



SOME SUBORDINATION CRITERIA CONCERNING THE SĂLĂGEAN OPERATOR

KAZUO KUROKI AND SHIGEYOSHI OWA

Department of Mathematics

Kinki University

Higashi-Osaka, Osaka 577-8502, Japan

EMail: freedom@sakai.zaq.ne.jp owa@math.kindai.ac.jp

Received: 14 September, 2008

Accepted: 11 January, 2009

Communicated by: **H.M. Srivastava**

2000 AMS Sub. Class.: 26D15.

Key words: Janowski function, Starlike, Convex, Univalent, Subordination.

Abstract: Applying Sălăgean operator, for the class \mathcal{A} of analytic functions $f(z)$ in the open unit disk \mathbb{U} which are normalized by $f(0) = f'(0) - 1 = 0$, the generalization of an analytic function to discuss the starlikeness is considered. Furthermore, from the subordination criteria for Janowski functions generalized by some complex parameters, some interesting subordination criteria for $f(z) \in \mathcal{A}$ are given.

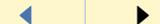
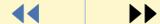
Subordination Criteria

Kazuo Kuroki and Shigeyoshi Owa

vol. 10, iss. 2, art. 36, 2009

[Title Page](#)

[Contents](#)



Page 1 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

Contents

- | | | |
|---|---|----|
| 1 | Introduction, Definition and Preliminaries | 3 |
| 2 | Subordinations for the Class Defined by the Sălăgean Operator | 11 |
| 3 | Subordination Criteria for Other Analytic Functions | 17 |



Subordination Criteria

Kazuo Kuroki and Shigeyoshi Owa

vol. 10, iss. 2, art. 36, 2009

[Title Page](#)

[Contents](#)



Page 2 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



1. Introduction, Definition and Preliminaries

Let \mathcal{A} denote the class of functions $f(z)$ of the form:

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also, let \mathcal{P} denote the class of functions $p(z)$ of the form:

$$(1.2) \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in \mathbb{U} . If $p(z) \in \mathcal{P}$ satisfies $\operatorname{Re}(p(z)) > 0$ ($z \in \mathbb{U}$), then we say that $p(z)$ is the Carathéodory function (cf. [1]).

By the familiar principle of differential subordination between analytic functions $f(z)$ and $g(z)$ in \mathbb{U} , we say that $f(z)$ is subordinate to $g(z)$ in \mathbb{U} if there exists an analytic function $w(z)$ satisfying the following conditions:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

Subordination Criteria

Kazuo Kuroki and Shigeyoshi Owa

vol. 10, iss. 2, art. 36, 2009

Title Page

Contents



Page 3 of 23

Go Back

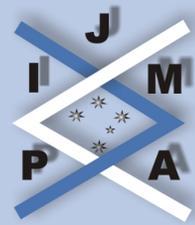
Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 4 of 23

Go Back

Full Screen

Close

In particular, if $g(z)$ is univalent in \mathbb{U} , then it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For the function $p(z) \in \mathcal{P}$, we introduce the following function

$$(1.3) \quad p(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1)$$

which has been investigated by Janowski [3]. Thus, the function $p(z)$ given by (1.3) is said to be the Janowski function. And, as a generalization of the Janowski function, Kuroki, Owa and Srivastava [2] have discussed the function

$$p(z) = \frac{1 + Az}{1 + Bz}$$

for some complex parameters A and B which satisfy one of following conditions

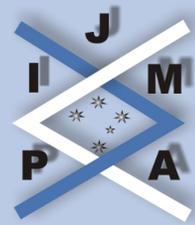
$$\left\{ \begin{array}{l} (i) \quad |A| \leq 1, \quad |B| < 1, \quad A \neq B, \quad \text{and} \quad \operatorname{Re}(1 - A\bar{B}) \geq |A - B| \\ (ii) \quad |A| \leq 1, \quad |B| = 1, \quad A \neq B, \quad \text{and} \quad 1 - A\bar{B} > 0. \end{array} \right.$$

