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ON CERTAIN SUBCLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS ASSOCIATED WITH THE GENERALIZED HYPERGEOMETRIC FUNCTION

JAGANNATH PATEL AND ASHIS KU. PALIT

DEPARTMENT OF MATHEMATICS
UTKAL UNIVERSITY, VANI VIHAR
BHUBANESWAR-751004, INDIA
jpatelmath@yahoo.co.in

DEPARTMENT OF MATHEMATICS
BHADRAK INSTITUTE OF ENGINEERING AND TECHNOLOGY
BHADRAK-756 113, INDIA
ashis_biet@rediffmail.com

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ABSTRACT. In the present paper, we investigate several inclusion relationships and other interesting properties of certain subclasses of meromorphically multivalent functions which are defined here by means of a linear operator involving the generalized hypergeometric function. Some interesting applications on Hadamard product concerning this and other classes of integral operators are also considered.

Key words and phrases: Meromorphic function, p-valent, Subordination, Hypergeometric function, Hadamard product.

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1. INTRODUCTION

For any integer $m > 1 - p$, let $\sum_{p,m}$ be the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic and p-valent in the *punctured* unit disk $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$. We also denote $\sum_{p,1-p} = \sum_p$. For $0 \leqq \alpha < p$, we denote by $\sum_S(p;\alpha)$, $\sum_K(p;\alpha)$ and $\sum_C(p;\alpha)$, the subclasses of $\sum_p$ consisting of all meromorphic functions which are, respectively, p-valently starlike of order α, p-valently convex of order α and p-valently close-to-convex of order α.

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or (more precisely) $f(z) \prec g(z)$ $z \in \mathbb{U}$, if there exists a function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and

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$|\omega(z)| < 1$ such that $f(z) = g(\omega(z)),\ z \in \mathbb{U}$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z)\ (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For a function $f \in \sum_{p,m}$, given by (1.1) and $g \in \sum_{p,m}$ defined by $g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k$, we define the Hadamard product (or convolution) of f and g by

$$f(z) * g(z) = (f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k \quad (p \in \mathbb{N}).$$

For real or complex numbers

$$\alpha_1, \alpha_2, \dots, \alpha_q \quad \text{and} \quad \beta_1, \beta_2, \dots, \beta_s \ \left(\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\};\ j = 1, 2, \dots, s\right),$$

we consider the *generalized hypergeometric function* ${}_qF_s$ (see, for example, [17]) defined as follows:

$$(1.2) \qquad {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{z^k}{k!}$$
$$(q \leqq s+1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in \mathbb{U}),$$

where $(x)_k$ denotes the *Pochhammer* symbol (or the *shifted factorial*) defined, in terms of the Gamma function Γ, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} x(x+1)(x+2)\cdots(x+k-1) & (k \in \mathbb{N}); \\ 1 & (k = 0). \end{cases}$$

Corresponding to the function $\phi_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ given by

$$(1.3) \qquad \phi_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z^{-p}\ {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z),$$

we introduce a function $\phi_{p,\mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ defined by

$$(1.4) \qquad \phi_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * \phi_{p,\mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$$
$$= \frac{1}{z^p(1-z)^{\mu+p}} \quad (\mu > -p;\ z \in \mathbb{U}^*).$$

We now define a linear operator $\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) : \sum_{p,m} \longrightarrow \sum_{p,m}$ by

$$(1.5) \qquad \mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z) = \phi_{p,\mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z)$$
$$\left(\alpha_i, \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ i = 1, 2 \dots, q;\ j = 1, 2, \dots, s;\ \mu > -p;\ f \in \sum\nolimits_{p,m};\ z \in \mathbb{U}^*\right).$$

For convenience, we write

$$\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) = \mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1) \quad \text{and}$$
$$\mathcal{H}_{p,q,s}^{1-p,\mu}(\alpha_1) = \mathcal{H}_{p,q,s}^{\mu}(\alpha_1)\ (\mu > -p).$$

If f is given by (1.1), then from (1.5), we deduce that

$$(1.6) \qquad \mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1) f(z) = z^{-p} + \sum_{k=m}^{\infty} \frac{(\mu+p)_{p+k} (\beta_1)_{p+k} \cdots (\beta_s)_{p+k}}{(\alpha_1)_{p+k} \cdots (\alpha_q)_{p+k}} a_k z^k$$
$$(\mu > -p;\ z \in \mathbb{U}^*).$$

and it is easily verified from (1.6) that

$$(1.7) \qquad z\left(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1) f\right)'(z) = (\mu+p)\ \mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1) f(z) - (\mu+2p)\ \mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1) f(z)$$

and

$$z\left(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1+1)f\right)'(z)=\alpha_1\,\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)f(z)-(p+\alpha_1)\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)f(z). \tag{1.8}$$

We note that the linear operator $H_{p,q,s}^{m,\mu}(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator [5] for analytic functions and is essentially motivated by the operators defined and studied in [3]. The linear operator $H_{1,q,s}^{0,\mu}(\alpha_1)$ was investigated recently by Cho and Kim [2], whereas $H_{p,2,1}^{1-p}(c,1;a;z)=\mathcal{L}_p(a,c)\ \ (c\in\mathbb{R},\ \ a\notin\mathbb{Z}_0^-)$ is the operator studied in [7]. In particular, we have the following observations:

(i) $\mathcal{H}_{p,s+1,s}^{m,0}(p+1,\beta_1,\dots,\beta_s;\beta_1,\dots,\beta_s)f(z)=\dfrac{p}{z^{2p}}\displaystyle\int_0^z t^{2p-1}f(t)\,dt;$

(ii) $\mathcal{H}_{p,s+1,s}^{m,0}(p,\beta_1,...,\beta_s;\beta_1,...,\beta_s)f(z)=\mathcal{H}_{p,s+1,s}^{m,1}(p+1,\beta_1,...,\beta_s;\beta_1,...,\beta_s)f(z)=f(z);$

(iii) $\mathcal{H}_{p,s+1,s}^{m,1}(p,\beta_1,\dots,\beta_s;\beta_1,\dots,\beta_s)f(z)=\dfrac{zf'(z)+2pf(z)}{p};$

(iv) $\mathcal{H}_{p,s+1,s}^{m,2}(p+1,\beta_1,\dots,\beta_s;\beta_1,\dots,\beta_s)f(z)=\dfrac{zf'(z)+(2p+1)f(z)}{p+1};$

(v) $H_{p,s+1,s}^{1-p,n}(\beta_1,\beta_2,\dots,\beta_s,1;\beta_1,\dots,\beta_s)f(z)=\dfrac{1}{z^p(1-z)^{n+p}}=\mathcal{D}^{n+p-1}f(z)$
$(n$ is an integer $>-p)$, the operator studied in [6], and

(vi) $H_{p,s+1,s}^{m,1-p}(\delta+1,\beta_2,\dots,\beta_s,1;\delta,\beta_2,\dots,\beta_s)f(z)=\dfrac{\delta}{z^{\delta+p}}\displaystyle\int_0^z t^{\delta+p-1}f(t)\,dt$
$(\delta>0;z\in\mathbb{U}^*)$, the integral operator defined by (3.6).

