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A CLASS OF MULTIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY CONVOLUTION

ROSIHAN M. ALI, M. HUSSAIN KHAN, V. RAVICHANDRAN, AND K.G. SUBRAMANIAN

SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITI SAINS MALAYSIA
11800 USM, PENANG, MALAYSIA
rosihan@cs.usm.my

DEPARTMENT OF MATHEMATICS
ISLAMIAH COLLEGE
VANIAMBADI 635 751, INDIA
khanhussaff@yahoo.co.in

SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITI SAINS MALAYSIA
11800 USM, PENANG, MALAYSIA
vravi@cs.usm.my

DEPARTMENT OF MATHEMATICS
MADRAS CHRISTIAN COLLEGE
TAMBARAM, CHENNAI- 600 059, INDIA
kgsmani@vsnl.net

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ABSTRACT. For a given p-valent analytic function g with positive coefficients in the open unit disk Δ, we study a class of functions $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n$, $a_n \geq 0$ satisfying

$$\frac{1}{p}\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) < \alpha \quad \left(z \in \Delta; 1 < \alpha < \frac{m+p}{2p}\right).$$

Coefficient inequalities, distortion and covering theorems, as well as closure theorems are determined. The results obtained extend several known results as special cases.

Key words and phrases: Starlike function, Ruscheweyh derivative, Convolution, Positive coefficients, Coefficient inequalities, Growth and distortion theorems.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ in the unit disk $\Delta := \{z \in \mathcal{C} : |z| < 1\}$ with $f(0) = 0 = f'(0) - 1$. The class $M(\alpha)$ defined by

$$M(\alpha) := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \quad \left(1 < \alpha < \frac{3}{2};\ z \in \Delta\right) \right\}$$

was investigated by Uralegaddi *et al.* [6]. A subclass of $M(\alpha)$ was recently investigated by Owa and Srivastava [3]. Motivated by $M(\alpha)$, we introduce a more general class $PM_g(p, m, \alpha)$ of analytic functions with positive coefficients. For two analytic functions

$$f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z^p + \sum_{n=m}^{\infty} b_n z^n,$$

the convolution (or Hadamard product) of f and g, denoted by $f * g$ or $(f * g)(z)$, is defined by

$$(f * g)(z) := z^p + \sum_{n=m}^{\infty} a_n b_n z^n.$$

Let $T(p, m)$ be the class of all analytic p-valent functions $f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n$ $(a_n \geq 0)$, defined on the unit disk Δ and let $T := T(1, 2)$. A function $f(z) \in T(p, m)$ is called a function with negative coefficients. The subclass of T consisting of starlike functions of order α, denoted by $TS^*(\alpha)$, was studied by Silverman [5]. Several other classes of starlike functions with negative coefficients were studied; for e.g. see [2].

Let $P(p, m)$ be the class of all analytic functions

$$f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{1.1}$$

and $P := P(1, 2)$.

Definition 1.1. Let

$$g(z) = z^p + \sum_{n=m}^{\infty} b_n z^n \quad (b_n > 0) \tag{1.2}$$

be a fixed analytic function in Δ. Define the class $PM_g(p, m, \alpha)$ by

$$PM_g(p, m, \alpha) := \left\{ f \in P(p, m) : \frac{1}{p}\Re\left(\frac{z(f * g)'(z)}{(f * g)(z)}\right) < \alpha, \quad \left(1 < \alpha < \frac{m + p}{2p}; z \in \Delta\right) \right\}.$$

When $g(z) = z/(1 - z)$, $p = 1$ and $m = 2$, the class $PM_g(p, m, \alpha)$ reduces to the subclass $PM(\alpha) := P \cap M(\alpha)$. When $g(z) = z/(1 - z)^{\lambda+1}$, $p = 1$ and $m = 2$, the class $PM_g(p, m, \alpha)$ reduces to the class:

$$P_\lambda(\alpha) = \left\{ f \in P : \Re\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}\right) < \alpha, \quad \left(\lambda > -1, 1 < \alpha < \frac{3}{2}; z \in \Delta\right) \right\},$$

where D^λ denotes the Ruscheweyh derivative of order λ. When

$$g(z) = z + \sum_{n=2}^{\infty} n^l z^n,$$

the class of functions $PM_g(1, 2, \alpha)$ reduces to the class $PM_l(\alpha)$ where