Here, for some complex numbers A and B which satisfy condition (i), the function $p(z)$ is analytic and univalent in \mathbb{U} and $p(z)$ maps the open unit disk \mathbb{U} onto the open disk given by

$$(1.4) \quad \left| p(z) - \frac{1 - A\bar{B}}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2}.$$

Thus, it is clear that

$$(1.5) \quad \operatorname{Re}(p(z)) > \frac{\operatorname{Re}(1 - A\bar{B}) - |A - B|}{1 - |B|^2} \geq 0 \quad (z \in \mathbb{U}).$$



Title Page

Contents



Page 5 of 23

Go Back

Full Screen

Close

Also, for some complex numbers A and B which satisfy condition (ii), the function $p(z)$ is analytic and univalent in \mathbb{U} and the domain $p(\mathbb{U})$ is the right half-plane satisfying

$$(1.6) \quad \operatorname{Re}(p(z)) > \frac{1 - |A|^2}{2(1 - A\bar{B})} \geq 0.$$

Hence, we see that the generalized Janowski function maps the open unit disk \mathbb{U} onto some domain which is on the right half-plane.

Remark 1. For the function

$$p(z) = \frac{1 + Az}{1 + Bz}$$

defined with the condition (i), the inequalities (1.4) and (1.5) give us that

$$p(z) \neq 0 \quad \text{namely,} \quad 1 + Az \neq 0 \quad (z \in \mathbb{U}).$$

Since, after a simple calculation, we see the condition $|A| \leq 1$, we can omit the condition $|A| \leq 1$ in (i).

Hence, the condition (i) is newly defined by the following conditions

$$(1.7) \quad |B| < 1, \quad A \neq B, \quad \text{and} \quad \operatorname{Re}(1 - A\bar{B}) \geq |A - B|.$$

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in \mathbb{U} if it satisfies

$$(1.8) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some α ($0 \leq \alpha < 1$). We denote by $\mathcal{S}^*(\alpha)$ the subclass of \mathcal{A} consisting of all functions $f(z)$ which are starlike of order α in \mathbb{U} .



Title Page

Contents



Page 6 of 23

Go Back

Full Screen

Close

Similarly, if $f(z) \in \mathcal{A}$ satisfies the following inequality

$$(1.9) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some α ($0 \leq \alpha < 1$), then $f(z)$ is said to be convex of order α in \mathbb{U} . We denote by $\mathcal{K}(\alpha)$ the subclass of \mathcal{A} consisting of all functions $f(z)$ which are convex of order α in \mathbb{U} .

As usual, in the present investigation, we write

$$\mathcal{S}^*(0) \equiv \mathcal{S}^* \quad \text{and} \quad \mathcal{K}(0) \equiv \mathcal{K}.$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [7].

We define the following differential operator due to Sălăgean [8].

For a function $f(z)$ and $j = 1, 2, 3, \dots$,

$$(1.10) \quad D^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$(1.11) \quad D^1 f(z) = Df(z) = zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n,$$

$$(1.12) \quad D^j f(z) = D(D^{j-1} f(z)) = z + \sum_{n=2}^{\infty} n^j a_n z^n.$$

Also, we consider the following differential operator

$$(1.13) \quad D^{-1} f(z) = \int_0^z \frac{f(\zeta)}{\zeta} d\zeta = z + \sum_{n=2}^{\infty} n^{-1} a_n z^n,$$



$$(1.14) \quad D^{-j}f(z) = D^{-1}(D^{-(j-1)}f(z)) = z + \sum_{n=2}^{\infty} n^{-j} a_n z^n$$

for any negative integers.