Let Ω be the class of all functions ϕ which are analytic, univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0)=1$ and $\Re\{\phi(z)\}>0$ in $\mathbb{U}$.

Next, by making use of the linear operator $H_{p,q,s}^{m,\mu}(\alpha_1)$, we introduce the following subclasses of $\sum_{p,m}$.

Definition 1.1. A function $f\in\sum_{p,m}$ is said to be in the class $\mathcal{MS}_{p,\alpha_1}^{\mu,m}(q,s;\eta;\phi)$, if it satisfies the following subordination condition:

$$-\frac{1}{p-\eta}\left\{\frac{z\left(H_{p,q,s}^{m,\mu}(\alpha_1)f\right)'(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}+\eta\right\}\prec\phi(z) \tag{1.9}$$
$$(\phi\in\Omega,\ 0\leqq\eta<p,\ \mu>-p;\ z\in\mathbb{U}).$$

In particular, for fixed parameters A and B $(-1\leqq B<A\leqq 1)$, we set

$$\mathcal{MS}_{p,\alpha_1}^{\mu,m}\left(q,s;\eta;\frac{1+Az}{1+Bz}\right)=\mathcal{MS}_{p,\alpha_1}^{\mu,m}(q,s;\eta;A,B).$$

It is easy to see that

$$\mathcal{MS}_{1,\alpha_1}^{\mu,0}(q,s;\eta;\phi)=\mathcal{MS}_{\mu+1,\alpha_1}(q,s;\eta;\phi)\text{ and}$$
$$\mathcal{MS}_{1,\alpha_1}^{\mu,0}(q,s;\eta;A,B)=\mathcal{MS}_{\mu+1,\alpha_1}(q,s;\eta;A,B)$$

are the function classes studied by Cho and Kim [2].

Definition 1.2. For fixed parameters A and B, a function $f \in \sum_{p,m}$ is said to be in the class $\mathcal{MC}^{\mu,m}_{p,\alpha_1}(q, s; \lambda; A, B)$, if it satisfies the following subordination condition:

$$-\frac{z^{p+1}\left\{(1-\lambda)(H^{m,\mu}_{p,q,s}(\alpha_1)f)'(z)+\lambda(H^{m,\mu+1}_{p,q,s}(\alpha_1)f)'(z)\right\}}{p} \prec \frac{1+Az}{1+Bz} \tag{1.10}$$

$$(-1 \leqq B < A \leqq 1,\ \lambda \geqq 0,\ \mu > -p;\ z \in \mathbb{U}).$$

To make the notation simple, we write $\mathcal{MC}^{\mu,m}_{p,\alpha_1}\left(q, s; 0; 1-\frac{2\eta}{p}, -1\right) = \mathcal{MC}^{\mu,m}_{p,\alpha_1}(q, s; \eta)$, the class of functions $f \in \sum_{p,m}$ satisfying the condition:

$$-\Re\left\{z^{p+1}\left(H^{m,\mu}_{p,q,s}(\alpha_1)f\right)'(z)\right\} > \eta \quad (0 \leqq \eta < p; z \in \mathbb{U}).$$

Meromorphically multivalent functions have been extensively studied by (for example) Liu and Srivastava [7], Cho et al. [4], Srivastava and Patel [18], Cho and Kim [2], Aouf [1], Srivastava et al. [19] and others.

The object of the present paper is to investigate several inclusion relationships and other interesting properties of certain subclasses of meromorphically multivalent functions which are defined here by means of the linear operator $H^{m,\mu}_{p,q,s}(\alpha_1)$ involving the generalized hypergeometric function. Some interesting applications of the Hadamard product concerning this and other classes of integral operators are also considered. Relevant connections of the results presented here with those obtained by earlier workers are also mentioned.

2. PRELIMINARIES

To prove our results, we need the following lemmas.

Lemma 2.1 ([8], see also [10])**.** *Let the function h be analytic and convex(univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function ϕ given by*

$$\phi(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + \cdots \quad (n \in \mathbb{N}) \tag{2.1}$$

is analytic in $\mathbb{U}$. If

$$\phi(z) + \frac{z\phi'(z)}{\kappa} \prec h(z) \quad (\Re(\kappa) \geqq 0,\ \kappa \neq 0;\ z \in \mathbb{U}),$$

then

$$\phi(z) \prec q(z) = \frac{\kappa}{n} z^{-\frac{\kappa}{n}} \int_0^z t^{\frac{\kappa}{n}-1} h(t)\, dt \prec h(z) \quad (z \in \mathbb{U})$$

and q is the best dominant.

The following identities are well-known [21, Chapter 14].

Lemma 2.2. *For real or complex numbers a, b, c $(c \notin \mathbb{Z}_0^-)$, we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z) \ \ (\Re(c) > \Re(b) > 0) \tag{2.2}$$

$${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z) \tag{2.3}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \tag{2.4}$$

$$(b+1){}_2F_1(1, b; b+1; z) = (b+1) + bz\, {}_2F_1(1, b+1; b+2; z) \tag{2.5}$$

and

$${}_2F_1\left(1, 1; 2; \frac{1}{2}\right) = 2\ln 2. \tag{2.6}$$

We denote by $\mathcal{P}(\gamma)$, the class of functions ψ of the form

$$\psi(z) = 1 + c_1 z + c_2 z^2 + \cdots, \tag{2.7}$$

which are analytic in $\mathbb{U}$ and satisfy the inequality:

$$\Re\{\psi(z)\} > \gamma \quad (0 \leqq \gamma < 1;\ z \in \mathbb{U}).$$

It is known [20] that if $f_j \in \mathcal{P}(\gamma_j)$ $(0 \leqq \gamma_j < 1;\ j = 1,2)$, then

$$(f_1 * f_2)(z) \in \mathcal{P}(\gamma_3) \quad (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)). \tag{2.8}$$

The result is the best possible.