$$PM_l(\alpha) = \left\{ f \in P : \Re\left(\frac{z(\mathcal{D}^l f(z))'}{\mathcal{D}^l f(z)}\right) < \alpha, \quad \left(1 < \alpha < \frac{3}{2}; l \geq 0;\ \ z \in \Delta\right) \right\},$$

where $\mathcal{D}^l$ denotes the Salagean derivative of order l. Also we have

$$PM(\alpha) \equiv P_0(\alpha) \equiv PM_0(\alpha).$$

A function $f \in \mathcal{A}(p,m)$ is in $PPC(p,m,\alpha,\beta)$ if

$$\frac{1}{p}\Re\left(\frac{(1-\beta)zf'(z)+\frac{\beta}{p}z(zf')'(z)}{(1-\beta)f(z)+\frac{\beta}{p}zf'(z)}\right) < \alpha \quad \left(\beta \geq 0;\ 0 \leq \alpha < \frac{m+p}{2p}\right)$$

This class is similar to the class of β-Pascu convex functions of order α and it unifies the class of $PM(\alpha)$ and the corresponding convex class.

For the newly defined class $PM_g(p,m,\alpha)$, we obtain coefficient inequalities, distortion and covering theorems, as well as closure theorems. As special cases, we obtain results for the classes $P_\lambda(\alpha)$, and $PM_l(\alpha)$. Similar results for the class $PPC(p,m,\alpha,\beta)$ also follow from our results, the details of which are omitted here.

2. COEFFICIENT INEQUALITIES

Throughout the paper, we assume that the function $f(z)$ is given by the equation (1.1) and $g(z)$ is given by by (1.2). We first prove a necessary and sufficient condition for functions to be in the class $PM_g(p,m,\alpha)$ in the following:

Theorem 2.1. *A function $f \in PM_g(p,m,\alpha)$ if and only if*

$$\sum_{n=m}^{\infty}(n-p\alpha)a_nb_n \leq p(\alpha-1) \quad \left(1 < \alpha < \frac{m+p}{2p}\right). \tag{2.1}$$

Proof. If $f \in PM_g(p,m,\alpha)$, then (2.1) follows from

$$\frac{1}{p}\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) < \alpha$$

by letting $z \to 1-$ through real values. To prove the converse, assume that (2.1) holds. Then by making use of (2.1), we obtain

$$\left|\frac{z(f*g)'(z)-p(f*g)(z)}{z(f*g)'(z)-(2\alpha-1)p(f*g)(z)}\right| \leq \frac{\sum_{n=m}^{\infty}(n-p)a_nb_n}{2(\alpha-1)p-\sum_{n=m}^{\infty}[n-(2\alpha-1)p]a_nb_n} \leq 1$$

or equivalently $f \in PM_g(p,m,\alpha)$. ☐

Corollary 2.2. *A function $f \in P_\lambda(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_nB_n(\lambda) \leq \alpha-1 \quad \left(1 < \alpha < \frac{3}{2}\right),$$

where

$$B_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{(n-1)!}. \tag{2.2}$$

Corollary 2.3. *A function $f \in PM_m(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_nn^m \leq \alpha-1 \quad \left(1 < \alpha < \frac{3}{2}\right).$$

Our next theorem gives an estimate for the coefficient of functions in the class $PM_g(p,m,\alpha)$.

Theorem 2.4. *If $f \in PM_g(p,m,\alpha)$, then*

$$a_n \leq \frac{p(\alpha-1)}{(n-p\alpha)b_n}$$

with equality only for functions of the form

$$f_n(z) = z^p + \frac{p(\alpha-1)}{(n-p\alpha)b_n} z^n.$$

Proof. Let $f \in PM_g(p,m,\alpha)$. By making use of the inequality (2.1), we have

$$(n-p\alpha)a_n b_n \leq \sum_{n=m}^{\infty} (n-p\alpha)a_n b_n \leq p(\alpha-1)$$

or

$$a_n \leq \frac{p(\alpha-1)}{(n-p\alpha)b_n}.$$

Clearly for

$$f_n(z) = z^p + \frac{p(\alpha-1)}{(n-p\alpha)b_n} z^n \in PM_g(p,m,\alpha),$$

we have

$$a_n = \frac{p(\alpha-1)}{(n-p\alpha)b_n}.$$

□

Corollary 2.5. *If $f \in P_\lambda(\alpha)$, then*

$$a_n \leq \frac{\alpha-1}{(n-\alpha)B_n(\lambda)}$$

with equality only for functions of the form

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)B_n(\lambda)} z^n,$$

where $B_n(\lambda)$ is given by (2.2).