Then, for $f(z) \in \mathcal{A}$ given by (1.1), we know that

$$(1.15) \quad D^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n \quad (j = 0, \pm 1, \pm 2, \dots).$$

We consider the subclass $\mathcal{S}_j^k(\alpha)$ as follows:

$$\mathcal{S}_j^k(\alpha) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \left(\frac{D^k f(z)}{D^j f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}.$$

In particular, putting $k = j + 1$, we also define $\mathcal{S}_j^{j+1}(\alpha)$ by

$$\mathcal{S}_j^{j+1}(\alpha) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \left(\frac{D^{j+1} f(z)}{D^j f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}.$$

Remark 2. Noting

$$\frac{D^1 f(z)}{D^0 f(z)} = \frac{z f'(z)}{f(z)}, \quad \frac{D^2 f(z)}{D^1 f(z)} = \frac{z(z f'(z))'}{z f'(z)} = 1 + \frac{z f''(z)}{f'(z)},$$

we see that

$$\mathcal{S}_0^1(\alpha) \equiv \mathcal{S}^*(\alpha), \quad \mathcal{S}_1^2(\alpha) \equiv \mathcal{K}(\alpha) \quad (0 \leq \alpha < 1).$$



[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 8 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

Furthermore, by applying subordination, we consider the following subclass

$$\mathcal{P}_j^k(A, B) = \left\{ f(z) \in \mathcal{A} : \frac{D^k f(z)}{D^j f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}; A \neq B, |B| \leq 1) \right\}.$$

In particular, putting $k = j + 1$, we also define

$$\mathcal{P}_j^{j+1}(A, B) = \left\{ f(z) \in \mathcal{A} : \frac{D^{j+1} f(z)}{D^j f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}; A \neq B, |B| \leq 1) \right\}.$$

Remark 3. Noting

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z} \iff \operatorname{Re} \left(\frac{D^k f(z)}{D^j f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1),$$

we see that

$$\mathcal{P}_0^1(1 - 2\alpha, -1) \equiv \mathcal{S}^*(\alpha), \quad \mathcal{P}_1^2(1 - 2\alpha, -1) \equiv \mathcal{K}(\alpha) \quad (0 \leq \alpha < 1).$$

In our investigation here, we need the following lemma concerning the differential subordination given by Miller and Mocanu [5] (see also [6, p. 132]).

Lemma 1.1. *Let the function $q(z)$ be analytic and univalent in \mathbb{U} . Also let $\phi(\omega)$ and $\psi(\omega)$ be analytic in a domain \mathcal{C} containing $q(\mathbb{U})$, with*

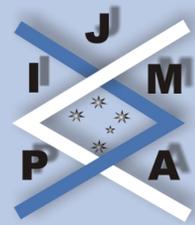
$$\psi(\omega) \neq 0 \quad (\omega \in q(\mathbb{U}) \subset \mathcal{C}).$$

Set

$$Q(z) = zq'(z)\psi(q(z)) \quad \text{and} \quad h(z) = \phi(q(z)) + Q(z),$$

and suppose that

(i) $Q(z)$ is starlike and univalent in \mathbb{U} ;



[Title Page](#)

[Contents](#)



Page 9 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

and

$$(ii) \quad \operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) = \operatorname{Re} \left(\frac{\phi'(q(z))}{\psi(q(z))} + \frac{zQ'(z)}{Q(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

If $p(z)$ is analytic in \mathbb{U} , with

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset \mathcal{C},$$

and

$$\phi(p(z)) + zp'(z)\psi(p(z)) \prec \phi(q(z)) + zq'(z)\psi(q(z)) =: h(z) \quad (z \in \mathbb{U}),$$

then

$$p(z) \prec q(z) \quad (z \in \mathbb{U})$$

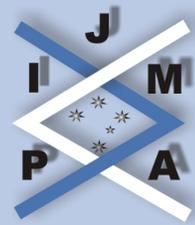
and $q(z)$ is the best dominant of this subordination.

By making use of Lemma 1.1, Kuroki, Owa and Srivastava [2] have investigated some subordination criteria for the generalized Janowski functions and deduced the following lemma.

