\|u\|_{C_k^\alpha(\mathbb{R}^d)} = \sup_{p\geq -1} 2^{p\alpha}\|\Delta_p u\|_{L_k^\infty(\mathbb{R}^d)} < \infty,$$

where $u = \sum_{p\geq -1}\Delta_p u$ is its Littlewood-Paley decomposition.

In the following proposition we give sufficient conditions so that the series $\sum_q u_q$ belongs to the Hölder spaces associated with the Dunkl operators.

Proposition 3.10.

i) *Let $\widetilde{C}$ be an annulus in $\mathbb{R}^d$ and $\alpha \in \mathbb{R}$. Let $(u_p)_{p\in\mathbb{N}}$ be a sequence of smooth functions. If the sequence $(u_p)_{p\in\mathbb{N}}$ satisfies*

$$\operatorname{supp}\mathcal{F}_{\mathrm{D}}(\mathrm{u_p}) \subset 2^{\mathrm{p}}\widetilde{\mathrm{C}} \quad \text{and} \quad \|\mathrm{u_p}\|_{\mathrm{L_k^\infty(\mathbb{R}^d)}} \leq \mathrm{C}2^{-\mathrm{p}\alpha},$$

then we have

$$u = \sum_{p\geq 0} u_p \in C_k^\alpha(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{C_k^\alpha(\mathbb{R}^d)} \leq C(\alpha)\sup_{p\geq 0} 2^{p\alpha}\|u_p\|_{L_k^\infty(\mathbb{R}^d)}.$$

ii) *Let $K > 0$ and $\alpha > 0$. Let $(u_p)_{p\in\mathbb{N}}$ be a sequence of smooth functions. If the sequence $(u_p)_{p\in\mathbb{N}}$ satisfies*

$$\operatorname{supp}\mathcal{F}_D(u_p) \subset B(0, K2^p) \quad \text{and} \quad \|u_p\|_{L_k^\infty(\mathbb{R}^d)} \leq C2^{-p\alpha},$$

then we have

$$u = \sum_{p\geq 0} u_p \in C_k^\alpha(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{C_k^\alpha(\mathbb{R}^d)} \leq C(\alpha)\sup_{p\geq 0} 2^{p\alpha}\|u_p\|_{L_k^\infty(\mathbb{R}^d)}.$$

Proof. The proof uses the same idea as for Propositions 3.6 and 3.7. □

Proposition 3.11. *The distribution defined by*

$$g(x) = \sum_{p \geq 0} K(ix, 2^p e), \quad \text{with } \; e = (1, \dots, 1),$$

belongs to $C^0_k(\mathbb{R}^d)$ *and does not belong to* $L^\infty_k(\mathbb{R}^d)$.

Proposition 3.12. *Let* $\varepsilon \in]0,1[$ *and* f *in* $C^\varepsilon_k(\mathbb{R}^d)$, *then there exists a positive constant* C *such that*

$$\|f\|_{L^\infty_k(\mathbb{R}^d)} \leq \frac{C}{\varepsilon} \|f\|_{C^0_k(\mathbb{R}^d)} \log\left(e + \frac{\|f\|_{C^\varepsilon_k(\mathbb{R}^d)}}{\|f\|_{C^0_k(\mathbb{R}^d)}}\right).$$

Proof. Since $f = \sum_{p \geq -1} \Delta_p f$,

$$\|f\|_{L^\infty_k(\mathbb{R}^d)} \leq \sum_{p \leq N-1} \|\Delta_p f\|_{L^\infty_k(\mathbb{R}^d)} + \sum_{p \geq N} \|\Delta_p f\|_{L^\infty_k(\mathbb{R}^d)},$$

with N is a positive integer that will be chosen later. Since $f \in C^\varepsilon_k(\mathbb{R}^d)$, using the definition of generalized Hölderien norms, we deduce that

$$\|f\|_{L^\infty_k(\mathbb{R}^d)} \leq (N+1)\|f\|_{C^0_k(\mathbb{R}^d)} + \frac{2^{-(N-1)\varepsilon}}{2^\varepsilon - 1} \|f\|_{C^\varepsilon_k(\mathbb{R}^d)}.$$

We take

$$N = 1 + \left[\frac{1}{\varepsilon} \log_2 \frac{\|f\|_{C^\varepsilon_k(\mathbb{R}^d)}}{\|f\|_{C^0_k(\mathbb{R}^d)}}\right],$$

we obtain

$$\|f\|_{L^\infty_k(\mathbb{R}^d)} \leq \frac{C}{\varepsilon} \|f\|_{C^0_k(\mathbb{R}^d)} \left[1 + \log\left(\frac{\|f\|_{C^\varepsilon_k(\mathbb{R}^d)}}{\|f\|_{C^0_k(\mathbb{R}^d)}}\right)\right].$$

This implies the result. □

Now we give the characterization of $L^p_k(\mathbb{R}^d)$ spaces by using the dyadic decomposition.

If $(f_j)_{j \in \mathbb{N}}$ is a sequence of $L^p_k(\mathbb{R}^d)$-functions, we set

$$\|(f_j)\|_{L^p_k(l^2)} = \left\| \left(\sum_{j \in \mathbb{N}} |f_j(x)|^2\right)^{\frac{1}{2}} \right\|_{L^p_k(\mathbb{R}^d)},$$

the norm in $L^p_k(\mathbb{R}^d, l^2(\mathbb{N}))$.

Theorem 3.13 (Littlewood-Paley decomposition of $L^p_k(\mathbb{R}^d)$)**.** *Let* f *be in* $\mathcal{S}'(\mathbb{R}^d)$ *and* $1 < p < \infty$. *Then the following assertions are equivalent*

i) $f \in L^p_k(\mathbb{R}^d)$,

ii) $S_0 f \in L^p_k(\mathbb{R}^d)$ *and* $\left(\sum_{j \in \mathbb{N}} |\Delta_j f(x)|^2\right)^{\frac{1}{2}} \in L^p_k(\mathbb{R}^d)$.

