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ON SOME INEQUALITIES FOR p-NORMS

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ABSTRACT. In this paper we establish several new inequalities including p-norms for functions whose absolute values aroused to the p-th power are convex functions.

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1. INTRODUCTION

Integral inequalities have become a major tool in the analysis of integral equations, so it is not surprising that many of them appear in the literature (see for example [2], [5], [3] and [1]).

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One of the most important inequalities in analysis is the integral Hölder's inequality which is stated as follows (for this variant see [3, p. 106]).

Theorem A. *Let* $p, q \in \mathbb{R} \setminus \{0\}$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and let* $f, g : [a, b] \to \mathbb{R}$, $a < b$, *be such that* $|f(x)|^p$ *and* $|g(x)|^q$ *are integrable on* $[a, b]$. *If* $p, q > 0$, *then*

$$\int_a^b |f(x) g(x)| \, dx \leq \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q \, dx \right)^{\frac{1}{q}}. \tag{1.1}$$

If $p < 0$ *and additionally* $f([a, b]) \subseteq \mathbb{R} \setminus \{0\}$, *or* $q < 0$ *and* $g([a, b]) \subseteq \mathbb{R} \setminus \{0\}$, *then the inequality in* (1.1) *is reversed.*

The Hermite-Hadamard inequalities for convex functions is also well known. This double inequality is stated as follows (see for example [3, p. 10]): Let f be a convex function on $[a, b] \subset \mathbb{R}$, where $a \neq b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \tag{1.2}$$

To prove our main result we need comparison inequalities between the power means defined by

$$M_n^{[r]}(\boldsymbol{x}; \boldsymbol{p}) = \begin{cases} \left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i^r \right)^{\frac{1}{r}}, & r \neq -\infty, 0, \infty; \\ \left(\prod_{i=1}^n x_i^{pi} \right)^{\frac{1}{P_n}}, & r = 0; \\ \min(x_1, \ldots, x_n), & r = -\infty; \\ \max(x_1, \ldots, x_n), & r = \infty, \end{cases}$$

where $\boldsymbol{x}, \boldsymbol{p}$ are positive n-tuples and $P_n = \sum_{i=1}^n p_i$. It is well known that for such means the following inequality holds:

$$M_n^{[r]}(\boldsymbol{x}; \boldsymbol{p}) \leq M_n^{[s]}(\boldsymbol{x}; \boldsymbol{p}) \tag{1.3}$$

whenever $r < s$ (see for example [3, p. 15]).

In this paper we also use the following result (see [5, p. 152]):

Theorem B. *Let* $\boldsymbol{\xi} \in [a, b]^n$, $0 < a < b$, *and* $\boldsymbol{p} \in [0, \infty)^n$ *be two* n*-tuples such that*

$$\sum_{i=1}^n p_i \xi_i \in [a, b], \qquad \sum_{i=1}^n p_i \xi_i \geq \xi_j, \quad j = 1, 2, \ldots, n.$$

If $f : [a, b] \to \mathbb{R}$ *is such that the function* $f(x)/x$ *is decreasing, then*

$$f\left(\sum_{i=1}^n p_i \xi_i \right) \leq \sum_{i=1}^n p_i f(\xi_i). \tag{1.4}$$

If $f(x)/x$ *is increasing, then the inequality in* (1.4) *is reversed.*

Our goal is to establish several new inequalities for functions whose absolute values raised to some real powers are convex functions.

2. RESULTS

In the literature, the following definition is well known.

Let $f : [a,b] \to \mathbb{R}$ and $p \in \mathbb{R}^+$. The p-norm of the function f on $[a,b]$ is defined by

$$\|f\|_p = \begin{cases} \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}}, & 0 < p < \infty; \\ \sup |f(x)|, & p = \infty, \end{cases}$$

and $L^p([a,b])$ is the set of all functions $f : [a,b] \to \mathbb{R}$ such that $\|f\|_p < \infty$.

Observe that if $|f|^p$ is convex (or concave) on $[a,b]$ it is also integrable on $[a,b]$, hence $0 \le \|f\|_p < \infty$, that is, f belongs to $L^p([a,b])$.