Lemma 1.2. *Let the function $f(z) \in \mathcal{A}$ be chosen so that $\frac{f(z)}{z} \neq 0$ ($z \in \mathbb{U}$). Also, let α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and some complex parameters A and B satisfy one of following conditions:*

(i) $|B| < 1$, $A \neq B$, and $\operatorname{Re}(1 - A\bar{B}) \geq |A - B|$ be such that

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)\{\operatorname{Re}(1 - A\bar{B}) - |A - B|\}}{1 - |B|^2} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0,$$



(ii) $|B| = 1$, $|A| \leq 1$, $A \neq B$, and $1 - A\bar{B} > 0$ be such that

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\bar{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geq 0.$$

If

$$(1.16) \quad \left(\frac{zf'(z)}{f(z)}\right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)}\right) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta-1} \left\{ (1-\alpha) \frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^2 + \alpha(A-B)z}{(1+Bz)^2} \right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

Subordination Criteria

Kazuo Kuroki and Shigeyoshi Owa

vol. 10, iss. 2, art. 36, 2009

Title Page

Contents



Page 10 of 23

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)



Page 11 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

2. Subordinations for the Class Defined by the Sălăgean Operator

First of all, by applying the Sălăgean operator for $f(z) \in \mathcal{A}$, we consider the following subordination criterion in the class $\mathcal{P}_j^k(A, B)$ for some complex parameters A and B .

Theorem 2.1. *Let the function $f(z) \in \mathcal{A}$ be chosen so that $\frac{D^j f(z)}{z} \neq 0$ ($z \in \mathbb{U}$). Also, let α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and some complex parameters A and B satisfy one of following conditions:*

(i) $|B| < 1$, $A \neq B$, and $\operatorname{Re}(1 - A\bar{B}) \geq |A - B|$ be so that

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)\{\operatorname{Re}(1 - A\bar{B}) - |A - B|\}}{1 - |B|^2} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0,$$

(ii) $|B| = 1$, $|A| \leq 1$, $A \neq B$, and $1 - A\bar{B} > 0$ be so that

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)(1 - |A|^2)}{2(1 - A\bar{B})} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0.$$

If

$$(2.1) \quad \left(\frac{D^k f(z)}{D^j f(z)}\right)^\beta \left\{ (1 - \alpha) + \alpha \left(\frac{D^k f(z)}{D^j f(z)} + \frac{D^{k+1} f(z)}{D^k f(z)} - \frac{D^{j+1} f(z)}{D^j f(z)} \right) \right\} \prec h(z),$$

where

$$h(z) = \left(\frac{1 + Az}{1 + Bz}\right)^{\beta-1} \left\{ (1 - \alpha) \frac{1 + Az}{1 + Bz} + \frac{\alpha(1 + Az)^2 + \alpha(A - B)z}{(1 + Bz)^2} \right\},$$

then

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

Proof. If we define the function $p(z)$ by

$$p(z) = \frac{D^k f(z)}{D^j f(z)} \quad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in \mathbb{U} with $p(0) = 1$. Further, since

$$zp'(z) = \left(\frac{D^k f(z)}{D^j f(z)} \right) \left(\frac{D^{k+1} f(z)}{D^k f(z)} - \frac{D^{j+1} f(z)}{D^j f(z)} \right),$$

the condition (2.1) can be written as follows:

$$\{p(z)\}^\beta \{(1 - \alpha) + \alpha p(z)\} + \alpha zp'(z) \{p(z)\}^{\beta-1} \prec h(z) \quad (z \in \mathbb{U}).$$

We also set

$$q(z) = \frac{1 + Az}{1 + Bz}, \quad \phi(z) = z^\beta(1 - \alpha + \alpha z), \quad \text{and} \quad \psi(z) = \alpha z^{\beta-1}$$

for $z \in \mathbb{U}$. Then, it is clear that the function $q(z)$ is analytic and univalent in \mathbb{U} and has a positive real part in \mathbb{U} for the conditions (i) and (ii).

Therefore, ϕ and ψ are analytic in a domain \mathcal{C} containing $q(\mathbb{U})$, with

$$\psi(\omega) \neq 0 \quad (\omega \in q(\mathbb{U}) \subset \mathcal{C}).$$

Also, for the function $Q(z)$ given by

$$Q(z) = zq'(z)\psi(q(z)) = \frac{\alpha(A - B)z(1 + Az)^{\beta-1}}{(1 + Bz)^{\beta+1}},$$



Subordination Criteria

Kazuo Kuroki and Shigeyoshi Owa

vol. 10, iss. 2, art. 36, 2009

Title Page

Contents



Page 12 of 23

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-575b

© 2007 Victoria University. All rights reserved.