Moreover, the following norms are equivalent :

$$\|f\|_{L^p_k(\mathbb{R}^d)} \quad \text{and} \quad \|S_0 f\|_{L^p_k(\mathbb{R}^d)} + \left\| \left(\sum_{j \in \mathbb{N}} |\Delta_j f(x)|^2\right)^{\frac{1}{2}} \right\|_{L^p_k(\mathbb{R}^d)}.$$

Proof. If f is in $L^2_k(\mathbb{R}^d)$, then from Proposition 3.4 we have

$$\left\| \left(\sum_{j\in\mathbb{N}} |\Delta_j f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^2_k(\mathbb{R}^d)} \leq \|f\|^2_{L^2_k(\mathbb{R}^d)}.$$

Thus the mapping

$$\Lambda_1 : f \mapsto (\Delta_j f)_{j\in\mathbb{N}},$$

is bounded from $L^2_k(\mathbb{R}^d)$ into $L^2_k(\mathbb{R}^d, l^2(\mathbb{N}))$.

On the other hand, from properties of φ we see that

$$\begin{aligned} \|(\widetilde{\varphi}_j(x))_j\|_{l^2} &\leq C\|x\|^{-(d+2\gamma)}, \quad \text{for } x \neq 0, \\ \|(\partial_{y_i}\widetilde{\varphi}_j(x))_j\|_{l^2} &\leq C\|x\|^{-(d+2\gamma)}, \quad \text{for } x \neq 0,\ i = 1, \dots, d, \end{aligned}$$

where

$$\widetilde{\varphi}_j(x) = 2^{j(d+2\gamma)} \mathcal{F}_D^{-1}(\varphi)(2^j x).$$

We may then apply the theory of singular integrals to this mapping Λ_1 (see [14]).

Thus we deduce that

$$\|\Delta_j f\|_{L^p_k(l^2)} \leq C_{p,k} \|f\|_{L^p_k(\mathbb{R}^d)}, \quad \text{for } 1 < p < \infty.$$

The converse uses the same idea. Indeed we put

$$\widetilde{\phi}_j = \sum_{i=-1}^{1} \widetilde{\varphi}_{j+i}.$$

From Proposition 3.4 the mapping

$$\Lambda_2 : (f_j)_{j\in\mathbb{N}} \mapsto \sum_{j\in\mathbb{N}} f_j *_D \widetilde{\phi}_j,$$

is bounded from $L^2_k(\mathbb{R}^d, l^2(\mathbb{N}))$ into $L^2_k(\mathbb{R}^d)$.

On the other hand, from properties of φ we see that

$$\begin{aligned} \|(\widetilde{\phi}_j(x))_j\|_{l^2} &\leq C\|x\|^{-(d+2\gamma)}, \quad \text{for } x \neq 0, \\ \|(\partial_{y_i}\widetilde{\phi}_j(x))_j\|_{l^2} &\leq C\|x\|^{-(d+2\gamma)}, \quad \text{for } x \neq 0,\ i = 1, \dots, d. \end{aligned}$$

We may then apply the theory of singular integrals to this mapping Λ_2 (see [14]).

Thus we obtain

$$\left\| \sum_{j\in\mathbb{N}} \Delta_j f \right\|_{L^p_k(\mathbb{R}^d)} \leq C_{p,k} \|\Delta_j f\|_{L^p_k(l^2)}.$$

□

4. APPLICATIONS

4.1. Estimates of the Product of Two Functions.

Proposition 4.1.

i) *Let* $u, v \in C_k^\alpha(\mathbb{R}^d)$ *and* $\alpha > 0$ *then* $uv \in C_k^\alpha(\mathbb{R}^d)$*, and*

$$\|uv\|_{C_k^\alpha(\mathbb{R}^d)} \leq C\left[\|u\|_{L_k^\infty(\mathbb{R}^d)}\|v\|_{C_k^\alpha(\mathbb{R}^d)} + \|v\|_{L_k^\infty(\mathbb{R}^d)}\|u\|_{C_k^\alpha(\mathbb{R}^d)}\right].$$

ii) *Let* $u, v \in H_k^s(\mathbb{R}^d) \bigcap L_k^\infty(\mathbb{R}^d)$ *and* $s > 0$ *then* $uv \in H_k^s(\mathbb{R}^d)$*, and*

$$\|uv\|_{H_k^s(\mathbb{R}^d)} \leq C\left[\|u\|_{L_k^\infty(\mathbb{R}^d)}\|v\|_{H_k^s(\mathbb{R}^d)} + \|v\|_{L_k^\infty(\mathbb{R}^d)}\|u\|_{H_k^s(\mathbb{R}^d)}\right].$$

Proof. Let $u = \sum_p \Delta_p u$ and $v = \sum_q \Delta_q v$ be their Littlewood-Paley decompositions. Then we have

$$\begin{aligned} uv &= \sum_{p,q} \Delta_p u \Delta_q v \\ &= \sum_q \sum_{p \leq q-1} \Delta_p u \Delta_q v + \sum_q \sum_{p \geq q} \Delta_p u \Delta_q v \\ &= \sum_q \sum_{p \leq q-1} \Delta_p u \Delta_q v + \sum_p \sum_{q \leq p} \Delta_p u \Delta_q v \\ &= \sum_q S_q u \Delta_q v + \sum_p S_{p+1} v \Delta_p u \\ &= \sum_1 + \sum_2 . \end{aligned}$$

We have

$$\operatorname{supp}(\mathcal{F}_D(S_q u \Delta_q v)) = \operatorname{supp}(\mathcal{F}_D(\Delta_q v) *_D \mathcal{F}_D(S_q u)).$$

Hence from Theorem 2.2 we deduce that $\operatorname{supp}(\mathcal{F}_D(S_q u \Delta_q v)) \subset B(0, C2^q)$.

i) If u and v are in $C_k^\alpha(\mathbb{R}^d)$, then we have

$$\begin{aligned} \|S_q u \Delta_q v\|_{L_k^\infty(\mathbb{R}^d)} &\leq \|S_q u\|_{L_k^\infty(\mathbb{R}^d)}\|\Delta_q v\|_{L_k^\infty(\mathbb{R}^d)}, \\ &\leq C\|u\|_{L_k^\infty(\mathbb{R}^d)}\|v\|_{C_k^\alpha(\mathbb{R}^d)}2^{-q\alpha}. \end{aligned}$$