We now state

Lemma 2.3 ([12]). *If the function ψ, given by (2.7) belongs to the class $\mathcal{P}(\gamma)$, then*

$$\Re\{\psi(z)\} \geqq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|} \quad (0 \leqq \gamma < 1;\ z \in \mathbb{U}).$$

Lemma 2.4 ([8, 10]). *Let the function $\Psi : \mathbb{C}^2 \times \mathbb{U} \longrightarrow \mathbb{C}$ satisfy the condition $\Re\{\Psi(ix, y; z)\} \leqq \varepsilon$ for $\varepsilon > 0$, all real x and $y \leqq -n(1+x^2)/2$, where $n \in \mathbb{N}$. If ϕ defined by (2.1) is analytic in $\mathbb{U}$ and $\Re\{\Psi(\phi(z), z\phi'(z); z)\} > \varepsilon$, then $\Re\{\phi(z)\} > 0$ in $\mathbb{U}$.*

We now recall the following result due to Singh and Singh [16].

Lemma 2.5. *Let the function Φ be analytic in $\mathbb{U}$ with $\Phi(0) = 1$ and $\Re\{\Phi(z)\} > 1/2$ in $\mathbb{U}$. Then for any function F, analytic in $\mathbb{U}$, $(\Phi * F)(\mathbb{U})$ is contained in the convex hull of $F(\mathbb{U})$.*

Lemma 2.6 ([13]). *The function $(1-z)^\beta = e^{\beta \log(1-z)}$, $\beta \neq 0$ is univalent in $\mathbb{U}$, if β satisfies either $|\beta + 1| \leqq 1$ or $|\beta - 1| \leqq 1$.*

Lemma 2.7 ([9]). *Let q be univalent in $\mathbb{U}$, θ and Φ be analytic in a domain $\mathcal{D}$ containing $q(\mathbb{U})$ with $\Phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) *Q is starlike(univalent) in $\mathbb{U}$ with $Q(0) = 0$, $Q'(0) \neq 0$ and*
(ii) *Q and h satisfy*

$$\Re\left\{\frac{zh(z)}{Q(z)}\right\} = \Re\left\{\frac{Q'(q(z))}{\Phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0.$$

If ϕ is analytic in $\mathbb{U}$ with $\phi(0) = q(0)$, $\phi(\mathbb{U}) \subset \mathcal{D}$ and

$$\theta(\phi(z)) + z\phi'(z)\Phi(\phi(z)) \prec \theta(q(z)) + zq'(z)\Phi(q(z)) = h(z) \quad (z \in \mathbb{U}), \tag{2.9}$$

then $\phi \prec q$ and q is the best dominant of (2.9).

3. MAIN RESULTS

Unless otherwise mentioned, we assume *throughout* the sequel that

$$\alpha_1 > 0,\ \alpha_i, \beta_j \in \mathbb{R} \setminus \mathbb{Z}_0^- \ (i = 2,3,\ldots,q; j = 1,2,\ldots,s),$$
$$\lambda > 0,\ \mu > -p \quad \text{and} \quad -1 \leqq B < A \leqq 1.$$

Following the lines of proof of Cho and Kim [2] (see, also [4]), we can prove the following theorem.

Theorem 3.1. *Let* $\phi \in \Omega$ *with*

$$\max_{z\in\mathbb{U}} \Re\{\phi(z)\} < \min\{(\mu+2p-\eta)/(p-\eta), (\alpha_1+p-\eta)/(p-\eta)\} \qquad (0 \leqq \eta < p).$$

Then

$$\mathcal{MS}^{\mu+1,m}_{p,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MS}^{\mu,m}_{p,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MS}^{\mu,m}_{p,\alpha_1+1}(q,s;\eta;\phi).$$

By carefully choosing the function ϕ in the above theorem, we obtain the following interesting consequences.

Example 3.1. The function

$$\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha} \quad (0 < \alpha \leqq 1;\ z \in \mathbb{U})$$

is analytic and convex univalent in $\mathbb{U}$. Moreover,

$$0 \leqq \left(\frac{1-A}{1-B}\right)^{\alpha} < \Re\{\phi(z)\} < \left(\frac{1+A}{1+B}\right)^{\alpha}$$
$$(0 < \alpha \leqq 1,\ -1 < B < A \leqq 1;\ z \in \mathbb{U}).$$

Thus, by Theorem 3.1 , we deduce that, if

$$\left(\frac{1+A}{1+B}\right)^{\alpha} < \min\left\{\frac{\mu+2p-\eta}{p-\eta}, \frac{\alpha_1+p-\eta}{p-\eta}\right\}$$
$$(0 < \alpha \leqq 1,\ -1 < B < A \leqq 1),$$

then

$$\mathcal{MS}^{\mu+1,m}_{p,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MS}^{\mu,m}_{p,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MS}^{\mu,m}_{p,\alpha_1+1}(q,s;\eta;\phi).$$

Example 3.2. The function

$$\phi(z) = 1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{\alpha\, z}}{1-\sqrt{\alpha\, z}}\right)\right]^2 \quad (0 < \alpha < 1;\ z \in \mathbb{U})$$

is in the class Ω (cf. [14]) and satisfies

$$\Re\{\phi(z)\} < 1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)\right]^2 \quad (z \in \mathbb{U}).$$

Thus, by using Theorem 3.1, we obtain that, if

$$1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)\right]^2 < \min\left\{\frac{\mu+2p-\eta}{p-\eta}, \frac{\alpha_1+p-\eta}{p-\eta}\right\} \quad (0 < \alpha < 1),$$

then

$$\mathcal{MS}^{\mu+1,m}_{p,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MS}^{\mu,m}_{p,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MS}^{\mu,m}_{p,\alpha_1+1}(q,s;\eta;\phi).$$

Example 3.3. The function

$$\phi(z) = 1 + \sum_{k=1}^{\infty}\left(\frac{\beta+1}{\beta+k}\right)\alpha^k z^k \quad (0 < \alpha < 1,\ \beta \geqq 0; z \in \mathbb{U})$$

belongs to the class Ω (cf. [15]) and satisfies

$$\Re\{\phi(z)\} < 1 + \sum_{k=1}^{\infty}\left(\frac{\beta+1}{\beta+k}\right)\alpha^k \quad (0 < \alpha < 1,\ \beta \geqq 0).$$