Corollary 2.6. *If $f \in PM_m(\alpha)$, then*

$$a_n \leq \frac{\alpha-1}{(n-\alpha)n^m}$$

with equality only for functions of the form

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)n^m} z^n.$$

3. GROWTH AND DISTORTION THEOREMS

We now prove the growth theorem for the functions in the class $PM_g(p,m,\alpha)$.

Theorem 3.1. *If $f \in PM_g(p,m,\alpha)$, then*

$$r^p - \frac{p(\alpha-1)}{(m-p\alpha)b_m} r^m \leq |f(z)| \leq r^p + \frac{p(\alpha-1)}{(m-p\alpha)b_m} r^m, \quad |z| = r < 1,$$

provided $b_n \geq b_m \geq 1$. The result is sharp for

$$f(z) = z^p + \frac{p(\alpha-1)}{(m-p\alpha)b_m} z^m. \tag{3.1}$$

Proof. By making use of the inequality (2.1) for $f \in PM_g(p,m,\alpha)$ together with

$$(m-p\alpha)b_m \leq (n-p\alpha)b_n,$$

we obtain

$$b_m(m-p\alpha)\sum_{n=m}^{\infty} a_n \leq \sum_{n=m}^{\infty}(n-p\alpha)a_nb_n \leq p(\alpha-1)$$

or

$$\sum_{n=m}^{\infty} a_n \leq \frac{p(\alpha-1)}{(m-p\alpha)b_m}. \tag{3.2}$$

By using (3.2) for the function $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \in PM_g(p,m,\alpha)$, we have for $|z| = r$,

$$\begin{aligned} |f(z)| &\leq r^p + \sum_{n=m}^{\infty} a_n r^n \\ &\leq r^p + r^m \sum_{n=m}^{\infty} a_n \\ &\leq r^p + \frac{p(\alpha-1)}{(m-p\alpha)b_m} r^m, \end{aligned}$$

and similarly,

$$|f(z)| \geq r^p - \frac{p(\alpha-1)}{(m-p\alpha)b_m} r^m.$$

□

Theorem 3.1 also shows that $f(\Delta)$ for every $f \in PM_g(p,m,\alpha)$ contains the disk of radius $1 - \frac{p(\alpha-1)}{(m-p\alpha)b_m}$.

Corollary 3.2. *If $f \in P_\lambda(\alpha)$, then*

$$r - \frac{\alpha-1}{(2-\alpha)(\lambda+1)} r^2 \leq |f(z)| \leq r + \frac{\alpha-1}{(2-\alpha)(\lambda+1)} r^2 \quad (|z| = r).$$

The result is sharp for

$$f(z) = z + \frac{\alpha-1}{(2-\alpha)(\lambda+1)} z^2. \tag{3.3}$$

Corollary 3.3. *If $f \in PM_m(\alpha)$, then*

$$r - \frac{\alpha-1}{(2-\alpha)2^m} r^2 \leq |f(z)| \leq r + \frac{\alpha-1}{(2-\alpha)2^m} r^2 \quad (|z| = r).$$

The result is sharp for

$$f(z) = z + \frac{\alpha-1}{(2-\alpha)2^m} z^2. \tag{3.4}$$

The distortion estimates for the functions in the class $PM_g(p,m,\alpha)$ is given in the following:

Theorem 3.4. *If $f \in PM_g(p,m,\alpha)$, then*

$$pr^{p-1} - \frac{mp(\alpha-1)}{(m-p\alpha)b_m} r^{m-1} \leq |f'(z)| \leq pr^{p-1} + \frac{mp(\alpha-1)}{(m-p\alpha)b_m} r^{m-1}, \quad |z| = r < 1,$$

provided $b_n \geq b_m$. The result is sharp for the function given by (3.1).