From Proposition 3.10 ii) we deduce

$$\left\|\sum_1\right\|_{C_k^\alpha(\mathbb{R}^d)} \leq C\|u\|_{L_k^\infty(\mathbb{R}^d)}\|v\|_{C_k^\alpha(\mathbb{R}^d)}.$$

Similarly we prove that

$$\left\|\sum_2\right\|_{C_k^\alpha(\mathbb{R}^d)} \leq C\|v\|_{L_k^\infty(\mathbb{R}^d)}\|u\|_{C_k^\alpha(\mathbb{R}^d)},$$

and this implies the result.

ii) If u and v are in $H_k^s(\mathbb{R}^d)$, then we have

$$\begin{aligned}\|S_q u \Delta_q v\|_{L^2_k(\mathbb{R}^d)} &\leq \|S_q u\|_{L^\infty_k(\mathbb{R}^d)} \|\Delta_q v\|_{L^2_k(\mathbb{R}^d)},\\ &\leq C\|u\|_{L^\infty_k(\mathbb{R}^d)} \|v\|_{H^s_k(\mathbb{R}^d)} c_q 2^{-qs}.\end{aligned}$$

Thus Proposition 3.7 gives

$$\left\|\sum_1\right\|_{H^s_k(\mathbb{R}^d)} \leq C\|u\|_{L^\infty_k(\mathbb{R}^d)} \|v\|_{H^s_k(\mathbb{R}^d)}.$$

Similarly, we prove that

$$\left\|\sum_2\right\|_{H^s_k(\mathbb{R}^d)} \leq C\|v\|_{L^\infty_k(\mathbb{R}^d)} \|u\|_{H^s_k(\mathbb{R}^d)},$$

and this implies the result. □

Corollary 4.2. *For $s > \frac{d}{2} + \gamma$, $H^s_k(\mathbb{R}^d)$ is an algebra.*

4.2. **Sobolev Embedding Theorem.** Using the Littlewood-Paley decomposition, we have a very simple proof of Sobolev embedding theorems:

Theorem 4.3. *For any $s > \gamma + \frac{d}{2}$, we have the continuous embedding*

$$H^s_k(\mathbb{R}^d) \hookrightarrow C_k^{s-\gamma-\frac{d}{2}}(\mathbb{R}^d).$$

Proof. Let u be in $H^s_k(\mathbb{R}^d)$, $u = \sum_{p\geq -1} \Delta_p u$ the Littlewood-Paley decomposition. Take ϕ in $D(\mathbb{R}^d)$ such that $\phi(\xi) = 1$ on C_0, and

$$\operatorname{supp}\phi \subset C'_0 = \left\{\xi \in \mathbb{R}^d, \frac{1}{3} \leq \|\xi\| \leq 3\right\}.$$

Setting $\phi_p(\xi) = \phi(2^{-p}\xi)$, we obtain

$$\mathcal{F}_D(\Delta_p u)(\xi) = \mathcal{F}_D(\Delta_p u)(\xi)\phi(2^{-p}\xi).$$

Hence

$$\begin{aligned}\Delta_p u(x) &= \int_{\mathbb{R}^d} \mathcal{F}_D(\Delta_p u)(\xi)\phi(2^{-p}\xi)K(ix,\xi)\omega_k(\xi)d\xi,\\ |\Delta_p u(x)| &\leq \int_{\mathbb{R}^d} |\mathcal{F}_D(\Delta_p u)(\xi)||\phi(2^{-p}\xi)|\omega_k(\xi)d\xi.\end{aligned}$$

The Cauchy-Schwartz inequality and Theorem 3.5 give that

$$\begin{aligned}\|\Delta_p u\|_{L^\infty_k(\mathbb{R}^d)} &\leq \left(\int_{\mathbb{R}^d} |\mathcal{F}_D(\Delta_p u)(\xi)|^2\omega_k(\xi)d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |\phi(2^{-p}\xi)|^2\omega_k(\xi)d\xi\right)^{\frac{1}{2}}\\ &\leq C2^{p(\gamma+\frac{d}{2})}\|\Delta_p u\|_{L^2_k(\mathbb{R}^d)}\\ &\leq C2^{-p(s-\gamma-\frac{d}{2})}c_p.\end{aligned}$$

Then from Definition 3.4 we deduce that $u \in C_k^{s-\gamma-\frac{d}{2}}(\mathbb{R}^d)$. □

Theorem 4.4. *For any* $0 < s < \gamma + \frac{d}{2}$, *we have the continuous embedding*

$$H_k^s(\mathbb{R}^d) \hookrightarrow L_k^p(\mathbb{R}^d),$$

where $p = \frac{2(2\gamma+d)}{2\gamma+d-2s}$.

Proof. Let f be in $\mathcal{S}(\mathbb{R}^d)$, we have, due to Fubini's theorem,

$$\|f\|^p_{L^p_k(\mathbb{R}^d)} = p\int_0^\infty \lambda^{p-1} m_k\Big\{|f| \geq \lambda\Big\}d\lambda, \tag{4.1}$$

where

$$m_k\Big\{|f| \geq \lambda\Big\} = \int_{\{x;\ |f(x)|\geq\lambda\}} \omega_k(x)dx.$$

For $A > 0$, we set $f = f_{1,A} + f_{2,A}$ with $f_{1,A} = \sum_{2^j<A} \Delta_j f$ and $f_{2,A} = \sum_{2^j\geq A} \Delta_j f$.

