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SOME SUBORDINATION CRITERIA CONCERNING THE SĂLĂGEAN OPERATOR

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ABSTRACT. Applying Sălăgean operator, for the class $\mathcal{A}$ of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ which are normalized by $f(0) = f'(0) - 1 = 0$, the generalization of an analytic function to discuss the starlikeness is considered. Furthermore, from the subordination criteria for Janowski functions generalized by some complex parameters, some interesting subordination criteria for $f(z) \in \mathcal{A}$ are given.

Key words and phrases: Janowski function, Starlike, Convex, Univalent, Subordination.

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1. INTRODUCTION, DEFINITION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.2}$$

which are analytic in $\mathbb{U}$. If $p(z) \in \mathcal{P}$ satisfies $\operatorname{Re}\big(p(z)\big) > 0 \quad (z \in \mathbb{U})$, then we say that $p(z)$ is the Carathéodory function (cf. [1]).

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By the familiar principle of differential subordination between analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ satisfying the following conditions:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(w(z)\big) \qquad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

In particular, if $g(z)$ is univalent in $\mathbb{U}$, then it is known that

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For the function $p(z) \in \mathcal{P}$, we introduce the following function

$$p(z) = \frac{1 + Az}{1 + Bz} \qquad (-1 \leqq B < A \leqq 1) \tag{1.3}$$

which has been investigated by Janowski [3]. Thus, the function $p(z)$ given by (1.3) is said to be the Janowski function. And, as a generalization of the Janowski function, Kuroki, Owa and Srivastava [2] have discussed the function

$$p(z) = \frac{1 + Az}{1 + Bz}$$

for some complex parameters A and B which satisfy one of following conditions

$$\begin{cases} (i)\ |A| \leqq 1,\ |B| < 1,\ A \neq B, \text{ and } \operatorname{Re}\big(1 - A\overline{B}\big) \geqq |A - B| \\ (ii)\ |A| \leqq 1,\ |B| = 1,\ A \neq B, \text{ and } 1 - A\overline{B} > 0. \end{cases}$$

Here, for some complex numbers A and B which satisfy condition (i), the function $p(z)$ is analytic and univalent in $\mathbb{U}$ and $p(z)$ maps the open unit disk $\mathbb{U}$ onto the open disk given by

$$\left| p(z) - \frac{1 - A\overline{B}}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2}. \tag{1.4}$$

Thus, it is clear that

$$\operatorname{Re}\big(p(z)\big) > \frac{\operatorname{Re}(1 - A\overline{B}) - |A - B|}{1 - |B|^2} \geqq 0 \qquad (z \in \mathbb{U}). \tag{1.5}$$

Also, for some complex numbers A and B which satisfy condition (ii), the function $p(z)$ is analytic and univalent in $\mathbb{U}$ and the domain $p(\mathbb{U})$ is the right half-plane satisfying

$$\operatorname{Re}\big(p(z)\big) > \frac{1 - |A|^2}{2(1 - A\overline{B})} \geqq 0. \tag{1.6}$$

Hence, we see that the generalized Janowski function maps the open unit disk $\mathbb{U}$ onto some domain which is on the right half-plane.

Remark 1. For the function

$$p(z) = \frac{1 + Az}{1 + Bz}$$

defined with the condition (i), the inequalities (1.4) and (1.5) give us that

$$p(z) \neq 0 \quad \text{namely,} \quad 1 + Az \neq 0 \qquad (z \in \mathbb{U}).$$

Since, after a simple calculation, we see the condition $|A| \leqq 1$, we can omit the condition $|A| \leqq 1$ in (i).
Hence, the condition (i) is newly defined by the following conditions

$$|B| < 1,\ A \neq B, \quad \text{and} \quad \operatorname{Re}\big(1 - A\overline{B}\big) \geqq |A - B|. \tag{1.7}$$

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.8}$$

for some α $(0 \leqq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are starlike of order α in $\mathbb{U}$.

Similarly, if $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.9}$$

for some α $(0 \leqq \alpha < 1)$, then $f(z)$ is said to be convex of order α in $\mathbb{U}$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are convex of order α in $\mathbb{U}$.

As usual, in the present investigation, we write

$$\mathcal{S}^*(0) \equiv \mathcal{S}^* \quad \text{and} \quad \mathcal{K}(0) \equiv \mathcal{K}.$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [7].

We define the following differential operator due to Sălăgean [8].