Thus, by Theorem 3.1, if

$$1+\sum_{k=1}^{\infty}\left(\frac{\beta+1}{\beta+k}\right)\alpha^k<\min\left\{\frac{\mu+2p-\eta}{p-\eta},\frac{\alpha_1+p-\eta}{p-\eta}\right\}\quad (0<\alpha<1,\ \beta\geqq 0),$$

then

$$\mathcal{MS}_{p,\alpha_1}^{\mu+1,m}(q,s;\eta;\phi)\subset\mathcal{MS}_{p,\alpha_1}^{\mu,m}(q,s;\eta;\phi)\subset\mathcal{MS}_{p,\alpha_1+1}^{\mu,m}(q,s;\eta;\phi).$$

Theorem 3.2. *If $f\in\mathcal{MC}_{p,\alpha_1}^{\mu,m}(q,s;\lambda;A,B)$, then*

$$-\frac{z^{p+1}\left(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)f\right)'(z)}{p}\prec\psi(z)\prec\frac{1+Az}{1+Bz}\quad (z\in\mathbb{U}),\tag{3.1}$$

where the function ψ given by

$$\psi(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\frac{\mu+p}{\lambda(p+m)}+1;\frac{Bz}{1+Bz}\right) & (B\neq 0);\\ 1+\frac{(\mu+p)A}{\mu+p+\lambda(p+m)}z & (B=0)\end{cases}$$

is the best dominant of (3.1). Further,

$$f\in\mathcal{MC}_{p,\alpha_1}^{\mu,m}(q,s;p\rho),\tag{3.2}$$

where

$$\rho=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\frac{\mu+p}{\lambda(p+m)}+1;\frac{B}{B-1}\right) & (B\neq 0);\\ 1-\frac{(\mu+p)A}{\mu+p+\lambda(p+m)} & (B=0).\end{cases}$$

The result is the best possible.

Proof. Setting

$$\varphi(z)=-\frac{z^{p+1}\left(H_{p,q,s}^{m,\mu}(\alpha_1)f\right)'(z)}{p}\quad (z\in\mathbb{U}),\tag{3.3}$$

we note that φ is of the form (2.1) and is analytic in $\mathbb{U}$. Making use of the identity (1.7) in (3.3) and differentiating the resulting equation, we get

$$\varphi(z)+\frac{z\varphi'(z)}{(\mu+p)/\lambda}=-\frac{z^{p+1}\left\{(1-\lambda)\left(H_{p,q,s}^{m,\mu}(\alpha_1)f\right)'(z)+\lambda\left(H_{p,q,s}^{m,\mu+1}(\alpha_1)f\right)'(z)\right\}}{p}\prec\frac{1+Az}{1+Bz}\quad (z\in\mathbb{U}).\tag{3.4}$$

Now, by applying Lemma 2.1 (with $\kappa=(\mu+p)/\lambda$) in (3.4), we deduce that

$$\begin{aligned}-\frac{z^{p+1}\left(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)f\right)'(z)}{p}&\prec\psi(z)=\frac{\mu+p}{\lambda(p+m)}z^{-\frac{\mu+p}{\lambda(p+m)}}\int_0^z t^{\frac{\mu+p}{\lambda(p+m)}-1}\left(\frac{1+Az}{1+Bz}\right)dt\\&=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\frac{\mu+p}{\lambda(p+m)}+1;\frac{Bz}{1+Bz}\right) & (B\neq 0)\\ 1+\frac{(\mu+p)A}{\mu+p+\lambda(p+m)}z & (B=0)\end{cases}\end{aligned}$$

by a change of variables followed by the use of the identities (2.2), (2.3), (2.4) and (2.5), respectively. This proves the assertion (3.3).

To prove (3.2), we follow the lines of proof of Theorem 1 in [18]. The result is the best possible as ψ is the best dominant. This completes the proof of Theorem 3.2. □

Setting $A = 1 - (2\eta/p)\,,\ B = -1,\ \mu = 0,\ m = 1 - p,\ \alpha_1 = \lambda = p$ and $\alpha_{i+1} = \beta_i\ (i = 1, 2, \ldots, s)$ in Theorem 3.2 followed by the use of the identity (2.6), we get

Corollary 3.3. *If $f \in \sum_p$ satisfies*

$$-\Re\left\{z^{p+1}\left((p+2)f'(z) + zf''(z)\right)\right\} > \eta \quad (0 \leqq \eta < p; z \in \mathbb{U}),$$

then

$$-\Re\{z^{p+1} f'(z)\} > \eta + 2(p - \eta)(\ln 2 - 1) \quad (z \in \mathbb{U}).$$

The result is the best possible.

Putting $A = 1 - (2\eta/p)\,,\ B = -1,\ \mu = 0,\ m = 2 - p,\ \alpha_1 = \lambda = p$ and $\alpha_{i+1} = \beta_i\ (i = 1, 2, \ldots, s)$ in Theorem 3.2, we obtain the following result due to Pap [11].

Corollary 3.4. *If $f \in \sum_{p,2-p}$ satisfies*

$$-\Re\left\{z^{p+1}\left((p+2)f'(z) + zf''(z)\right)\right\} > -\frac{p(\pi - 2)}{4 - \pi} \quad (z \in \mathbb{U}),$$

then

$$-\Re\{z^{p+1} f'(z)\} > 0 \quad (z \in \mathbb{U}).$$

The result is the best possible.

The proof of the following result is much akin to that of Theorem 2 in [18] and we choose to omit the details.

Theorem 3.5. *If $f \in \mathcal{MC}^{\mu,m}_{p,\alpha_1}(q, s; \eta)\ \ (0 \leqq \eta < p)$, then*

$$-\Re\left[z^{p+1}\left\{(1-\lambda)\ \left(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f\right)'(z) + \lambda\ \left(\mathcal{H}^{m,\mu+1}_{p,q,s}(\alpha_1)f\right)'(z)\right\}\right] > \eta$$
$$(|z| < R(p, \mu, \lambda, m)),$$

where

$$R(p, \mu, \lambda, m) = \left[\frac{\sqrt{(\mu + p)^2 + \lambda^2(p + m)^2} - \lambda(p + m)}{\mu + p}\right]^{\frac{1}{p+m}}.$$

The result is the best possible.

Upon replacing $\varphi(z)$ by $z^p\,\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z)$ in (3.3) and using the same techniques as in the proof of Theorem 3.2, we get the following result.

Theorem 3.6. *If $f \in \sum_{p,m}$ satisfies*

$$z^p\left\{(1-\lambda)\ \mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z) + \lambda\ \mathcal{H}^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)\right\} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),$$

then

$$z^p\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z) \prec \psi(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})$$

and

$$\Re\left\{z^p\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right\} > \rho \quad (z \in \mathbb{U}),$$

where ψ and ρ are given as in Theorem 3.2. The result is the best possible.