Title Page

Contents



Page 13 of 23

Go Back

Full Screen

Close

we obtain

$$(2.2) \quad \frac{zQ'(z)}{Q(z)} = \frac{1-\beta}{1+Az} + \frac{1+\beta}{1+Bz} - 1.$$

Furthermore, we have

$$\begin{aligned} h(z) &= \phi(q(z)) + Q(z) \\ &= \left(\frac{1+Az}{1+Bz}\right)^\beta \left(1 - \alpha + \alpha \frac{1+Az}{1+Bz}\right) + \frac{\alpha(A-B)z(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}} \end{aligned}$$

and

$$(2.3) \quad \frac{zh'(z)}{Q(z)} = \frac{\beta(1-\alpha)}{\alpha} + (1+\beta)q(z) + \frac{zQ'(z)}{Q(z)}.$$

Hence,

(i) For the complex numbers A and B such that

$$|B| < 1, \quad A \neq B, \quad \text{and} \quad \operatorname{Re}(1 - A\bar{B}) \geq |A - B|,$$

it follows from (2.2) and (2.3) that

$$\operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) > \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geq 0,$$

and

$$\begin{aligned} \operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) &> \frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\operatorname{Re}(1 - A\bar{B}) - |A - B|\}}{1 - |B|^2} \\ &\quad + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geq 0 \quad (z \in \mathbb{U}). \end{aligned}$$



[Title Page](#)

[Contents](#)



Page 14 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

(ii) For the complex numbers A and B such that

$$|B| = 1, |A| \leq 1, A \neq B, \quad \text{and} \quad 1 - A\bar{B} > 0,$$

from (2.2) and (2.3), we get

$$\operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) > \frac{1 - \beta}{1 + |A|} + \frac{1}{2}(1 + \beta) - 1 = \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0,$$

and

$$\operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) > \frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)(1 - |A|^2)}{2(1 - A\bar{B})} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0 \quad (z \in \mathbb{U}).$$

Since all the conditions of Lemma 1.1 are satisfied, we conclude that

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),$$

which completes the proof of Theorem 2.1. \square

Remark 4. We know that a function $f(z)$ satisfying the conditions in Theorem 2.1 belongs to the class $\mathcal{P}_j^k(A, B)$.

Letting $k = j + 1$ in Theorem 2.1, we obtain the following theorem.

Theorem 2.2. Let the function $f(z) \in \mathcal{A}$ be chosen so that $\frac{D^j f(z)}{z} \neq 0$ ($z \in \mathbb{U}$). Also, let α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and some complex parameters A and B satisfy one of following conditions

(i) $|B| < 1$, $A \neq B$, and $\operatorname{Re}(1 - A\bar{B}) \geq |A - B|$ be so that

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)\{\operatorname{Re}(1 - A\bar{B}) - |A - B|\}}{1 - |B|^2} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0,$$



[Title Page](#)

[Contents](#)



Page 15 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

(ii) $|B| = 1$, $|A| \leq 1$, $A \neq B$, and $1 - A\bar{B} > 0$ be so that

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)(1 - |A|^2)}{2(1 - A\bar{B})} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0.$$

If

$$(2.4) \quad \left(\frac{D^{j+1}f(z)}{D^j f(z)} \right)^\beta \left(1 - \alpha + \alpha \frac{D^{j+2}f(z)}{D^{j+1}f(z)} \right) \prec h(z),$$

where

$$h(z) = \left(\frac{1 + Az}{1 + Bz} \right)^{\beta-1} \left\{ (1 - \alpha) \frac{1 + Az}{1 + Bz} + \frac{\alpha(1 + Az)^2 + \alpha(A - B)z}{(1 + Bz)^2} \right\},$$

then

$$\frac{D^{j+1}f(z)}{D^j f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

Remark 5. A function $f(z)$ satisfying the conditions in Theorem 2.2 belongs to the class $\mathcal{P}_j^{j+1}(A, B)$. Setting $j = 0$ in Theorem 2.2, we obtain Lemma 1.2 proven by Kuroki, Owa and Srivastava [2].