For a function $f(z)$ and $j = 1, 2, 3, \dots,$

$$D^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.10}$$

$$D^1 f(z) = Df(z) = zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n, \tag{1.11}$$

$$D^j f(z) = D\big(D^{j-1} f(z)\big) = z + \sum_{n=2}^{\infty} n^j a_n z^n. \tag{1.12}$$

Also, we consider the following differential operator

$$D^{-1} f(z) = \int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta = z + \sum_{n=2}^{\infty} n^{-1} a_n z^n, \tag{1.13}$$

$$D^{-j} f(z) = D^{-1}\big(D^{-(j-1)} f(z)\big) = z + \sum_{n=2}^{\infty} n^{-j} a_n z^n \tag{1.14}$$

for any negative integers.

Then, for $f(z) \in \mathcal{A}$ given by (1.1), we know that

$$D^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n \quad (j = 0,\ \pm 1,\ \pm 2,\ \dots). \tag{1.15}$$

We consider the subclass $\mathcal{S}_j^k(\alpha)$ as follows:

$$\mathcal{S}_j^k(\alpha) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(\frac{D^k f(z)}{D^j f(z)}\right) > \alpha \quad (z \in \mathbb{U}\,;\ 0 \leqq \alpha < 1) \right\}.$$

In particular, putting $k = j+1$, we also define $\mathcal{S}_j^{j+1}(\alpha)$ by

$$\mathcal{S}_j^{j+1}(\alpha) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(\frac{D^{j+1}f(z)}{D^j f(z)}\right) > \alpha \quad (z \in \mathbb{U}\,;\, 0 \leqq \alpha < 1) \right\}.$$

Remark 2. Noting

$$\frac{D^1 f(z)}{D^0 f(z)} = \frac{zf'(z)}{f(z)}, \quad \frac{D^2 f(z)}{D^1 f(z)} = \frac{z\big(zf'(z)\big)'}{zf'(z)} = 1 + \frac{zf''(z)}{f'(z)},$$

we see that

$$\mathcal{S}_0^1(\alpha) \equiv \mathcal{S}^*(\alpha), \quad \mathcal{S}_1^2(\alpha) \equiv \mathcal{K}(\alpha) \qquad (0 \leqq \alpha < 1).$$

Furthermore, by applying subordination, we consider the following subclass

$$\mathcal{P}_j^k(A,B) = \left\{ f(z) \in \mathcal{A} : \frac{D^k f(z)}{D^j f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}\,;\, A \neq B,\, |B| \leqq 1) \right\}.$$

In particular, putting $k = j+1$, we also define

$$\mathcal{P}_j^{j+1}(A,B) = \left\{ f(z) \in \mathcal{A} : \frac{D^{j+1} f(z)}{D^j f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U};\, A \neq B,\, |B| \leqq 1) \right\}.$$

Remark 3. Noting

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad \iff \quad \operatorname{Re}\left(\frac{D^k f(z)}{D^j f(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1),$$

we see that

$$\mathcal{P}_0^1(1-2\alpha,-1) \equiv \mathcal{S}^*(\alpha), \quad \mathcal{P}_1^2(1-2\alpha,-1) \equiv \mathcal{K}(\alpha) \qquad (0 \leqq \alpha < 1).$$

In our investigation here, we need the following lemma concerning the differential subordination given by Miller and Mocanu [5] (see also [6, p. 132]).

Lemma 1.1. *Let the function $q(z)$ be analytic and univalent in $\mathbb{U}$. Also let $\phi(\omega)$ and $\psi(\omega)$ be analytic in a domain $\mathcal{C}$ containing $q(\mathbb{U})$, with*

$$\psi(\omega) \neq 0 \qquad \big(\omega \in q(\mathbb{U}) \subset \mathcal{C}\big).$$

Set

$$Q(z) = zq'(z)\psi\big(q(z)\big) \quad \textit{and} \quad h(z) = \phi\big(q(z)\big) + Q(z),$$

and suppose that

(i) $$Q(z)\ is\ starlike\ and\ univalent\ in\ \mathbb{U};$$

and

(ii) $$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\phi'\big(q(z)\big)}{\psi\big(q(z)\big)} + \frac{zQ'(z)}{Q(z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

If $p(z)$ is analytic in $\mathbb{U}$, with

$$p(0) = q(0) \quad \textit{and} \quad p(\mathbb{U}) \subset \mathcal{C},$$

and

$$\phi\big(p(z)\big) + zp'(z)\psi\big(p(z)\big) \prec \phi\big(q(z)\big) + zq'(z)\psi\big(q(z)\big) =: h(z) \qquad (z \in \mathbb{U}),$$

then

$$p(z) \prec q(z) \qquad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant of this subordination.