We have

$$\|f_{1,A}\|_{L^\infty_k(\mathbb{R}^d)} \leq \sum_{2^j<A} \|\Delta_j f\|_{L^\infty_k(\mathbb{R}^d)} \leq \sum_{2^j<A} \|\mathcal{F}_D(\Delta_j f)\|_{L^1_k(\mathbb{R}^d)}.$$

Using the Cauchy-Schwartz inequality, the Parseval's identity associated with the Dunkl operators and Theorem 3.5, we obtain

$$\|f_{1,A}\|_{L^\infty_k(\mathbb{R}^d)} \leq \sum_{2^j<A} 2^{j(\gamma+\frac{d}{2}-s)} c_j \|f\|_{H^s_k(\mathbb{R}^d)} \leq CA^{\gamma+\frac{d}{2}-s}\|f\|_{H^s_k(\mathbb{R}^d)}.$$

On the other hand for all $\lambda > 0$, we have

$$m_k\Big\{|f| \geq \lambda\Big\} \leq m_k\left\{|f_{1,A}| \geq \frac{\lambda}{2}\right\} + m_k\left\{|f_{2,A}| \geq \frac{\lambda}{2}\right\}. \tag{4.2}$$

From (4.2) we infer that if we take

$$A = A_\lambda = \left(\frac{\lambda}{4C\|f\|_{H^s_k(\mathbb{R}^d)}}\right)^{\frac{1}{\gamma+\frac{d}{2}-s}},$$

then

$$\|f_{1,A_\lambda}\|_{L^\infty_k(\mathbb{R}^d)} \leq \frac{\lambda}{4}.$$

Hence

$$m_k\left\{|f_{1,A_\lambda}| \geq \frac{\lambda}{2}\right\} = 0.$$

From (4.1) and (4.2) we deduce that

$$\|f\|^p_{L^p_k(\mathbb{R}^d)} \leq p\int_0^\infty \lambda^{p-1} m_k\Big\{2|f_{2,A_\lambda}| \geq \lambda\Big\}d\lambda.$$

Moreover the Bienaymé-Tchebytchev inequality yields

$$m_k\Big\{2|f_{2,A_\lambda}| \geq \lambda\Big\} \leq \frac{4}{\lambda^2}\|f_{2,A_\lambda}\|^2_{L^2_k(\mathbb{R}^d)}.$$

Thus we obtain

$$\|f\|^p_{L^p_k(\mathbb{R}^d)} \leq p\int_0^\infty \lambda^{p-3}\|f_{2,A_\lambda}\|^2_{L^2_k(\mathbb{R}^d)}d\lambda.$$

On the other hand, by using the Cauchy-Schwartz inequality for all $\varepsilon > 0$, we have

$$\begin{aligned}\|f_{2,A_\lambda}\|^2_{L^2_k(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\left|\sum_{2^j\geq A_\lambda}\Delta_j f(x)\right|^2\omega_k(x)dx\\ &\leq \left(\int_{\mathbb{R}^d}\sum_{2^j\geq A_\lambda}2^{2j\varepsilon}|\Delta_j f(x)|^2\omega_k(x)dx\right)\left(\sum_{2^j\geq A_\lambda}2^{-2j\varepsilon}\right)\\ &\leq A_\lambda^{-2\varepsilon}\sum_{2^j\geq A_\lambda}2^{2j\varepsilon}\|\Delta_j f\|^2_{L^2_k(\mathbb{R}^d)}.\end{aligned}$$

So by using the definition of A_λ and the Fubini theorem, we can write

$$\begin{aligned}&\|f\|^p_{L^p_k(\mathbb{R}^d)}\\ &\leq p\int_0^\infty \lambda^{p-3}A_\lambda^{-2\varepsilon}\sum_{2^j\geq A_\lambda}2^{2j\varepsilon}\|\Delta_j f\|^2_{L^2_k(\mathbb{R}^d)}d\lambda\\ &\leq C\sum_{j\geq -1}\int_0^{4C2^{j(\gamma+\frac{d}{2}-s)}\|f\|_{H^s_k(\mathbb{R}^d)}}\lambda^{p-3-\frac{2\varepsilon}{\gamma+\frac{d}{2}-s}}d\lambda\left(4C\|f\|_{H^s_k(\mathbb{R}^d)}\right)^{\frac{2\varepsilon}{\gamma+\frac{d}{2}-s}}2^{2j\varepsilon}\|\Delta_j f\|^2_{L^2_k(\mathbb{R}^d)}\\ &\leq C\|f\|^{p-2}_{H^s_k(\mathbb{R}^d)}\sum_{j\geq -1}2^{j(p-2)(\gamma+\frac{d}{2}-s)}\|\Delta_j f\|^2_{L^2_k(\mathbb{R}^d)}\\ &\leq C\|f\|^{p-2}_{H^s_k(\mathbb{R}^d)}\sum_{j\geq -1}2^{2js}\|\Delta_j f\|^2_{L^2_k(\mathbb{R}^d)}\\ &\leq C\|f\|^p_{H^s_k(\mathbb{R}^d)}.\end{aligned}$$

This implies the result. □

Definition 4.1. We define the space BMO_k as the set of functions $u \in L^1_{loc,k}(\mathbb{R}^d)$ satisfying

$$\sup_B \frac{1}{\mathrm{mes}_k(B)}\int_B |u(x)-u_B|\omega_k(x)dx < \infty,$$

where

$$B = B(x_0,R), \quad u_B = \frac{1}{\mathrm{mes}_k(B)}\int_B u(x)\omega_k(x)dx$$

denote the average of u on B and $\mathrm{mes}_k(B) = \int_B \omega_k(x)dx$.

Theorem 4.5. *We have the continuous embedding*

$$H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d) \hookrightarrow BMO_k.$$

Proof. For $R > 0$ small enough, let N be such that $2^N = [\frac{1}{R}]$. Let u be in $H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)$. Set $u = u^{(1)} + u^{(2)}$ with

$$u^{(1)} = \sum_{p=-1}^{N-1}\Delta_p u \qquad \text{and} \qquad u^{(2)} = \sum_{p\geq N}\Delta_p u.$$

From the Cauchy-Schwartz inequality we have

$$\left(\frac{1}{\mathrm{mes}_k(B)}\int_B |u(x)-u_B|\omega_k(x)dx\right)^2 \le \frac{1}{\mathrm{mes}_k(B)}\int_B |u(x)-u_B|^2\omega_k(x)dx.$$

It is easy to see that this implies

$$\left(\frac{1}{\mathrm{mes}_k(B)}\int_B |u(x)-u_B|\omega_k(x)dx\right)^2 \le \frac{2}{\mathrm{mes}_k(B)}\left[\int_B \left|u^{(1)}(x)-u_B^{(1)}\right|^2\omega_k(x)dx + \int_B \left|u^{(2)}(x)-u_B^{(2)}\right|^2\omega_k(x)dx\right].$$