Letting

$$A=\left\{{}_2F_1\left(1,1;\frac{p}{\lambda(p+m)}+1;\frac{1}{2}\right)-1\right\}\left\{2-\ {}_2F_1\left(1,1;\frac{p}{\lambda(p+m)}+1;\frac{1}{2}\right)\right\}^{-1},$$

$B=-1,\ \mu=0,\ \alpha_1=p$ and $\alpha_{i+1}=\beta_i\ (i=1,2,\dots,s)$ in Theorem 3.6, we obtain

Corollary 3.7. *If $f\in\sum_{p,m}$ satisfies*

$$\Re\left\{(1+\lambda)f(z)+\frac{\lambda}{p}z^{p+1}f'(z)\right\}>\frac{3-2\ {}_2F_1\left(1,1;\frac{p}{\lambda(p+m)}+1;\frac{1}{2}\right)}{2\left\{2-{}_2F_1\left(1,1;\frac{p}{\lambda(p+m)}+1;\frac{1}{2}\right)\right\}}\quad(z\in\mathbb{U}),\tag{3.5}$$

then

$$\Re\{z^pf(z)\}>\frac{1}{2}\quad(z\in\mathbb{U}).$$

The result is the best possible.

For a function $f\in\sum_{p,m}$, we consider the integral operator $\mathcal{F}_{\delta,p}$ defined by

$$\begin{aligned}\mathcal{F}_{\delta,p}(z)=\mathcal{F}_{\delta,p}(f)(z)&=\frac{\delta}{z^{\delta+p}}\int_0^z t^{\delta+p-1}f(t)\,dt\\&=\left(z^{-p}+\sum_{k=m}^{\infty}\frac{\delta}{\delta+p+k}z^k\right)*f(z)\quad(\delta>0;\ z\in\mathbb{U}^*).\end{aligned}\tag{3.6}$$

It follows from (3.6) that $\mathcal{F}_{\delta,p}(f)\in\sum_{p,m}$ and

$$z\left(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\mathcal{F}_{\delta,p}(f)\right)'(z)=\delta\ \mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z)-(\delta+p)\ \mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\mathcal{F}_{\delta,p}(f)(z).\tag{3.7}$$

Using (3.7) and the lines of proof of Theorem 1 [2], we obtain the following inclusion relation.

Theorem 3.8. *Let $\phi\in\Omega$ with $\max_{z\in\mathbb{U}}\Re\{\phi(z)\}<(\delta+p-\eta)/(p-\eta)\ (0\leqq\eta<p;\ \delta>0)$. If $f\in\mathcal{MS}^{\mu,m}_{p,\alpha_1}(q,s;\eta;\phi)$, then $\mathcal{F}_{\delta,p}(f)\in\mathcal{MS}^{\mu,m}_{p,\alpha_1}(q,s;\eta;\phi)$.*

Theorem 3.9. *If $f\in\sum_{p,m}$ and the function $\mathcal{F}_{\delta,p}(f)$, defined by (3.6) satisfies*

$$-\frac{z^{p+1}\left\{(1-\lambda)\ \left(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\mathcal{F}_{\delta,p}(f)\right)'(z)+\lambda\ \left(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f\right)'(z)\right\}}{p}\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}),$$

then

$$-\Re\left\{\frac{z^{p+1}\left(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\mathcal{F}_{\delta,p}(f)\right)'(z)}{p}\right\}>\varrho\quad(z\in\mathbb{U}),$$

where

$$\varrho=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\frac{\delta}{\lambda(p+m)}+1;\frac{B}{B-1}\right) & (B\neq0)\\ 1-\frac{\delta A}{\mu+p+\lambda(p+m)} & (B=0).\end{cases}$$

The result is the best possible.

Proof. If we let

$$\varphi(z)=-\frac{z^{p+1}\left(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\mathcal{F}_{\delta,p}(f)\right)'(z)}{p}\quad(z\in\mathbb{U}),\tag{3.8}$$

then φ is of the form (2.1) and is analytic in $\mathbb{U}$. Using the identity (3.7) in (3.8) followed by differentiation of the resulting equation, we get

$$\varphi(z) + \frac{z\varphi'(z)}{\delta/\lambda} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

The proof of the remaining part follows by employing the techniques that proved Theorem 3.2. □

Upon setting $A = 1 - (2\eta/p)\,,\ B = -1,\ \lambda = \mu = 1,\ \alpha_1 = p+1$ and $\alpha_{i+1} = \beta_i\ (i = 1, 2, \ldots, s)$ in Theorem 3.9, we have

Corollary 3.10. *If $f \in \sum_C(p;\eta)\ (0 \leqq \eta < p)$, then the function $\mathcal{F}_{\delta,p}(f)$ defined by (3.6) belongs to the class $\sum_C(p;\varkappa)$, where*

$$\varkappa = \eta + (p-\eta)\left\{ {}_2F_1\left(1, 1; \frac{\delta}{p+m} + 1; \frac{1}{2}\right) - 1 \right\}.$$

The result is the best possible.

Remark 1. Under the hypothesis of Theorem 3.9 and using the fact that

$$z^{p+1}\left(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\mathcal{F}_{\delta,p}(f)\right)'(z) = \frac{\delta}{z^\delta}\int_0^z t^{\delta+p}\left(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f\right)'(t)\,dt \quad (\delta > 0; z \in \mathbb{U}),$$

we obtain

$$-\Re\left\{\frac{\delta}{z^\delta}\int_0^z t^{\delta+p}\left(\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f\right)'(t)\,dt\right\} > \varrho \quad (z \in \mathbb{U}),$$

where ϱ is given as in Theorem 3.9.

Following the same lines of proof as in Theorem 3.9, we obtain

Theorem 3.11. *If $f \in \sum_{p,m}$ and the function $\mathcal{F}_{\delta,p}(f)$ defined by (3.6) satisfies*

$$z^p\left\{(1-\lambda)\,\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\mathcal{F}_{\delta,p}(f)(z) + \lambda\,\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right\} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

then

$$\Re\left\{z^p\mathcal{H}^{m,\mu}_{p,q,s}(\alpha_1)\mathcal{F}_{\delta,p}(f)(z)\right\} > \varrho \quad (z \in \mathbb{U}),$$

where ϱ is given as in Theorem 3.9. The result is the best possible.