Although p-norms are not defined for $p < 0$, for the sake of the simplicity we will use the same notation $\|f\|_p$ when $p \in \mathbb{R} \setminus \{0\}$.

In order to prove our results we need the following two lemmas.

Lemma 2.1. *Let $\boldsymbol{x}$ and $\boldsymbol{p}$ be two n-tuples such that*

$$x_i > 0, \; p_i \ge 1, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

If $r < s < 0$ or $0 < r < s$, then

$$\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{1}{s}} \le \left(\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, \tag{2.2}$$

and if $r < 0 < s$, then

$$\left(\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}} \le \left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{1}{s}}.$$

If the n-tuple $\boldsymbol{x}$ is only nonnegative, then (2.2) holds whenever $0 < r < s$.

Proof. Suppose that $\boldsymbol{x}$ and $\boldsymbol{p}$ are such that the inequalities in (2.1) hold. It can be easily seen that in this case for any $q \in \mathbb{R}$

$$\sum_{i=1}^n p_i x_i^q \ge x_j^q > 0, \quad j = 1, 2, \ldots, n.$$

To prove the lemma we must consider three cases: (i) $r < s < 0$, (ii) $0 < r < s$ and (iii) $r < 0 < s$. In case (i) we define the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(x) = x^{\frac{s}{r}}$. Since in this case we have $(s-r)/r < 0$, the function

$$f(x)/x = x^{\frac{s}{r}-1} = x^{\frac{s-r}{r}}$$

is decreasing. Applying Theorem B on f, $\boldsymbol{\xi} = (x_1^r, \ldots, x_n^r)$ and $\boldsymbol{p}$ we obtain

$$\left(\sum_{i=1}^n p_i x_i^r\right)^{\frac{s}{r}} \le \sum_{i=1}^n p_i \left(x_i^r\right)^{\frac{s}{r}} = \sum_{i=1}^n p_i x_i^s,$$

i.e.,

$$\left(\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}} \ge \left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{1}{s}}$$

since s is negative.

In case (ii) for the same f as in (i) we have $(s-r)/r > 0$, so similarly as before from Theorem B we obtain

$$\left(\sum_{i=1}^{n} p_i x_i^r\right)^{\frac{s}{r}} \geq \sum_{i=1}^{n} p_i \left(x_i^r\right)^{\frac{s}{r}} = \sum_{i=1}^{n} p_i x_i^s,$$

and since s is positive, (2.2) immediately follows.

And in the end, in case (iii) we have $(s-r)/r < 0$, so using again Theorem B we obtain (2.2) reversed. □

Remark 2.2. In this paper we will use Lemma 2.1 only in a special case when all weights are equal to 1. Then for $r < s < 0$ or $0 < r < s$, (2.2) becomes

$$\left(\sum_{i=1}^{n} x_i^s\right)^{\frac{1}{s}} \leq \left(\sum_{i=1}^{n} x_i^r\right)^{\frac{1}{r}} \tag{2.3}$$

and for $r < 0 < s$,

$$\left(\sum_{i=1}^{n} x_i^s\right)^{\frac{1}{s}} \geq \left(\sum_{i=1}^{n} x_i^r\right)^{\frac{1}{r}}.$$

In the rest of the paper we denote

$$C_p = \begin{cases} 2^{-\frac{1}{p}}, & p \leq -1 \text{ or } p \geq 1; \\ 2, & -1 < p < 0; \\ 2^{-1}, & 0 < p < 1; \end{cases} \qquad \widetilde{C}_p = \begin{cases} 2, & p \leq -1; \\ 2^{-\frac{1}{p}}, & -1 < p < 1, \ p \neq 0; \\ 2^{-1}, & p \geq 1. \end{cases}.$$