Proof. By making use of the inequality (2.1) for $f \in PM_g(p,m,\alpha)$, we obtain

$$\sum_{n=m}^{\infty} a_n b_n \le \frac{p(\alpha-1)}{(m-p\alpha)}$$

and therefore, again using the inequality (2.1), we get

$$\sum_{n=m}^{\infty} n a_n \le \frac{mp(\alpha-1)}{(m-p\alpha)b_m}.$$

For the function $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \in PM_g(p,m,\alpha)$, we now have

$$\begin{aligned} |f'(z)| &\le pr^{p-1} + \sum_{n=m}^{\infty} n a_n r^{n-1} \quad (|z| = r) \\ &\le pr^{p-1} + r^{m-1} \sum_{n=m}^{\infty} n a_n \\ &\le pr^{p-1} + \frac{mp(\alpha-1)}{(m-p\alpha)b_m} r^{m-1} \end{aligned}$$

and similarly we have

$$|f'(z)| \ge pr^{p-1} - \frac{mp(\alpha-1)}{(m-p\alpha)b_m} r^{m-1}.$$

□

Corollary 3.5. *If $f \in P_\lambda(\alpha)$, then*

$$1 - \frac{2(\alpha-1)}{(2-\alpha)(\lambda+1)} r \le |f'(z)| \le 1 + \frac{2(\alpha-1)}{(2-\alpha)(\lambda+1)} r \quad (|z| = r).$$

The result is sharp for the function given by (3.3)

Corollary 3.6. *If $f \in PM_m(\alpha)$, then*

$$1 - \frac{2(\alpha-1)}{(2-\alpha)2^m} r \le |f'(z)| \le 1 + \frac{2(\alpha-1)}{(2-\alpha)2^m} r \quad (|z| = r).$$

The result is sharp for the function given by (3.4)

4. CLOSURE THEOREMS

We shall now prove the following closure theorems for the class $PM_g(p,m,\alpha)$. Let the functions $F_k(z)$ be given by

$$F_k(z) = z^p + \sum_{n=m}^{\infty} f_{n,k} z^n, \quad (k = 1, 2, \ldots, M). \tag{4.1}$$

Theorem 4.1. *Let $\lambda_k \ge 0$ for $k = 1, 2, \ldots, M$ and $\sum_{k=1}^{M} \lambda_k \le 1$. Let the function $F_k(z)$ defined by (4.1) be in the class $PM_g(p,m,\alpha)$ for every $k = 1, 2, \ldots, M$. Then the function $f(z)$ defined by*

$$f(z) = z^p + \sum_{n=m}^{\infty} \left(\sum_{k=1}^{M} \lambda_k f_{n,k} \right) z^n$$

belongs to the class $PM_g(p,m,\alpha)$.

Proof. Since $F_k(z) \in PM_g(p,m,\alpha)$, it follows from Theorem 2.1 that

$$\sum_{n=m}^{\infty}(n-p\alpha)b_n f_{n,k} \leq p(\alpha-1) \tag{4.2}$$

for every $k = 1, 2, \dots, M$. Hence

$$\begin{aligned}\sum_{n=m}^{\infty}(n-p\alpha)b_n\left(\sum_{k=1}^{M}\lambda_k f_{n,k}\right) &= \sum_{k=1}^{M}\lambda_k\left(\sum_{n=m}^{\infty}(n-p\alpha)b_n f_{n,k}\right)\\ &\leq \sum_{k=1}^{M}\lambda_k p(\alpha-1)\\ &\leq p(\alpha-1).\end{aligned}$$

By Theorem 2.1, it follows that $f(z) \in PM_g(p,m,\alpha)$. □

Corollary 4.2. *The class $PM_g(p,m,\alpha)$ is closed under convex linear combinations.*

Theorem 4.3. *Let*

$$F_p(z) = z^p \text{ and } F_n(z) = z^p + \frac{p(\alpha-1)}{(n-p\alpha)b_n}z^n$$

for $n = m, m+1, \dots$. Then $f(z) \in PM_g(p,m,\alpha)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \lambda_p z^p + \sum_{n=m}^{\infty}\lambda_n F_n(z), \tag{4.3}$$

where each $\lambda_j \geq 0$ and $\lambda_p + \sum_{n=m}^{\infty}\lambda_n = 1$.

Proof. Let $f(z)$ be of the form (4.3). Then

$$f(z) = z^p + \sum_{n=m}^{\infty}\frac{\lambda_n p(\alpha-1)}{(n-p\alpha)b_n}z^n$$

and therefore

$$\sum_{n=m}^{\infty}\frac{\lambda_n p(\alpha-1)}{(n-p\alpha)b_n}\frac{(n-p\alpha)b_n}{p(\alpha-1)} = \sum_{n=m}^{\infty}\lambda_n = 1-\lambda_p \leq 1.$$

By Theorem 2.1, we have $f(z) \in PM_g(p,m,\alpha)$.