Moreover, from the mean value theorem, we have

$$\frac{1}{\mathrm{mes}_k(B)}\int_B \left|u^{(1)}(x)-u_B^{(1)}\right|^2\omega_k(x)dx \le \frac{R^2}{\mathrm{mes}_k(B)}\int_B \left|Du^{(1)}(x)\right|^2\omega_k(x)dx, \le R^2\|Du^{(1)}\|^2_{L^\infty_k(\mathbb{R}^d)}.$$

By (2.7) we deduce that

$$\|Du^{(1)}\|_{L^\infty_k(\mathbb{R}^d)} \le \int_{\mathbb{R}^d}\|\xi\||\mathcal{F}_D(u^{(1)})(\xi)|\omega_k(\xi)d\xi.$$

By recalling that $\mathrm{supp}\,\mathcal{F}_D(\Delta_p u)\subset C_p$ and $|\mathcal{F}_D(\Delta_p u)(\xi)|\le|\mathcal{F}_D(u)(\xi)|$, we apply the Parseval identity associated with the Dunkl operators and the Cauchy-Schwartz inequality. We deduce that

$$\begin{aligned}\frac{1}{\mathrm{mes}_k(B)}\int_B |u^{(1)}(x)-u_B^{(1)}|^2\omega_k(x)dx &\le R^2\left(\sum_{p=-1}^{N-1}\int_{C_p}\|\xi\|\;|\mathcal{F}_D(\Delta_p u)(\xi)|\omega_k(\xi)d\xi\right)^2\\ &\le R^2\left(\int_{B(0,2^N)}\|\xi\|^{2-2\gamma-d}\omega_k(\xi)d\xi\right)\|u\|^2_{H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)}\\ &\le C2^{2N}R^2\|u\|^2_{H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)}.\end{aligned}$$

For the second term, we have

$$\begin{aligned}\frac{1}{\mathrm{mes}_k(B)}\int_B |u^{(2)}(x)-u_B^{(2)}|^2\omega_k(x)dx &\le \frac{2}{\mathrm{mes}_k(B)}\int_B |u^{(2)}(x)|^2\omega_k(x)dx\\ &\le CR^{-d-2\gamma}\int_{\|\xi\|\ge 2^N}|\mathcal{F}_D(u)(\xi)|^2\omega_k(\xi)d\xi\\ &\le C(2^N R)^{-d-2\gamma}\|u\|^2_{H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)}.\end{aligned}$$

Hence,

$$\left(\frac{1}{\mathrm{mes}_k(B)}\int_B |u(x)-u_B|\omega_k(x)dx\right)^2 \le C\|u\|^2_{H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)}. \tag{4.3}$$

We have proved (4.3) for small R, since $u\in H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)\subset L^2_k(\mathbb{R}^d)$, (4.3) is evident for $R>R_0$ with constant $C=C(R_0)$. This implies the continuous embedding

$$H_k^{\frac{d}{2}+\gamma}(\mathbb{R}^d)\hookrightarrow BMO_k.$$

□

4.3. **Gagliardo-Nirenberg Inequality.** We will use the generalized Sobolev space $W_k^{s,r}(\mathbb{R}^d)$ associated with the Dunkl operators defined as

$$W_k^{s,r}(\mathbb{R}^d) = \left\{u \in \mathcal{S}'(\mathbb{R}^d) : (-\triangle_k)^{\frac{s}{2}} u \in L_k^r(\mathbb{R}^d)\right\},$$

with

$$\triangle_k u = \sum_{j=1}^{d} T_j^2 u.$$

The main result of this subsection is the following theorem.

Theorem 4.6. *Let f be in $W_k^{s,r}(\mathbb{R}^d) \cap L_k^q(\mathbb{R}^d)$ with $q, r \in [1, \infty]$ and $s \geq 0$. Then f belongs to $W_k^{t,p}(\mathbb{R}^d)$, and we have*

$$\left\|(-\triangle_k)^{\frac{t}{2}} f\right\|_{L_k^p(\mathbb{R}^d)} \leq C\|f\|^{\theta}_{L_k^q(\mathbb{R}^d)} \left\|(-\triangle_k)^{\frac{s}{2}} f\right\|^{1-\theta}_{L_k^r(\mathbb{R}^d)},$$

where $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$, $t = (1-\theta)s$ and $\theta \in]0, 1[$.

Proof. First, we prove this theorem for q and r in $]1, \infty]$. Let f be in $\mathcal{S}(\mathbb{R}^d)$. It is easy to see that

$$(-\triangle_k)^{\frac{t}{2}} f = \sum_{j \leq A} (-\triangle_k)^{\frac{t}{2}} \Delta_j f + \sum_{j > A} (-\triangle_k)^{\frac{t-s}{2}} \Delta_j \left((-\triangle_k)^{\frac{s}{2}} f\right),$$

where A will be chosen later.

On the other hand, by a simple calculation, if a is a homogenous function in $C^\infty(\mathbb{R}^*)$ of degree m, we can write

$$a\left((-\triangle_k)^{\frac{1}{2}}\right) \Delta_j f = 2^{jm + j(d+2\gamma)} b(\delta_{2^j}) *_D \sum_{|j-j'| \leq 1} \Delta_{j'} f, \tag{4.4}$$

where δ_{2^j} is defined by $\delta_{2^j} x = 2^j x,\ x \in \mathbb{R}^d$ and b is in $\mathcal{S}(\mathbb{R}^d)$ such that

$$\mathcal{F}_D(b)(\xi) = \varphi(\xi) a(\|\xi\|).$$

We proceed as in [12, p. 21] to obtain

$$\left|a\left((-\triangle_k)^{\frac{1}{2}}\right) \Delta_j f(x)\right| \leq C 2^{jm} M_k f(x), \tag{4.5}$$

where $M_k(f)$ is a maximal function of f associated with the Dunkl operators (see [13]).