In the special case when $A = 1 - 2\eta,\ B = -1,\ \lambda = 1,\ \mu = 1-p,\ \alpha_1 = \delta+1,\ \beta_1 = \delta,\ \alpha_i = \beta_i\ (i = 2, 3, \ldots, s)$ and $\alpha_{s+1} = 1$ in Theorem 3.11, we get

Corollary 3.12. *If $f \in \sum_{p,m}$ satisfies*

$$\Re\{z^p f(z)\} > \eta \quad (0 \leqq \eta < 1;\ z \in \mathbb{U}),$$

then

$$\Re\left\{\frac{\delta}{z^\delta}\int_0^z t^{\delta+p-1}f(t)\,dt\right\} > \eta + (1-\eta)\left\{ {}_2F_1\left(1, 1; \frac{\delta}{p+m}+1; \frac{1}{2}\right) - 1\right\}$$
$$(\delta > 0;\ z \in \mathbb{U}).$$

The result is the best possible.

Theorem 3.13. *Let $-1 \leqq B_j < A_j \leqq 1\ (j = 1, 2)$. If $f_j \in \sum_p$ satisfies the following subordination condition:*

$$z^p\left\{(1-\lambda)\mathcal{H}^{\mu}_{p,q,s}(\alpha_1)f_j(z) + \lambda\mathcal{H}^{\mu+1}_{p,q,s}(\alpha_1)f_j(z)\right\} \prec \frac{1+A_jz}{1+B_jz} \tag{3.9}$$
$$(j = 1, 2;\ z \in \mathbb{U}),$$

then

$$\Re\left[z^p\left\{(1-\lambda)\mathcal{H}^{\mu}_{p,q,s}(\alpha_1)\mathfrak{g}(z)+\lambda\mathcal{H}^{\mu+1}_{p,q,s}(\alpha_1)\mathfrak{g}(z)\right\}\right]>\tau \quad (z\in\mathbb{U}), \tag{3.10}$$

where

$$\mathfrak{g}(z)=\mathcal{H}^{\mu}_{p,q,s}(\alpha_1)(f_1*f_2)(z) \quad (z\in\mathbb{U}^*) \tag{3.11}$$

and

$$\tau=1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left\{1-\frac{1}{2}\,{}_2F_1\left(1,1;\frac{\mu+p}{\lambda}+1;\frac{1}{2}\right)\right\}.$$

The result is the best possible when $B_1=B_2=-1$.

Proof. Setting

$$\varphi_j(z)=z^p\left\{(1-\lambda)\mathcal{H}^{\mu}_{p,q,s}(\alpha_1)f_j(z)+\lambda\mathcal{H}^{\mu+1}_{p,q,s}(\alpha_1)f_j(z)\right\} \quad (j=1,2;\ z\in\mathbb{U}), \tag{3.12}$$

we note that φ_j is of the form (2.7) for each $j=1,2$ and using (3.9), we obtain

$$\varphi_j\in\mathcal{P}(\gamma_j) \quad \left(\gamma_j=\frac{1-A_j}{1-B_j};\ j=1,2\right)$$

so that by (2.8),

$$\varphi_1*\varphi_2\in\mathcal{P}(\gamma_3) \quad (\gamma_3=1-2(1-\gamma_1)(1-\gamma_2)). \tag{3.13}$$

Using the identity (1.7) in (3.12), we conclude that

$$\mathcal{H}^{\mu}_{p,q,s}(\alpha_1)f_j(z)=\frac{\mu+p}{\lambda}z^{-p-\frac{\mu+p}{\lambda}}\int_0^z t^{\frac{\mu+p}{\lambda}-1}\varphi_j(t)dt \quad (j=1,2;\ z\in\mathbb{U}^*)$$

which, in view of (3.11) yields

$$\mathcal{H}^{\mu}_{p,q,s}(\alpha_1)\mathfrak{g}(z)=\frac{\mu+p}{\lambda}z^{-p-\frac{\mu+p}{\lambda}}\int_0^z t^{\frac{\mu+p}{\lambda}-1}\varphi_0(t)dt \quad (z\in\mathbb{U}^*),$$

where, for convenience

$$\begin{aligned}\varphi_0(z)&=z^p\left\{(1-\lambda)\mathcal{H}^{\mu}_{p,q,s}(\alpha_1)\mathfrak{g}(z)+\lambda\mathcal{H}^{\mu+1}_{p,q,s}(\alpha_1)\mathfrak{g}(z)\right\}\\&=\frac{\mu+p}{\lambda}z^{-\frac{\mu+p}{\lambda}}\int_0^z t^{\frac{\mu+p}{\lambda}-1}\,(\varphi_1*\varphi_2)(t)\,dt \quad (z\in\mathbb{U}).\end{aligned} \tag{3.14}$$

Now, by using (3.13) in (3.14) and by appealing to Lemma 2.3 and Lemma 2.5, we get

$$\begin{aligned}\Re\{\varphi_0(z)\}&=\frac{\mu+p}{\lambda}\int_0^1 s^{\frac{\mu+p}{\lambda}-1}\,\Re(\varphi_1*\varphi_2)(sz)\,ds\\&\geqq\frac{\mu+p}{\lambda}\int_0^1 s^{\frac{\mu+p}{\lambda}-1}\left(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+s|z|}\right)ds\\&>\frac{\mu+p}{\lambda}\int_0^1 s^{\frac{\mu+p}{\lambda}-1}\left(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+s}\right)ds\\&=1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1-\frac{\mu+p}{\lambda}\int_0^1 s^{\frac{\mu+p}{\lambda}-1}\,(1+s)^{-1}\,ds\right)\end{aligned}$$

$$= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left\{ 1 - \frac{1}{2}\, {}_2F_1\left(1, 1; \frac{\mu + p}{\lambda} + 1; \frac{1}{2}\right) \right\}$$
$$= \tau \quad (z \in \mathbb{U}).$$

This proves the assertion (3.10).

When $B_1 = B_2 = -1$, we consider the functions $f_j \in \sum_p$ defined by

$$\mathcal{H}^{\mu}_{p,q,s}(\alpha_1) f_j(z) = \frac{\mu + p}{\lambda} z^{-p-\frac{\mu+p}{\lambda}} \int_0^z t^{\frac{\mu+p}{\lambda}-1} \left(\frac{1 + A_j t}{1 - t} \right) dt$$
$$(j = 1, 2;\ z \in \mathbb{U}^*).$$

Then it follows from (3.14) that and Lemma 2.2 that

$$\varphi_0(z) = \frac{\mu + p}{\lambda} \int_0^1 s^{\frac{\mu+p}{\lambda}-1} \left(1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - sz} \right) ds$$
$$= 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)(1 - z)^{-1}\, {}_2F_1\left(1, 1; \frac{\mu + p}{\lambda} + 1; \frac{z}{z - 1}\right)$$
$$\longrightarrow 1 - (1 + A_1)(1 + A_2) + \frac{1}{2}(1 + A_1)(1 + A_2)\, {}_2F_1\left(1, 1; \frac{\mu + p}{\lambda} + 1; \frac{1}{2}\right) \text{ as } z \to 1^-,$$

which evidently completes the proof of Theorem 3.13. □

By taking $A_j = 1 - 2\eta_j,\ B_j = -1\ (j = 1, 2),\ \mu = 0,\ \alpha_1 = p$ and $\alpha_{i+1} = \beta_i\ (i = 1, 2, \ldots, s)$ in Theorem 3.13, we get the following result which refines the corresponding work of Yang [22, Theorem 4].