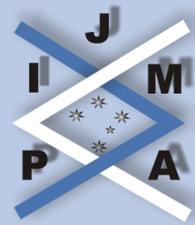
Also, if we assume that

$$\alpha = 1, \beta = A = 0, \quad \text{and} \quad B = \frac{1 - \mu}{1 + \mu} e^{i\theta} \quad (0 \leq \mu < 1, 0 \leq \theta < 2\pi),$$

Theorem 2.2 becomes the following corollary.

Corollary 2.3. If $f(z) \in \mathcal{A} \left(\frac{D^j f(z)}{z} \neq 0 \text{ in } \mathbb{U} \right)$ satisfies

$$\frac{D^{j+2}f(z)}{D^{j+1}f(z)} \prec \frac{1 + \mu - (1 - \mu)e^{i\theta}z}{1 + \mu + (1 - \mu)e^{i\theta}z} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi)$$



for some μ ($0 \leq \mu < 1$), then

$$\frac{D^{j+1}f(z)}{D^j f(z)} \prec \frac{1 + \mu}{1 + \mu + (1 - \mu)e^{i\theta}z} \quad (z \in \mathbb{U}).$$

From the above corollary, we have

$$\operatorname{Re} \left(\frac{D^{j+2}f(z)}{D^{j+1}f(z)} \right) > \mu \implies \operatorname{Re} \left(\frac{D^{j+1}f(z)}{D^j f(z)} \right) > \frac{1 + \mu}{2} \quad (z \in \mathbb{U}; 0 \leq \mu < 1).$$

In particular, making $j = 0$, we get

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \mu \implies \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \frac{1 + \mu}{2} \quad (z \in \mathbb{U}; 0 \leq \mu < 1),$$

namely

$$f(z) \in \mathcal{K}(\mu) \implies f(z) \in \mathcal{S}^* \left(\frac{1 + \mu}{2} \right) \quad (z \in \mathbb{U}; 0 \leq \mu < 1).$$

And, taking $\mu = 0$, we find that every convex function is starlike of order $\frac{1}{2}$. This fact is well-known as the Marx-Strohhäcker theorem in Univalent Function Theory (cf. [4, 9]).

Subordination Criteria

Kazuo Kuroki and Shigeyoshi Owa

vol. 10, iss. 2, art. 36, 2009

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 16 of 23

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents



Page 17 of 23

Go Back

Full Screen

Close

3. Subordination Criteria for Other Analytic Functions

In this section, by making use of Lemma 1.1, we consider some subordination criteria concerning the analytic function $\frac{D^j f(z)}{z}$ for $f(z) \in \mathcal{A}$.

Theorem 3.1. Let α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and some complex parameters A and B which satisfy one of following conditions

(i) $|B| < 1$, $A \neq B$, and $\operatorname{Re}(1 - A\bar{B}) \geq |A - B|$ be so that

$$\frac{\beta}{\alpha} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0,$$

(ii) $|B| = 1$, $|A| \leq 1$, $A \neq B$, and $1 - A\bar{B} > 0$ be so that

$$\frac{\beta}{\alpha} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0.$$

If $f(z) \in \mathcal{A}$ satisfies

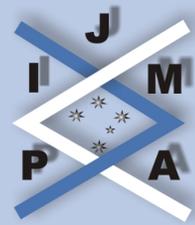
$$(3.1) \quad \left(\frac{D^j f(z)}{z} \right)^\beta \left(1 - \alpha + \alpha \frac{D^{j+1} f(z)}{D^j f(z)} \right) \prec h(z),$$

where

$$h(z) = \left(\frac{1 + Az}{1 + Bz} \right)^\beta + \frac{\alpha(A - B)z(1 + Az)^{\beta-1}}{(1 + Bz)^{\beta+1}},$$

then

$$\frac{D^j f(z)}{z} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$



[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 18 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