By making use of Lemma 1.1, Kuroki, Owa and Srivastava [2] have investigated some subordination criteria for the generalized Janowski functions and deduced the following lemma.

Lemma 1.2. *Let the function* $f(z) \in \mathcal{A}$ *be chosen so that* $\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$. *Also, let* α $(\alpha \neq 0)$, β $(-1 \leqq \beta \leqq 1)$, *and some complex parameters* A *and* B *satisfy one of following conditions:*

(i) $|B| < 1$, $A \neq B$, *and* $\mathrm{Re}(1 - A\overline{B}) \geqq |A - B|$ *be such that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\mathrm{Re}(1-A\overline{B}) - |A-B|\}}{1-|B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

(ii) $|B| = 1$, $|A| \leqq 1$, $A \neq B$, *and* $1 - A\overline{B} > 0$ *be such that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\overline{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

If

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec h(z) \qquad (z \in \mathbb{U}), \tag{1.16}$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta-1}\left\{(1-\alpha)\frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^2 + \alpha(A-B)z}{(1+Bz)^2}\right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

2. SUBORDINATIONS FOR THE CLASS DEFINED BY THE SĂLĂGEAN OPERATOR

First of all, by applying the Sălăgean operator for $f(z) \in \mathcal{A}$, we consider the following subordination criterion in the class $\mathcal{P}_j^k(A,B)$ for some complex parameters A and B.

Theorem 2.1. *Let the function* $f(z) \in \mathcal{A}$ *be chosen so that* $\frac{D^j f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$. *Also, let* α $(\alpha \neq 0)$, β $(-1 \leqq \beta \leqq 1)$, *and some complex parameters* A *and* B *satisfy one of following conditions:*

(i) $|B| < 1$, $A \neq B$, *and* $\mathrm{Re}(1 - A\overline{B}) \geqq |A - B|$ *be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\mathrm{Re}(1-A\overline{B}) - |A-B|\}}{1-|B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

(ii) $|B| = 1$, $|A| \leqq 1$, $A \neq B$, *and* $1 - A\overline{B} > 0$ *be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\overline{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

If

$$\left(\frac{D^k f(z)}{D^j f(z)}\right)^{\beta}\left\{(1-\alpha) + \alpha\left(\frac{D^k f(z)}{D^j f(z)} + \frac{D^{k+1} f(z)}{D^k f(z)} - \frac{D^{j+1} f(z)}{D^j f(z)}\right)\right\} \prec h(z), \tag{2.1}$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta-1}\left\{(1-\alpha)\frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^2 + \alpha(A-B)z}{(1+Bz)^2}\right\},$$

then

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Proof. If we define the function $p(z)$ by

$$p(z) = \frac{D^k f(z)}{D^j f(z)} \qquad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Further, since

$$zp'(z) = \left(\frac{D^k f(z)}{D^j f(z)}\right)\left(\frac{D^{k+1} f(z)}{D^k f(z)} - \frac{D^{j+1} f(z)}{D^j f(z)}\right),$$

the condition (2.1) can be written as follows:

$$\big\{p(z)\big\}^{\beta}\big\{(1-\alpha) + \alpha p(z)\big\} + \alpha z p'(z)\big\{p(z)\big\}^{\beta-1} \prec h(z) \qquad (z \in \mathbb{U}).$$

We also set

$$q(z) = \frac{1+Az}{1+Bz}, \quad \phi(z) = z^{\beta}(1-\alpha+\alpha z), \quad \text{and} \quad \psi(z) = \alpha z^{\beta-1}$$

for $z \in \mathbb{U}$. Then, it is clear that the function $q(z)$ is analytic and univalent in $\mathbb{U}$ and has a positive real part in $\mathbb{U}$ for the conditions (i) and (ii).
Therefore, ϕ and ψ are analytic in a domain $\mathcal{C}$ containing $q(\mathbb{U})$, with

$$\psi(\omega) \neq 0 \qquad \big(\omega \in q(\mathbb{U}) \subset \mathcal{C}\big).$$

Also, for the function $Q(z)$ given by

$$Q(z) = zq'(z)\psi\big(q(z)\big) = \frac{\alpha(A-B)z(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}},$$

we obtain

$$\frac{zQ'(z)}{Q(z)} = \frac{1-\beta}{1+Az} + \frac{1+\beta}{1+Bz} - 1. \tag{2.2}$$