Corollary 3.14. *If each of the functions $f_j \in \sum_p$ satisfies*

$$\Re \left[z^p \left\{ (1 + \lambda)\, f_j(z) + \frac{\lambda}{p}\, z f_j'(z) \right\} \right] > \eta_j$$
$$(0 \leqq \eta_j < 1,\ j = 1, 2;\ z \in \mathbb{U}),$$

then

$$\Re \left[z^p \left\{ (1 + \lambda)\, (f_1 * f_2)(z) + \frac{\lambda}{p}\, z (f_1 * f_2)'(z) \right\} \right] > \sigma \quad (z \in \mathbb{U}),$$

where

$$\sigma = 1 - 4(1 - \eta_1)(1 - \eta_2) \left\{ 1 - \frac{1}{2}\, {}_2F_1\left(1, 1; \frac{p}{\lambda} + 1; \frac{1}{2}\right) \right\}.$$

The result is the best possible.

For $A_j = 1 - 2\eta_j,\ B_j = -1\ (j = 1, 2),\ \mu = 0,\ \lambda = 1,\ \alpha_1 = p + 1$ and $\alpha_{i+1} = \beta_i\ (i = 1, 2, \ldots, s)$ in Theorem 3.13, we obtain

Corollary 3.15. *If each of the functions $f_j \in \sum_p$ satisfies*

$$\Re \left\{ z^p\, f_j(z) \right\} > \eta_j \quad (0 \leqq \eta_j < 1,\ j = 1, 2;\ z \in \mathbb{U}),$$

then

$$\Re \left\{ z^p (f_1 * f_2)(z) \right\} > 1 - 4(1 - \eta_1)(1 - \eta_2) \left\{ 1 - \frac{1}{2}\, {}_2F_1\left(1, 1; p + 1; \frac{1}{2}\right) \right\} \quad (z \in \mathbb{U}).$$

The result is the best possible.

Theorem 3.16. *Let* $-1 \leqq B_j < A_j \leqq 1$ $(j = 1, 2)$. *If each of the functions* $f_j \in \sum_{p,m}$ *satisfies*

$$z^p \mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1) f_j(z) \prec \frac{1 + A_j z}{1 + B_j z} \quad (j = 1, 2;\ z \in \mathbb{U}), \tag{3.15}$$

then the function $\mathfrak{h} = \mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)(f_1 * f_2)$ *satisfies*

$$\Re \left\{ \frac{\mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)\mathfrak{h}(z)}{\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)\mathfrak{h}(z)} \right\} > 0 \quad (z \in \mathbb{U}),$$

provided

$$\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} < \frac{2\mu + 3p + m}{2\left[\left\{ (p+m)\, {}_2F_1\left(1, 1; \frac{\mu+p}{p+m}; \frac{1}{2}\right) - 2 \right\}^2 + 2(\mu + p) \right]}. \tag{3.16}$$

Proof. From (3.15), we have

$$z^p \mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1) f_j(z) \in \mathcal{P}(\gamma_j) \quad \left(\gamma_j = \frac{1 - A_j}{1 - B_j};\ j = 1, 2 \right).$$

Thus, it follows from (2.8) that

$$\begin{aligned} \Re \left\{ z^p \mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)\mathfrak{h}(z) + \frac{z \left(z^p \mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)\mathfrak{h} \right)'(z)}{\mu + p} \right\} & \\ = \Re \left\{ z^p \mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1) f_1(z) * z^p \mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1) f_2(z) \right\} & \\ > 1 - \frac{2(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \quad (z \in \mathbb{U}), & \end{aligned} \tag{3.17}$$

which in view of Lemma 2.1 for

$$A = -1 + 4\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)},$$
$$B = -1,\ n = p + m \quad \text{and} \quad \kappa = \mu + p$$

yields

$$\Re \left\{ z^p \mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)\mathfrak{h}(z) \right\} > 1 + \frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left\{ {}_2F_1\left(1, 1; \frac{\mu + p}{p + m}; \frac{1}{2}\right) - 2 \right\} \ (z \in \mathbb{U}). \tag{3.18}$$

From (3.18), by using Theorem 3.6 for

$$A = -1 - 4\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left\{ {}_2F_1\left(1, 1; \frac{\mu + p}{p + m}; \frac{1}{2}\right) - 2 \right\},$$
$$B = -1 \quad \text{and} \quad \lambda = 1,$$

we deduce that

$$\Re \left\{ z^p \vartheta(z) \right\} > 1 - 2\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left\{ {}_2F_1\left(1, 1; \frac{\mu + p}{p + m}; \frac{1}{2}\right) - 2 \right\}^2 \quad (z \in \mathbb{U}), \tag{3.19}$$

where $\vartheta(z) = z^p \mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)\mathfrak{h}(z)$. If we set

$$\varphi(z) = \frac{\mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)\mathfrak{h}(z)}{\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)\mathfrak{h}(z)} \quad (z \in \mathbb{U}),$$

then φ is of the form (2.1), analytic in $\mathbb{U}$ and a simple calculation gives

$$z^p\mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)\mathfrak{h}(z)+\frac{z\left(z^p\mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)\mathfrak{h}\right)'(z)}{\mu+p} = \vartheta(z)\left\{(\varphi(z))^2+\frac{z\varphi'(z)}{\mu+p}\right\} = \Psi\left(\varphi(z), z\varphi'(z); z\right) \quad (z\in\mathbb{U}), \tag{3.20}$$

where $\Psi(u,v;z) = \vartheta(z)\left\{u^2+(v/(\mu+p))\right\}$. Thus, by applying (3.17) in (3.20), we get

$$\Re\left\{\Psi\left(\varphi(z), z\varphi'(z); z\right)\right\} > 1-2\frac{(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \quad (z\in\mathbb{U}).$$