Proof. If we define the function $p(z)$ by

$$p(z) = \frac{D^j f(z)}{z} \quad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in \mathbb{U} with $p(0) = 1$ and the condition (3.1) can be written as follows:

$$\{p(z)\}^\beta + \alpha z p'(z) \{p(z)\}^{\beta-1} \prec h(z) \quad (z \in \mathbb{U}).$$

We also set

$$q(z) = \frac{1 + Az}{1 + Bz}, \quad \phi(z) = z^\beta, \quad \text{and} \quad \psi(z) = \alpha z^{\beta-1}$$

for $z \in \mathbb{U}$. Then, the function $q(z)$ is analytic and univalent in \mathbb{U} and satisfies

$$\operatorname{Re}(q(z)) > 0 \quad (z \in \mathbb{U})$$

for the condition (i) and (ii).

Thus, the functions ϕ and ψ satisfy the conditions required by Lemma 1.1.

Further, for the functions $Q(z)$ and $h(z)$ given by

$$Q(z) = zq'(z)\psi(q(z)) \quad \text{and} \quad h(z) = \phi(q(z)) + Q(z),$$

we have

$$\frac{zQ'(z)}{Q(z)} = \frac{1-\beta}{1+Az} + \frac{1+\beta}{1+Bz} - 1 \quad \text{and} \quad \frac{zh'(z)}{Q(z)} = \frac{\beta}{\alpha} + \frac{zQ'(z)}{Q(z)}.$$

Then, similarly to the proof of Theorem 2.1, we see that

$$\operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) > 0 \quad \text{and} \quad \operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) > 0 \quad (z \in \mathbb{U})$$



for the conditions (i) and (ii).

Thus, by applying Lemma 1.1, we conclude that $p(z) \prec q(z)$ ($z \in \mathbb{U}$).
The proof of the theorem is completed. \square

Letting $j = 0$ in Theorem 3.1, we obtain the following theorem.

Theorem 3.2. Let α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and some complex parameters A and B satisfy one of following conditions:

(i) $|B| < 1$, $A \neq B$, and $\operatorname{Re}(1 - A\bar{B}) \geq |A - B|$ be so that

$$\frac{\beta}{\alpha} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0,$$

(ii) $|B| = 1$, $|A| \leq 1$, $A \neq B$, and $1 - A\bar{B} > 0$ be so that

$$\frac{\beta}{\alpha} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0.$$

If $f(z) \in \mathcal{A}$ satisfies

$$(3.2) \quad \left(\frac{f(z)}{z}\right)^{\beta-1} \left\{ (1 - \alpha)\frac{f(z)}{z} + \alpha f'(z) \right\} \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta} + \frac{\alpha(A - B)z(1 + Az)^{\beta-1}}{(1 + Bz)^{\beta+1}},$$

then

$$\frac{f(z)}{z} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$



Title Page

Contents



Page 20 of 23

Go Back

Full Screen

Close

Also, taking

$$\alpha = 1, \beta = A = 0, \quad \text{and} \quad B = \frac{1-\nu}{\nu} e^{i\theta} \quad \left(\frac{1}{2} \leq \nu < 1, 0 \leq \theta < 2\pi \right)$$

in Theorem 3.2, we have

Corollary 3.3. *If $f(z) \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi)$$

for some ν ($\frac{1}{2} \leq \nu < 1$), then

$$\frac{f(z)}{z} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \quad (z \in \mathbb{U}).$$

Further, making

$$\alpha = \beta = 1, A = 0, \quad \text{and} \quad B = \frac{1-\nu}{\nu} e^{i\theta} \quad \left(\frac{1}{2} \leq \nu < 1, 0 \leq \theta < 2\pi \right)$$

in Theorem 3.2, we get

Corollary 3.4. *If $f(z) \in \mathcal{A}$ satisfies*

$$f'(z) \prec \left(\frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \right)^2 \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi)$$

for some ν ($\frac{1}{2} \leq \nu < 1$), then

$$\frac{f(z)}{z} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \quad (z \in \mathbb{U}).$$



Title Page

Contents



Page 21 of 23

Go Back

Full Screen

Close

The above corollaries give:

$$(3.3) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \nu \implies \operatorname{Re} \left(\frac{f(z)}{z} \right) > \nu \quad \left(z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right),$$

and

$$(3.4) \quad \operatorname{Re} \sqrt{f'(z)} > \nu \implies \operatorname{Re} \left(\frac{f(z)}{z} \right) > \nu \quad \left(z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right).$$

Here, taking $\nu = \frac{1}{2}$, we find some results that are known as the Marx-Strohhäcker theorem in Univalent Function Theory (cf. [4], [9]).