Lemma 2.3. *Let $f : [a,b] \to \mathbb{R}$, $a < b$. If $|f|^p$ is convex on $[a,b]$ for some $p > 0$, then*

$$\left|f\left(\frac{a+b}{2}\right)\right| \leq (b-a)^{-\frac{1}{p}} \|f\|_p \leq \left(\frac{|f(a)|^p + |f(b)|^p}{2}\right)^{\frac{1}{p}} \leq C_p \left(|f(a)| + |f(b)|\right),$$

and if $|f|^p$ is concave on $[a,b]$, then

$$\widetilde{C}_p \left(|f(a)| + |f(b)|\right) \leq \left(\frac{|f(a)|^p + |f(b)|^p}{2}\right)^{\frac{1}{p}} \leq (b-a)^{-\frac{1}{p}} \|f\|_p \leq \left|f\left(\frac{a+b}{2}\right)\right|.$$

Proof. Suppose first that $|f|^p$ is convex on $[a,b]$ for some $p > 0$. We have

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}} \left(\frac{1}{b-a}\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

From (1.2) we obtain

$$\left|f\left(\frac{a+b}{2}\right)\right|^p \leq \frac{1}{b-a}\int_a^b |f(x)|^p \, dx \leq \frac{|f(a)|^p + |f(b)|^p}{2}, \tag{2.4}$$

hence

$$\left|f\left(\frac{a+b}{2}\right)\right| \leq (b-a)^{-\frac{1}{p}} \|f\|_p \leq \left(\frac{|f(a)|^p + |f(b)|^p}{2}\right)^{\frac{1}{p}}.$$

Now we must consider two cases. If $p \geq 1$ we can use (2.3) to obtain

$$\left(|f(a)|^p + |f(b)|^p\right)^{\frac{1}{p}} \leq |f(a)| + |f(b)|,$$

hence

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \le C_p\left(|f(a)|+|f(b)|\right), \tag{2.5}$$

where $C_p = 2^{-\frac{1}{p}}$.

In the other case, when $0<p<1$, from (1.3) we have

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \le \frac{|f(a)|+|f(b)|}{2},$$

so again we obtain (2.5), where $C_p = 2^{-1}$. This completes the proof for $|f|^p$ convex.

Suppose now that $|f|^p$ is concave on $[a,b]$ for some $p>0$. In that case $-|f|^p$ is convex on $[a,b]$, hence (1.2) implies

$$\frac{|f(a)|^p+|f(b)|^p}{2} \le \frac{1}{b-a}\int_a^b |f(x)|^p\,dx \le \left|f\left(\frac{a+b}{2}\right)\right|^p.$$

If $p\ge 1$ from (1.3) we obtain

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \ge \frac{|f(a)|+|f(b)|}{2},$$

hence

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \ge \widetilde{C}_p\left(|f(a)|+|f(b)|\right),$$

where $\widetilde{C}_p = 2^{-1}$.

In the other case, when $0<p<1$, from (2.3) we have

$$\left(|f(a)|^p+|f(b)|^p\right)^{\frac{1}{p}} \ge |f(a)|+|f(b)|,$$

hence

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \ge \widetilde{C}_p\left(|f(a)|+|f(b)|\right),$$

where $\widetilde{C}_p = 2^{-\frac{1}{p}}$. This completes the proof. □

Lemma 2.4. *Let $f:[a,b]\to\mathbb{R}\setminus\{0\}$, $a<b$. If $|f|^p$ is convex on $[a,b]$ for some $p<0$, then*

$$C_p\frac{|f(a)f(b)|}{|f(a)|+|f(b)|} \le \left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \le (b-a)^{-\frac{1}{p}}\|f\|_p \le \left|f\left(\frac{a+b}{2}\right)\right|$$

and if $|f|^p$ is concave on $[a,b]$, then

$$\left|f\left(\frac{a+b}{2}\right)\right| \le (b-a)^{-\frac{1}{p}}\|f\|_p \le \left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \le \widetilde{C}_p\frac{|f(a)f(b)|}{|f(a)|+|f(b)|}.$$

Proof. Suppose that $|f|^p$ is convex on $[a,b]$ for some $p<0$. From (2.4), using the fact that $p<0$, we obtain

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \le (b-a)^{-\frac{1}{p}}\|f\|_p \le \left|f\left(\frac{a+b}{2}\right)\right|.$$

Again we consider two cases. If $-1<p<0$, then from (1.3) we have

$$\left(\frac{|f(a)|^{-1}+|f(b)|^{-1}}{2}\right)^{-1} \le \left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}},$$

hence

$$C_p \frac{|f(a) f(b)|}{|f(a)| + |f(b)|} \leq \left(\frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}},$$

where $C_p = 2$.