Hence by applying (4.5) for $a(r) = r^t$ and $a(r) = r^{\frac{t-s}{2}}$, we get

$$\begin{aligned}\left|(-\triangle_k)^{\frac{t}{2}} f(x)\right| &\leq C\left(\sum_{j \leq A} 2^{jt} M_k f(x) + \sum_{j > A} 2^{j(t-s)} M_k((-\triangle_k f)^{\frac{s}{2}})(x)\right) \\ &\leq C 2^{tA} M_k f(x) + C 2^{(t-s)A} M_k\left((-\triangle_k)^{\frac{s}{2}} f\right)(x).\end{aligned}$$

We minimize over A to obtain

$$\left|(-\triangle_k)^{\frac{t}{2}} f(x)\right| \leq C\Big(M_k f(x)\Big)^{1-\frac{t}{s}} \left(M_k\left((-\triangle_k)^{\frac{s}{2}} f\right)(x)\right)^{\frac{t}{s}}.$$

By this inequality and the Hölder inequality, we have

$$\left\|(-\triangle_k)^{\frac{t}{2}} f\right\|_{L_k^p(\mathbb{R}^d)} \leq C\|M_k f\|^{\theta}_{L_k^q(\mathbb{R}^d)} \left\|M_k((-\triangle_k)^{\frac{s}{2}} f)\right\|^{1-\theta}_{L_k^r(\mathbb{R}^d)},$$

with $\theta = 1 - \frac{t}{s}$.

Now, we apply Theorem 6.1 of [13] to deduce the result if q and $r \in]1, \infty]$.

Now, we assume $q = r = 1$. Let f be in $\mathcal{S}(\mathbb{R}^d)$. We have

$$\left\|(-\triangle_k)^{\frac{t}{2}} f\right\|_{L^1_k(\mathbb{R}^d)} \leq \left\|\sum_{j\leq A}(-\triangle_k)^{\frac{t}{2}}\Delta_j f\right\|_{L^1_k(\mathbb{R}^d)} + \left\|\sum_{j>A}(-\triangle_k)^{\frac{t-s}{2}}\Delta_j\left((-\triangle_k)^{\frac{s}{2}} f\right)\right\|_{L^1_k(\mathbb{R}^d)}$$
$$\leq C2^{(1-\theta)sA}\|f\|_{L^1_k(\mathbb{R}^d)} + C2^{-\theta sA}\left\|(-\triangle_k)^{\frac{s}{2}} f\right\|_{L^1_k(\mathbb{R}^d)}.$$

By minimizing over A, we obtain the result. □

5. PARAPRODUCT ASSOCIATED WITH THE DUNKL OPERATORS

In this section, we are going to study how the product acts on Sobolev and Hölder spaces associated with the Dunkl operators. This could be very useful in nonlinear partial differential-difference equations. Of course, we shall use the Littlewood-Paley decomposition associated with the Dunkl operators. Let us consider two temperate distributions u and v. We write

$$u = \sum_p \Delta_p u \quad \text{and} \quad v = \sum_q \Delta_q v.$$

Formally, the product can be written as

$$uv = \sum_{p,q} \Delta_p u \Delta_q v.$$

Now we introduce the paraproduct operator associated with the Dunkl operators.

Definition 5.1. We define the paraproduct operator $\Pi_a : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$\Pi_a u = \sum_{q\geq 1}(S_{q-2}\, a)\Delta_q u,$$

where $u \in \mathcal{S}'(\mathbb{R}^d)$; $\{\Delta_q a\}$ and $\{\Delta_q u\}$ are the Littlewood-Paley decompositions and $S_q a = \sum_{p\leq q-1}\Delta_p a$.

Let R indicate the following bilinear symmetric operator on $\mathcal{S}'(\mathbb{R}^d)$ defined by

$$R(u, v) = \sum_{|p-q|\leq 1} \Delta_p u \Delta_q v, \qquad \text{for all } u, v \in \mathcal{S}'(\mathbb{R}^d).$$

Obviously from Definition 5.1 it is clear that

$$uv = \Pi_u v + \Pi_v u + R(u, v).$$

The following theorems describe the action of the paraproduct and remainder on the Sobolov and the Hölder spaces associated with the Dunkl operators.

Theorem 5.1. *There exists a positive constant C such that the operator Π has the following properties:*

(1) $\|\Pi\|_{\mathcal{L}(L^\infty_k(\mathbb{R}^d)\times C^s_k(\mathbb{R}^d), C^s_k(\mathbb{R}^d))} \leq C^{s+1}$, *for all* $s > 0$.
(2) $\|\Pi\|_{\mathcal{L}(L^\infty_k(\mathbb{R}^d)\times H^s_k(\mathbb{R}^d), H^s_k(\mathbb{R}^d))} \leq C^{s+1}$, *for all* $s > 0$.

(3) $\|\Pi\|_{\mathcal{L}(C^t_k(\mathbb{R}^d)\times H^s_k(\mathbb{R}^d),H^{s+t}_k(\mathbb{R}^d))} \leq \frac{C^{s+t+1}}{-t}$, *for all* s,t *with* $s+t>0$ *and* $t<0$.

(4) $\|\Pi\|_{\mathcal{L}(C^t_k(\mathbb{R}^d)\times C^s_k(\mathbb{R}^d),C^{s+t}_k(\mathbb{R}^d))} \leq \frac{C^{s+t+1}}{-t}$, *for all* s,t *with* $s+t>0$ *and* $t<0$.

(5) $\|\Pi\|_{\mathcal{L}(H^s_k(\mathbb{R}^d)\times H^t_k(\mathbb{R}^d),H^{s+t-\gamma-\frac{d}{2}}_k(\mathbb{R}^d))} \leq C^{s+t-\gamma-\frac{d}{2}+1}$, *with* $s+t>\gamma+\frac{d}{2}$ *and* $s<\frac{d}{2}+\gamma$.