Conversely, let $f(z) \in PM_g(p,m,\alpha)$. From Theorem 2.4, we have

$$a_n \leq \frac{p(\alpha-1)}{(n-p\alpha)b_n} \quad \text{for} \quad n = m, m+1, \dots.$$

Therefore we may take

$$\lambda_n = \frac{(n-p\alpha)b_n a_n}{p(\alpha-1)} \quad \text{for} \quad n = m, m+1, \dots$$

and

$$\lambda_p = 1 - \sum_{n=m}^{\infty}\lambda_n.$$

Then

$$f(z) = \lambda_p z^p + \sum_{n=m}^{\infty}\lambda_n F_n(z).$$

□

We now prove that the class $PM_g(p,m,\alpha)$ is closed under convolution with certain functions and give an application of this result to show that the class $PM_g(p,m,\alpha)$ is closed under the familiar Bernardi integral operator.

Theorem 4.4. *Let* $h(z) = z^p + \sum_{n=m}^{\infty} h_n z^n$ *be analytic in* Δ *with* $0 \leq h_n \leq 1$. *If* $f(z) \in PM_g(p,m,\alpha)$, *then* $(f*h)(z) \in PM_g(p,m,\alpha)$.

Proof. The result follows directly from Theorem 2.1. □

The generalized Bernardi integral operator is defined by the following integral:

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \quad (c > -1;\ z \in \Delta). \tag{4.4}$$

Since

$$F(z) = f(z) * \left(z^p + \sum_{n=m}^{\infty} \frac{c+p}{c+n} z^n \right),$$

we have the following:

Corollary 4.5. *If* $f(z) \in PM_g(p,m,\alpha)$, *then* $F(z)$ *given by (4.4) is also in* $PM_g(p,m,\alpha)$.

5. ORDER AND RADIUS RESULTS

Let $PS_h^*(p,m,\beta)$ be the subclass of $P(m,p)$ consisting of functions f for which $f*h$ is starlike of order β.

Theorem 5.1. *Let* $h(z) = z^p + \sum_{n=m}^{\infty} h_n z^n$ *with* $h_n > 0$. *Let* $(\alpha-1)nh_n \leq (n-p\alpha)b_n$. *If* $f \in PM_g(p,m,\alpha)$, *then* $f \in PS_h^*(p,m,\beta)$, *where*

$$\beta := \inf_{n \geq m} \left[\frac{(n-p\alpha)b_n - (\alpha-1)nh_n}{(n-p\alpha)b_n - (\alpha-1)ph_n} \right].$$

Proof. Let us first note that the condition $(\alpha-1)nh_n \leq (n-p\alpha)b_n$ implies $f \in PS_h^*(p,m,0)$. From the definition of β, it follows that

$$\beta \leq \frac{(n-p\alpha)b_n - (\alpha-1)nh_n}{(n-p\alpha)b_n - (\alpha-1)ph_n}$$

or

$$\frac{(n-p\beta)h_n}{1-\beta} \leq \frac{(n-p\alpha)b_n}{\alpha-1}$$

and therefore, in view of (2.1),

$$\sum_{n=m}^{\infty} \frac{(n-p\beta)}{p(1-\beta)} a_n h_n \leq \sum_{n=m}^{\infty} \frac{(n-p\alpha)}{p(\alpha-1)} a_n b_n \leq 1.$$

Thus

$$\left| \frac{1}{p} \cdot \frac{z(f*h)'(z)}{(f*h)(z)} - 1 \right| \leq \frac{\sum_{n=m}^{\infty}(n/p-1)a_n h_n}{1-\sum_{n=m}^{\infty} a_n h_n} \leq 1-\beta$$

and therefore $f \in PS_h^*(p,m,\beta)$. □

Similarly we can prove the following:

Theorem 5.2. *If* $f \in PM_g(p,m,\alpha)$, *then* $f \in PM_h(p,m,\beta)$ *in* $|z| < r(\alpha,\beta)$ *where*

$$r(\alpha,\beta) := \min\left\{1; \inf_{n\geq m}\left[\frac{(n-p\alpha)}{(n-p\alpha)}\frac{(\beta-1)}{(\alpha-1)}\frac{b_n}{h_n}\right]^{\frac{1}{n-p}}\right\}.$$

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