In the other case, when $p \leq -1$, from (2.3) we have

$$\left(|f(a)|^{-1} + |f(b)|^{-1} \right)^{-1} \leq \left(|f(a)|^p + |f(b)|^p \right)^{\frac{1}{p}},$$

hence

$$C_p \frac{|f(a) f(b)|}{|f(a)| + |f(b)|} \leq \left(\frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}},$$

where $C_p = 2^{-\frac{1}{p}}$.

In the other case, when $|f|^p$ is concave on $[a,b]$ for some $p < 0$, the proof is similar. □

Theorem 2.5. *Let $p, q > 0$ and let $f, g : [a,b] \to \mathbb{R}$, $a < b$, be such that*

$$m(|g(a)| + |g(b)|) \leq |f(a)| + |f(b)| \leq M(|g(a)| + |g(b)|) \tag{2.6}$$

for some $0 < m \leq M$.
If $|f|^p$ and $|g|^q$ are convex on $[a,b]$, then

$$\|f\|_p + \|g\|_q \leq \left[\frac{M}{M+1} C_p (b-a)^{\frac{1}{p}} + \frac{1}{m+1} C_q (b-a)^{\frac{1}{q}} \right] K(f,g), \tag{2.7}$$

where

$$K(f,g) = |f(a)| + |f(b)| + |g(a)| + |g(b)|.$$

If $|f|^p$ and $|g|^q$ are concave on $[a,b]$, then

$$\|f\|_p + \|g\|_q \geq \left[\frac{m}{m+1} \widetilde{C}_p (b-a)^{\frac{1}{p}} + \frac{1}{M+1} \widetilde{C}_q (b-a)^{\frac{1}{q}} \right] K(f,g). \tag{2.8}$$

Proof. Suppose that $|f|^p$ and $|g|^q$ are convex on $[a,b]$ for some fixed $p, q > 0$. From Lemma 2.3 we have that

$$\begin{aligned} &\|f\|_p + \|g\|_q \\ &\leq \left(\frac{b-a}{2} \right)^{\frac{1}{p}} \left(|f(a)|^p + |f(b)|^p \right)^{\frac{1}{p}} + \left(\frac{b-a}{2} \right)^{\frac{1}{q}} \left(|g(a)|^q + |g(b)|^q \right)^{\frac{1}{q}} \\ &\leq C_p (b-a)^{\frac{1}{p}} (|f(a)| + |f(b)|) + C_q (b-a)^{\frac{1}{q}} (|g(a)| + |g(b)|). \end{aligned} \tag{2.9}$$

Using (2.6) we can write

$$|f(a)| + |f(b)| \leq M(|f(a)| + |f(b)| + |g(a)| + |g(b)|) - M(|f(a)| + |f(b)|),$$

i.e.,

$$|f(a)| + |f(b)| \leq \frac{M}{M+1} (|f(a)| + |f(b)| + |g(a)| + |g(b)|) = \frac{M}{M+1} K(f,g), \tag{2.10}$$

and analogously

$$|g(a)| + |g(b)| \leq \frac{1}{m+1} K(f,g). \tag{2.11}$$

Combining (2.10) and (2.11) with (2.9) we obtain (2.7).

Suppose now that $|f|^p$ and $|g|^q$ are concave on $[a,b]$ for some fixed $p, q > 0$. From Lemma 2.3 we have that

$$\|f\|_p + \|g\|_q \geq \widetilde{C}_p (b-a)^{\frac{1}{p}} (|f(a)| + |f(b)|) + \widetilde{C}_q (b-a)^{\frac{1}{q}} (|g(a)| + |g(b)|).$$

Using again (2.6) we can write

$$|f(a)|+|f(b)| \geq m(|f(a)|+|f(b)|+|g(a)|+|g(b)|)-m(|f(a)|+|f(b)|),$$

i.e.,

$$|f(a)|+|f(b)| \geq \frac{m}{m+1} K(f,g),$$

and analogously

$$|g(a)|+|g(b)| \geq \frac{1}{M+1} K(f,g),$$

from which (2.8) easily follows. □

Remark 2.6. A similar type of condition as in (2.6) was used in [1, Theorem 1.1] where a variant of the reversed Minkowski's integral inequality for $p>1$ was proved.