Furthermore, we have

$$\begin{aligned} h(z) &= \phi\big(q(z)\big) + Q(z) \\ &= \left(\frac{1+Az}{1+Bz}\right)^{\beta}\left(1-\alpha+\alpha\frac{1+Az}{1+Bz}\right) + \frac{\alpha(A-B)z(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}} \end{aligned}$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\beta(1-\alpha)}{\alpha} + (1+\beta)q(z) + \frac{zQ'(z)}{Q(z)}. \tag{2.3}$$

Hence,
(i) For the complex numbers A and B such that

$$|B| < 1,\ A \neq B, \quad \text{and} \quad \mathrm{Re}(1-A\overline{B}) \geqq |A-B|,$$

it follows from (2.2) and (2.3) that

$$\mathrm{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

and

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > \frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\operatorname{Re}(1-A\overline{B}) - |A-B|\}}{1-|B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0 \qquad (z \in \mathbb{U}).$$

(ii) For the complex numbers A and B such that

$$|B| = 1,\ |A| \leqq 1,\ A \neq B, \quad \text{and} \quad 1 - A\overline{B} > 0,$$

from (2.2) and (2.3), we get

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > \frac{1-\beta}{1+|A|} + \frac{1}{2}(1+\beta) - 1 = \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0,$$

and

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > \frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\overline{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0 \qquad (z \in \mathbb{U}).$$

Since all the conditions of Lemma 1.1 are satisfied, we conclude that

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

which completes the proof of Theorem 2.1. □

Remark 4. We know that a function $f(z)$ satisfying the conditions in Theorem 2.1 belongs to the class $\mathcal{P}_j^k(A,B)$.

Letting $k = j+1$ in Theorem 2.1, we obtain the following theorem.

Theorem 2.2. *Let the function $f(z) \in \mathcal{A}$ be chosen so that $\frac{D^j f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$. Also, let α $(\alpha \neq 0)$, β $(-1 \leqq \beta \leqq 1)$, and some complex parameters A and B satisfy one of following conditions*

(i) $|B| < 1$, $A \neq B$, *and* $\operatorname{Re}(1-A\overline{B}) \geqq |A-B|$ *be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\operatorname{Re}(1-A\overline{B}) - |A-B|\}}{1-|B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

(ii) $|B| = 1$, $|A| \leqq 1$, $A \neq B$, *and* $1 - A\overline{B} > 0$ *be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\overline{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

If

$$\left(\frac{D^{j+1}f(z)}{D^j f(z)}\right)^{\beta}\left(1-\alpha+\alpha\frac{D^{j+2}f(z)}{D^{j+1}f(z)}\right) \prec h(z), \tag{2.4}$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta-1}\left\{(1-\alpha)\frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^2 + \alpha(A-B)z}{(1+Bz)^2}\right\},$$

then

$$\frac{D^{j+1}f(z)}{D^j f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Remark 5. A function $f(z)$ satisfying the conditions in Theorem 2.2 belongs to the class $\mathcal{P}_j^{j+1}(A,B)$. Setting $j = 0$ in Theorem 2.2, we obtain Lemma 1.2 proven by Kuroki, Owa and Srivastava [2].

Also, if we assume that

$$\alpha = 1,\ \beta = A = 0, \quad \text{and} \quad B = \frac{1-\mu}{1+\mu}e^{i\theta} \quad (0 \leqq \mu < 1,\ 0 \leqq \theta < 2\pi),$$

Theorem 2.2 becomes the following corollary.

Corollary 2.3. *If* $f(z) \in \mathcal{A}$ $\left(\frac{D^j f(z)}{z} \neq 0 \text{ in } \mathbb{U}\right)$ *satisfies*

$$\frac{D^{j+2}f(z)}{D^{j+1}f(z)} \prec \frac{1+\mu-(1-\mu)e^{i\theta}z}{1+\mu+(1-\mu)e^{i\theta}z} \qquad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi)$$

for some μ $(0 \leqq \mu < 1)$*, then*

$$\frac{D^{j+1}f(z)}{D^j f(z)} \prec \frac{1+\mu}{1+\mu+(1-\mu)e^{i\theta}z} \qquad (z \in \mathbb{U}).$$