Now, for all real $x, y \leqq -(p+m)(1+x^2)/2$, we have

$$\begin{aligned}\Re\left\{\Psi(ix,y;z)\right\} &= \left(\frac{y}{\mu+p}-x^2\right)\Re\{\vartheta(z)\} \\ &\leqq -\frac{p+m}{2(\mu+p)}\left\{1+x^2+\frac{2(\mu+p)}{p+m}x^2\right\}\Re\{\vartheta(z)\} \\ &\leqq -\frac{p+m}{2(\mu+p)}\Re\{\vartheta(z)\} \leqq 1-2\frac{(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \quad (z\in\mathbb{U}),\end{aligned}$$

by (3.16) and (3.19). Thus, by Lemma 2.4, we get $\Re\{\varphi(z)\}>0$ in $\mathbb{U}$. This completes the proof of Theorem 3.16. □

Taking $A_j = 1-2\eta_j,\ B_j=-1\ (j=1,2),\ \mu=0,\ \lambda=1,\ \alpha_1=p+1$ and $\alpha_{i+1}=\beta_i\ (i=1,2,\dots,s)$ in Theorem 3.16, we have

Corollary 3.17. *If each of the functions $f_j \in \sum_{p,m}$ satisfies*

$$\Re\{z^pf_j(z)\}>\eta_j \quad (0\leqq\eta_j<1,\ j=1,2;\ z\in\mathbb{U}),$$

then

$$\Re\left\{\frac{z^{2p}(f_1*f_2)(z)}{\int_0^z t^{2p-1}(f_1*f_2)(t)\,dt}\right\}>0 \quad (z\in\mathbb{U}),$$

provided

$$(1-\eta_1)(1-\eta_2)<\frac{3p+m}{2\left[\left\{(p+m)\,{}_2F_1\left(1,1;\frac{p}{p+m};\frac{1}{2}\right)-2\right\}^2+2p\right]}.$$

Theorem 3.18. *If $f\in\mathcal{MC}_{p,\alpha_1}^{\mu,m}(q,s;\lambda;A,B)$ and $g\in\sum_{p,m}$ satisfies (3.5), then $f*g\in\mathcal{MC}_{p,\alpha_1}^{\mu,m}(q,s;\lambda;A,B)$.*

Proof. From Corollary 3.7, it follows that $\Re\{g(z)\}>1/2$ in $\mathbb{U}$. Since

$$\begin{aligned}&-\frac{z^{p+1}\left\{(1-\lambda)(\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)(f*g))'(z)+\lambda(\mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)(f*g))'(z)\right\}}{p} \\ &\qquad = \frac{z^{p+1}\left\{(1-\lambda)\mathcal{H}_{p,q,s}^{m,\mu}(\alpha_1)f)'(z)+\lambda(\mathcal{H}_{p,q,s}^{m,\mu+1}(\alpha_1)f)'(z)\right\}}{p}*g(z) \quad (z\in\mathbb{U})\end{aligned}$$

and the function $(1+Az)/(1+Bz)$ is convex(univalent) in $\mathbb{U}$, the assertion of the theorem follows from (1.10) and Lemma 2.5. □

Theorem 3.19. *Let $0 \neq \beta \in \mathbb{C}$ and $0 < \gamma \leqq p$ be such that either $|1 + 2\beta\gamma| \leqq 1$ or $|1 - 2\beta\gamma| \leqq 1$. If $f \in \sum_p$ satisfies*

$$\Re\left\{\frac{\mathcal{H}_{p,q,s}^{\mu+1}(\alpha_1)f(z)}{\mathcal{H}_{p,q,s}^{\mu}(\alpha_1)f(z)}\right\} < 1 + \frac{\gamma}{\mu + p} \quad (z \in \mathbb{U}), \tag{3.21}$$

then

$$\left\{z^p\mathcal{H}_{p,q,s}^{\mu}(\alpha_1)f(z)\right\}^{\beta} \prec q(z) = (1 - z)^{2\beta\gamma} \quad (z \in \mathbb{U})$$

and q is the best dominant.

Proof. Letting

$$\varphi(z) = \left\{z^p\mathcal{H}_{p,q,s}^{\mu}(\alpha_1)f(z)\right\}^{\beta} \quad (z \in \mathbb{U}) \tag{3.22}$$

and choosing the principal branch in (3.22), we note that φ is analytic in $\mathbb{U}$ with $\varphi(0) = 1$. Differentiating (3.22) logarithmically, we deduce that

$$\frac{z\varphi'(z)}{\varphi(z)} = \beta\left\{p + \frac{z\left(\mathcal{H}_{p,q,s}^{\mu}(\alpha_1)f\right)'(z)}{\mathcal{H}_{p,q,s}^{\mu}(\alpha_1)f(z)}\right\} \quad (z \in \mathbb{U}),$$

which in view of the identities (1.7) and (3.21) give

$$-p + \frac{z\varphi'(z)}{\beta\,\varphi(z)} \prec -p\frac{1 - \left(1 - \frac{2\gamma}{p}\right)z}{1 - z} \quad (z \in \mathbb{U}). \tag{3.23}$$

If we take $q(z) = (1 - z)^{2\beta\gamma},\ \theta(z) = -p,\ \Phi(z) = 1/\beta z$ in Lemma 3.11, then by Lemma 2.6, q is univalent in $\mathbb{U}$. Further, it is easy to see that $q,\ \theta$ and Φ satisfy the hypothesis of Lemma 2.7. Since

$$Q(z) = zq'(z)\Phi(q(z)) = -\frac{2\gamma\, z}{1 - z}$$

is starlike (univalent) in $\mathbb{U}$,

$$h(z) = \frac{-p + (p - 2\gamma)z}{1 - z} \quad \text{and} \quad \Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{1}{1 - z}\right\} > 0 \quad (z \in \mathbb{U}),$$

it is readily seen that the conditions (i) and (ii) of Lemma 2.7 are satisfied. Thus, the assertion of the theorem follows from (3.23) and Lemma 2.7. □

Putting $\mu = 0,\ \gamma = p(1 - \eta),\ \beta = -1/2\gamma,\ \alpha_1 = p$ and $\alpha_{i+1} = \beta_i\ (i = 1, 2, \dots, s)$ in Theorem 3.19, we deduce that

Corollary 3.20. *If $f \in \sum_p$ satisfies*

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > p\,\eta \quad (0 \leqq \eta < 1;\ z \in \mathbb{U}),$$

then

$$\Re\left\{z^p f(z)\right\}^{-\frac{1}{2p(1-\eta)}} > \frac{1}{2} \quad (z \in \mathbb{U}).$$

The result is the best possible.

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