Proof. Let u and v be in $\mathcal{S}'(\mathbb{R}^d)$. We have

$$\operatorname{supp}(\mathcal{F}_D(S_{q-2}u\Delta_q v)) = \operatorname{supp}(\mathcal{F}_D(\Delta_q v) *_D \mathcal{F}_D(S_{q-2}u)) \subset B(0, C2^q).$$

1) If u in $L^\infty_k(\mathbb{R}^d)$ and v in $C^s_k(\mathbb{R}^d)$, then we have

$$\begin{aligned}\|S_{q-2}u\Delta_q v\|_{L^\infty_k(\mathbb{R}^d)} &\leq \|S_{q-2}u\|_{L^\infty_k(\mathbb{R}^d)}\|\Delta_q v\|_{L^\infty_k(\mathbb{R}^d)}\\ &\leq C\|u\|_{L^\infty_k(\mathbb{R}^d)}\|v\|_{C^s_k(\mathbb{R}^d)}2^{-qs}.\end{aligned}$$

Applying Proposition 3.10 ii), we obtain

$$\|\Pi_u v\|_{C^s_k(\mathbb{R}^d)} \leq C\|u\|_{L^\infty_k(\mathbb{R}^d)}\|v\|_{C^s_k(\mathbb{R}^d)}.$$

2) If u is in $L^\infty_k(\mathbb{R}^d)$ and v is in $H^s_k(\mathbb{R}^d)$, then we have

$$\begin{aligned}\|S_{q-2}u\Delta_q v\|_{L^2_k(\mathbb{R}^d)} &\leq \|S_{q-2}u\|_{L^\infty_k(\mathbb{R}^d)}\|\Delta_q v\|_{L^2_k(\mathbb{R}^d)}\\ &\leq C\|u\|_{L^\infty_k(\mathbb{R}^d)}\|v\|_{H^s_k(\mathbb{R}^d)}c_q 2^{-qs}.\end{aligned}$$

Thus Proposition 3.7 gives

$$\|\Pi_u v\|_{H^s_k(\mathbb{R}^d)} \leq C\|u\|_{L^\infty_k(\mathbb{R}^d)}\|v\|_{H^s_k(\mathbb{R}^d)},$$

this implies the result.
3) Let u be in $C^t_k(\mathbb{R}^d)$ and v in $H^s_k(\mathbb{R}^d)$. We have

$$\|S_{q-2}u\Delta_q v\|_{L^2_k(\mathbb{R}^d)} \leq \|S_{q-2}u\|_{L^\infty_k(\mathbb{R}^d)}\|\Delta_q v\|_{L^2_k(\mathbb{R}^d)}.$$

Since $t<0$, we estimate $\|S_{q-2}u\|_{L^\infty_k(\mathbb{R}^d)}$ in a different way. In fact, $S_{q-2}u = \sum_{p\leq q-3}\Delta_p u$. Since $u\in C^t_k(\mathbb{R}^d)$ and $t<0$, we obtain

$$\|S_{q-2}u\|_{L^\infty_k(\mathbb{R}^d)} \leq \|u\|_{C^t_k(\mathbb{R}^d)}\sum_{p\leq q-3}2^{-pt} \leq \frac{C}{-t}2^{-qt}\|u\|_{C^t_k(\mathbb{R}^d)}.$$

Hence

$$\|S_{q-2}u\Delta_q v\|_{L^2_k(\mathbb{R}^d)} \leq \frac{C}{-t}2^{-q(t+s)}c_q\|u\|_{C^t_k(\mathbb{R}^d)}\|v\|_{H^s_k(\mathbb{R}^d)}, \qquad c_q\in l^2.$$

Thus Proposition 3.7 gives the result.

The proof of 4) uses the same idea.

5) We have

$$\begin{aligned}\|S_{q-2}u\|_{L^\infty_k(\mathbb{R}^d)} &\leq \|\mathcal{F}_D(S_{q-2}u)\|_{L^1_k(\mathbb{R}^d)}\\ &\leq \int_{\mathbb{R}^d}|\psi(2^{-(q-2)}\xi)|(1+\|\xi\|^2)^{\frac{-s}{2}}|\mathcal{F}_D(u)(\xi)|(1+\|\xi\|^2)^{\frac{s}{2}}\omega_k(\xi)d\xi.\end{aligned}$$

The Cauchy-Schwartz inequality implies that

$$\begin{aligned}\|S_{q-2}u\|_{L^\infty_k(\mathbb{R}^d)} &\leq \left(\int_{\mathbb{R}^d}|\psi(2^{-(q-2)}\xi)|^2(1+\|\xi\|^2)^{-s}\omega_k(\xi)d\xi\right)^{\frac{1}{2}}\|u\|_{H^s_k(\mathbb{R}^d)},\\ &\leq C2^{q(\frac{d}{2}+\gamma)}\left(\int_{B(0,1)}|\psi(t)|^2(1+2^{2(q-2)}\|t\|^2)^{-s}\omega_k(t)dt\right)^{\frac{1}{2}}\|u\|_{H^s_k(\mathbb{R}^d)}.\end{aligned}$$

If $s\geq 0$,

$$\begin{aligned}\|S_{q-2}u\|_{L^\infty_k(\mathbb{R}^d)} &\leq C2^{q(\frac{d}{2}+\gamma-s)}\left(\int_{B(0,1)}|\psi(t)|^2\|t\|^{-2s}\omega_k(t)dt\right)^{\frac{1}{2}}\\ &\leq C2^{q(\frac{d}{2}+\gamma-s)}.\end{aligned}$$

If $s\leq 0$,

$$\begin{aligned}\|S_{q-2}u\|_{L^\infty_k(\mathbb{R}^d)} &\leq C2^{q(\frac{d}{2}+\gamma-s)}\left(\int_{B(0,1)}|\psi(t)|^2(1+\|t\|^2)^{-s}\omega_k(t)dt\right)^{\frac{1}{2}}\\ &\leq C2^{q(\frac{d}{2}+\gamma-s)}.\end{aligned}$$

By proceeding as in the previous cases we deduce the result. □

Theorem 5.2. *There exists a positive constant C such that the operator Π has the following properties:*

(1) *If $a\in L^\infty_k(\mathbb{R}^d)$ is radial, then for any s in $\mathbb{R}$, we have*

$$\|\Pi_a\|_{\mathcal{L}(C^s_k(\mathbb{R}^d),C^s_k(\mathbb{R}^d))}\leq C\|a\|_{L^\infty_k(\mathbb{R}^d)},\qquad \|\Pi_a\|_{\mathcal{L}(H^s_k(\mathbb{R}^d),H^s_k(\mathbb{R}^d))}\leq C\|a\|_{L^\infty_k(\mathbb{R}^d)}.$$

(2) *If $a\in C^t_k(\mathbb{R}^d)$ is radial with $t<0$, then for all s, we have*

$$\|\Pi_a\|_{\mathcal{L}(H^s_k(\mathbb{R}^d),H^{s+t}_k(\mathbb{R}^d))}\leq C\|a\|_{C^t_k(\mathbb{R}^d)},\qquad \|\Pi_a\|_{\mathcal{L}(C^s_k(\mathbb{R}^d),C^{s+t}_k(\mathbb{R}^d))}\leq C\|a\|_{C^t_k(\mathbb{R}^d)}.$$