From the above corollary, we have

$$\operatorname{Re}\left(\frac{D^{j+2}f(z)}{D^{j+1}f(z)}\right) > \mu \quad \Longrightarrow \quad \operatorname{Re}\left(\frac{D^{j+1}f(z)}{D^j f(z)}\right) > \frac{1+\mu}{2} \qquad (z \in \mathbb{U};\ 0 \leqq \mu < 1).$$

In particular, making $j = 0$, we get

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \mu \quad \Longrightarrow \quad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{1+\mu}{2} \qquad (z \in \mathbb{U};\ 0 \leqq \mu < 1),$$

namely

$$f(z) \in \mathcal{K}(\mu) \quad \Longrightarrow \quad f(z) \in \mathcal{S}^*\left(\frac{1+\mu}{2}\right) \qquad (z \in \mathbb{U};\ 0 \leqq \mu < 1).$$

And, taking $\mu = 0$, we find that every convex function is starlike of order $\frac{1}{2}$. This fact is well-known as the Marx-Strohhäcker theorem in Univalent Function Theory (cf. [4, 9]).

3. Subordination Criteria for Other Analytic Functions

In this section, by making use of Lemma 1.1, we consider some subordination criteria concerning the analytic function $\frac{D^j f(z)}{z}$ for $f(z) \in \mathcal{A}$.

Theorem 3.1. *Let* α $(\alpha \neq 0)$, β $(-1 \leqq \beta \leqq 1)$*, and some complex parameters* A *and* B *which satisfy one of following conditions*

(i) $|B| < 1$, $A \neq B$*, and* $\operatorname{Re}(1 - A\overline{B}) \geqq |A - B|$ *be so that*

$$\frac{\beta}{\alpha} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

(ii) $|B| = 1$, $|A| \leqq 1$, $A \neq B$*, and* $1 - A\overline{B} > 0$ *be so that*

$$\frac{\beta}{\alpha} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

If $f(z) \in \mathcal{A}$ *satisfies*

$$\left(\frac{D^j f(z)}{z}\right)^{\beta}\left(1 - \alpha + \alpha\frac{D^{j+1}f(z)}{D^j f(z)}\right) \prec h(z), \tag{3.1}$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta} + \frac{\alpha(A-B)z(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}},$$

then

$$\frac{D^j f(z)}{z} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Proof. If we define the function $p(z)$ by

$$p(z) = \frac{D^j f(z)}{z} \qquad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and the condition (3.1) can be written as follows:

$$\{p(z)\}^{\beta} + \alpha z p'(z)\{p(z)\}^{\beta-1} \prec h(z) \qquad (z \in \mathbb{U}).$$

We also set

$$q(z) = \frac{1+Az}{1+Bz}, \quad \phi(z) = z^{\beta}, \quad \text{and} \quad \psi(z) = \alpha z^{\beta-1}$$

for $z \in \mathbb{U}$. Then, the function $q(z)$ is analytic and univalent in $\mathbb{U}$ and satisfies

$$\mathrm{Re}\big(q(z)\big) > 0 \qquad (z \in \mathbb{U})$$

for the condition (i) and (ii).

Thus, the functions ϕ and ψ satisfy the conditions required by Lemma 1.1.

Further, for the functions $Q(z)$ and $h(z)$ given by

$$Q(z) = zq'(z)\psi\big(q(z)\big) \quad \text{and} \quad h(z) = \phi\big(q(z)\big) + Q(z),$$

we have

$$\frac{zQ'(z)}{Q(z)} = \frac{1-\beta}{1+Az} + \frac{1+\beta}{1+Bz} - 1 \quad \text{and} \quad \frac{zh'(z)}{Q(z)} = \frac{\beta}{\alpha} + \frac{zQ'(z)}{Q(z)}.$$

Then, similarly to the proof of Theorem 2.1, we see that

$$\mathrm{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > 0 \quad \text{and} \quad \mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0 \qquad (z \in \mathbb{U})$$

for the conditions (i) and (ii).

Thus, by applying Lemma 1.1, we conclude that $p(z) \prec q(z) \quad (z \in \mathbb{U})$.
The proof of the theorem is completed. □

Letting $j = 0$ in Theorem 3.1, we obtain the following theorem.