Theorem 2.7. *Let $p,q<0$ and let $f,g:[a,b]\to\mathbb{R}\setminus\{0\}$, $a<b$, be such that*

$$m\frac{|g(a)\,g(b)|}{|g(a)|+|g(b)|} \leq \frac{|f(a)\,f(b)|}{|f(a)|+|f(b)|} \leq M\frac{|g(a)\,g(b)|}{|g(a)|+|g(b)|}$$

for some $0<m\leq M$.
If $|f|^p$ and $|g|^q$ are concave on $[a,b]$, then

$$\|f\|_p+\|g\|_q \leq \left[\frac{M}{M+1}\widetilde{C}_p(b-a)^{\frac{1}{p}}+\frac{1}{m+1}\widetilde{C}_q(b-a)^{\frac{1}{q}}\right]H(f,g),$$

where

$$H(f,g)=\frac{|f(a)\,f(b)|}{|f(a)|+|f(b)|}+\frac{|g(a)\,g(b)|}{|g(a)|+|g(b)|}.$$

If $|f|^p$ and $|g|^q$ are convex on $[a,b]$, then

$$\|f\|_p+\|g\|_q \geq \left[\frac{m}{m+1}C_p(b-a)^{\frac{1}{p}}+\frac{1}{M+1}C_q(b-a)^{\frac{1}{q}}\right]H(f,g).$$

Proof. Similar to that of Theorem 2.5. □

Theorem 2.8. *Let $f,g:[a,b]\to\mathbb{R}$, $a<b$, be such that $|f|^p$ and $|g|^q$ are convex on $[a,b]$ for some fixed $p,q>1$, where $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \leq \frac{b-a}{2}\left(|f(a)|^p+|f(b)|^p\right)^{\frac{1}{p}}\left(|g(a)|^q+|g(b)|^q\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2}\left[M(f,g)+N(f,g)\right],$$

where

$$M(f,g)=|f(a)|\,|g(a)|+|f(b)|\,|g(b)|, \qquad N(f,g)=|f(a)|\,|g(b)|+|f(b)|\,|g(a)|.$$

Proof. First note that since $|f|^p$ and $|g|^q$ are convex on $[a,b]$ we have $f\in L^p([a,b])$ and $g\in L^q([a,b])$, and since $\frac{1}{p}+\frac{1}{q}=1$ we know that $fg\in L^1([a,b])$, that is, fg is integrable on $[a,b]$. Using Hölder's integral inequality (1.1) we obtain

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \leq \int_a^b |f(x)\,g(x)|\,dx \leq \|f\|_p\,\|g\|_q.$$

From Lemma 2.4 we have that

$$\|f\|_p \leq \left(\frac{b-a}{2}\right)^{\frac{1}{p}}\left(|f(a)|^p+|f(b)|^p\right)^{\frac{1}{p}} \leq \left(\frac{b-a}{2}\right)^{\frac{1}{p}}\left(|f(a)|+|f(b)|\right)$$

and

$$\|g\|_q \leq \left(\frac{b-a}{2}\right)^{\frac{1}{q}} \left(|g(a)|^q + |g(b)|^q\right)^{\frac{1}{q}} \leq \left(\frac{b-a}{2}\right)^{\frac{1}{q}} \left(|g(a)| + |g(b)|\right),$$

hence

$$\begin{aligned} \left| \int_a^b f(x)\, g(x)\, dx \right| &\leq \frac{b-a}{2} \left(|f(a)|^p + |f(b)|^p\right)^{\frac{1}{p}} \left(|g(a)|^q + |g(b)|^q\right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2} \left(|f(a)| + |f(b)|\right) \left(|g(a)| + |g(b)|\right) \\ &= \frac{b-a}{2} \left[M(f,g) + N(f,g)\right]. \end{aligned}$$

□

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