(3) *If $a\in H^t_k(\mathbb{R}^d)$ is radial, then for all s,t with $s<\frac{d}{2}+\gamma$, we have*

$$\|\Pi_a\|_{\mathcal{L}(H^s_k(\mathbb{R}^d),H^{s+t-\gamma-\frac{d}{2}}_k(\mathbb{R}^d))}\leq C\|a\|_{H^t_k(\mathbb{R}^d)}.$$

Proof. From the relation (2.19) and Definition 2.2 we deduce that there exists an annulus $\widetilde{C}_0$ such that $\operatorname{supp}\mathcal{F}_D(S_{q-2}a\Delta_q u)\subset 2^q\widetilde{C}_0$. Thus we proceed as in the proof of Theorem 4.3 and using Propositions 3.6 and 3.10 i), we obtain the result. □

Remark 5. In the case $W=\mathbb{Z}^d_2$ the assumption that a is radial is not necessary.

Theorem 5.3. *There exists a positive constant C such that the operator R has the following properties:*

(1) $\|R\|_{\mathcal{L}(C^t_k(\mathbb{R}^d)\times H^s_k(\mathbb{R}^d),H^{s+t}_k(\mathbb{R}^d))}\leq\frac{C^{s+t+1}}{s+t}$, *for all s,t with $s+t>0$.*
(2) $\|R\|_{\mathcal{L}(C^t_k(\mathbb{R}^d)\times C^s_k(\mathbb{R}^d),C^{s+t}_k(\mathbb{R}^d))}\leq\frac{C^{s+t+1}}{s+t}$, *for all s,t with $s+t>0$.*
(3) $\|R\|_{\mathcal{L}(H^t_k(\mathbb{R}^d)\times H^s_k(\mathbb{R}^d),H^{s+t-\gamma-\frac{d}{2}}_k(\mathbb{R}^d))}\leq\frac{C^{s+t+1}}{s+t-\gamma-\frac{d}{2}}$, *for all s,t with $s+t>\gamma+\frac{d}{2}$.*

Proof. By the definition of the remainder operator

$$R(u,v)=\sum_q R_q \quad \text{with} \quad R_q=\sum_{i=-1}^{1}\Delta_{q-i}u\Delta_q v.$$

By the definition of Δ_q, the support of the Dunkl transform of R_q is included in $B(0,C2^q)$. Then, to prove 1) it is sufficient to estimate $\|R_q\|_{L^2_k(\mathbb{R}^d)}$. In fact, we have

$$\|R_q\|_{L^2_k(\mathbb{R}^d)}\le \|\Delta_q v\|_{L^2_k(\mathbb{R}^d)}\sum_{i=-1}^{1}\|\Delta_{q-i}u\|_{L^\infty_k(\mathbb{R}^d)}.$$

Using the facts that $u\in C^t_k(\mathbb{R}^d)$ and $v\in H^s_k(\mathbb{R}^d)$, we obtain

$$\begin{aligned}\|R_q\|_{L^2_k(\mathbb{R}^d)}&\le 2^{-qs}c_q\|v\|_{H^s_k(\mathbb{R}^d)}\sum_{i=-1}^{1}2^{-(q-i)t}\|u\|_{C^t_k(\mathbb{R}^d)}, \quad c_q\in l^2\\ &\le Cc_q2^{-q(s+t)}\|u\|_{C^t_k(\mathbb{R}^d)}\|v\|_{H^s_k(\mathbb{R}^d)}.\end{aligned}$$

Now we apply Proposition 3.7 to conclude the proof. The proof of the second case uses the same idea. We want to prove 3). We have $R(u,v)=\sum_q R_q$. We proceed as in 1), so

$$\begin{aligned}\|R_q\|_{L^2_k(\mathbb{R}^d)}&\le C\|\mathcal{F}_D(\Delta_q v)\|_{L^2_k(\mathbb{R}^d)}\sum_{i=-1}^{1}\|\Delta_{q-i}u\|_{L^\infty_k(\mathbb{R}^d)},\\ &\le C\|\mathcal{F}_D(\Delta_q v)\|_{L^2_k(\mathbb{R}^d)}\sum_{i=-1}^{1}\|\mathcal{F}_D(\Delta_{q-i}u)\|_{L^1_k(\mathbb{R}^d)}.\end{aligned}$$

Using the fact that $v\in H^s_k(\mathbb{R}^d)$, we obtain

$$\|R_q\|_{L^2_k(\mathbb{R}^d)}\le Cc_q2^{-qt}\|v\|_{H^t_k(\mathbb{R}^d)}\sum_{i=-1}^{1}\|\mathcal{F}_D(\Delta_{q-i}u)\|_{L^1_k(\mathbb{R}^d)}, \quad c_q\in l^2.$$

On the other hand, by the Cauchy-Schwartz inequality we have

$$\begin{aligned}\|\mathcal{F}_D(\Delta_{q-i}u)\|_{L^1_k(\mathbb{R}^d)}&\le \int_{\mathbb{R}^d}|\varphi(2^{-(q-i)}\xi)|(1+\|\xi\|^2)^{\frac{-s}{2}}(1+\|\xi\|^2)^{\frac{s}{2}}\mathcal{F}_D(u)(\xi)\omega_k(\xi)d\xi\\ &\le \|u\|_{H^s_k(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}|\varphi(2^{-(q-i)}\xi)|^2(1+\|\xi\|^2)^{-s}\omega_k(\xi)d\xi\right)^{\frac12}\\ &\le C2^{q(\gamma+\frac{d}{2})}\|u\|_{H^s_k(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}|\varphi(t)|^2(1+2^{2(q-i)}\|t\|^2)^{-s}\omega_k(t)dt\right)^{\frac12}\\ &\le C2^{q(\gamma+\frac{d}{2}-s)}.\end{aligned}$$

Hence

$$\|R_q\|_{L^2_k(\mathbb{R}^d)}\le C2^{q(\gamma+\frac{d}{2}-s-t)}c_q, \quad c_q\in l^2.$$

Then we conclude the result by using Proposition 3.7. □

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