Theorem 3.2. *Let α $(\alpha \neq 0)$, β $(-1 \leqq \beta \leqq 1)$, and some complex parameters A and B satisfy one of following conditions:*

(i) $|B| < 1$, $A \neq B$, *and* $\mathrm{Re}(1 - A\overline{B}) \geqq |A - B|$ *be so that*

$$\frac{\beta}{\alpha} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

(ii) $|B| = 1$, $|A| \leqq 1$, $A \neq B$, *and* $1 - A\overline{B} > 0$ *be so that*

$$\frac{\beta}{\alpha} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

If $f(z) \in \mathcal{A}$ satisfies

$$\left(\frac{f(z)}{z}\right)^{\beta-1}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta} + \frac{\alpha(A-B)z(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}}, \tag{3.2}$$

then

$$\frac{f(z)}{z} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Also, taking

$$\alpha = 1,\ \beta = A = 0, \quad \text{and} \quad B = \frac{1-\nu}{\nu}e^{i\theta} \quad \left(\frac{1}{2} \leqq \nu < 1,\ 0 \leqq \theta < 2\pi\right)$$

in Theorem 3.2, we have

Corollary 3.3. *If $f(z) \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \qquad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi)$$

for some ν $\left(\frac{1}{2} \leqq \nu < 1\right)$, then

$$\frac{f(z)}{z} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \qquad (z \in \mathbb{U}).$$

Further, making

$$\alpha = \beta = 1,\ A = 0, \quad \text{and} \quad B = \frac{1-\nu}{\nu}e^{i\theta} \quad \left(\frac{1}{2} \leqq \nu < 1,\ 0 \leqq \theta < 2\pi\right)$$

in Theorem 3.2, we get

Corollary 3.4. *If $f(z) \in \mathcal{A}$ satisfies*

$$f'(z) \prec \left(\frac{\nu}{\nu + (1-\nu)e^{i\theta}z}\right)^2 \qquad (z \in \mathbb{U}\,;\ 0 \leqq \theta < 2\pi)$$

for some ν $\left(\frac{1}{2} \leqq \nu < 1\right)$, then

$$\frac{f(z)}{z} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \qquad (z \in \mathbb{U}).$$

The above corollaries give:

$$\text{(3.3)} \qquad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \nu \quad \Longrightarrow \quad \operatorname{Re}\left(\frac{f(z)}{z}\right) > \nu \qquad \left(z \in \mathbb{U}\,;\ \frac{1}{2} \leqq \nu < 1\right),$$

and

$$\text{(3.4)} \qquad \operatorname{Re}\sqrt{f'(z)} > \nu \quad \Longrightarrow \quad \operatorname{Re}\left(\frac{f(z)}{z}\right) > \nu \qquad \left(z \in \mathbb{U}\,;\ \frac{1}{2} \leqq \nu < 1\right).$$

Here, taking $\nu = \frac{1}{2}$, we find some results that are known as the Marx-Strohhäcker theorem in Univalent Function Theory (cf. [4], [9]).

Setting $j = 1$ in Theorem 3.1, we obtain the following theorem.

Theorem 3.5. *Let α $(\alpha \neq 0)$, β $(-1 \leqq \beta \leqq 1)$, and some complex parameters A and B satisfy one of following conditions*

(i) $|B| < 1$, $A \neq B$, *and* $\operatorname{Re}(1 - A\overline{B}) \geqq |A - B|$ *be so that*

$$\frac{\beta}{\alpha} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

(ii) $|B| = 1$, $|A| \leqq 1$, $A \neq B$, *and* $1 - A\overline{B} > 0$ *be so that*

$$\frac{\beta}{\alpha} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

If $f(z) \in \mathcal{A}$ *satisfies*

$$(f'(z))^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta} + \frac{\alpha(A-B)z(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}}, \tag{3.5}$$

then

$$f'(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Here, making

$$\alpha = 1,\ \beta = A = 0, \quad \text{and} \quad B = \frac{1-\nu}{\nu}e^{i\theta} \qquad \left(\frac{1}{2} \leqq \nu < 1,\ 0 \leqq \theta < 2\pi\right)$$

in Theorem 3.5, we have:

Corollary 3.6. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \qquad (z \in \mathbb{U}\,;\ 0 \leqq \theta < 2\pi)$$

for some ν $\left(\frac{1}{2} \leqq \nu < 1\right)$, *then*

$$f'(z) \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \qquad (z \in \mathbb{U}).$$

Also, from Corollary 3.6 we have:

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \nu \quad \Longrightarrow \quad \mathrm{Re}\,(f'(z)) > \nu \qquad \left(z \in \mathbb{U};\ \frac{1}{2} \leqq \nu < 1\right). \tag{3.6}$$

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