Setting $j = 1$ in Theorem 3.1, we obtain the following theorem.

Theorem 3.5. *Let α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and some complex parameters A and B satisfy one of following conditions*

(i) $|B| < 1$, $A \neq B$, and $\operatorname{Re}(1 - A\bar{B}) \geq |A - B|$ be so that

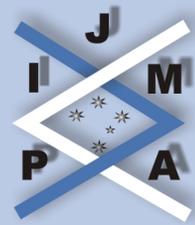
$$\frac{\beta}{\alpha} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0,$$

(ii) $|B| = 1$, $|A| \leq 1$, $A \neq B$, and $1 - A\bar{B} > 0$ be so that

$$\frac{\beta}{\alpha} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0.$$

If $f(z) \in \mathcal{A}$ satisfies

$$(3.5) \quad (f'(z))^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \prec \left(\frac{1 + Az}{1 + Bz} \right)^\beta + \frac{\alpha(A - B)z(1 + Az)^{\beta-1}}{(1 + Bz)^{\beta+1}},$$



then

$$f'(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

Here, making

$$\alpha = 1, \beta = A = 0, \quad \text{and} \quad B = \frac{1 - \nu}{\nu} e^{i\theta} \quad \left(\frac{1}{2} \leq \nu < 1, 0 \leq \theta < 2\pi \right)$$

in Theorem 3.5, we have:

Corollary 3.6. *If $f(z) \in \mathcal{A}$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\nu}{\nu + (1 - \nu)e^{i\theta}z} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi)$$

for some ν ($\frac{1}{2} \leq \nu < 1$), then

$$f'(z) \prec \frac{\nu}{\nu + (1 - \nu)e^{i\theta}z} \quad (z \in \mathbb{U}).$$

Also, from Corollary 3.6 we have:

$$(3.6) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \nu \implies \operatorname{Re} (f'(z)) > \nu \quad \left(z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right).$$

Subordination Criteria

Kazuo Kuroki and Shigeyoshi Owa

vol. 10, iss. 2, art. 36, 2009

Title Page

Contents



Page 22 of 23

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

References

- [1] P.L. DUREM, *Univalent Functions*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
- [2] K. KUROKI, S. OWA AND H.M. SRIVASTAVA, Some subordination criteria for analytic functions, *Bull. Soc. Sci. Lett. Lodz*, Vol., **52** (2007), 27–36.
- [3] W. JANOWSKI, Extremal problem for a family of functions with positive real part and for some related families, *Ann. Polon. Math.*, **23** (1970), 159–177.
- [4] A. MARX, Untersuchungen uber schlichte Abbildungen, *Math. Ann.*, **107** (1932/33), 40–67.
- [5] S.S. MILLER AND P.T. MOCANU, On some classes of first-order differential subordinations, *Michigan Math. J.*, **32** (1985), 185–195.
- [6] S.S. MILLER AND P.T. MOCANU, *Differential Subordinations*, Pure and Applied Mathematics **225**, Marcel Dekker, 2000.
- [7] M.S. ROBERTSON, On the theory of univalent functions, *Ann. Math.*, **37** (1936), 374–408.
- [8] G.S. SĂLĂGEAN, *Subclass of univalent functions*, Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1(Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, pp. 362–372.
- [9] E. STROHHÄCKER, Beitrage zur Theorie der schlichten Funktionen, *Math. Z.*, **37** (1933), 356–380.



Subordination Criteria

Kazuo Kuroki and Shigeyoshi Owa

vol. 10, iss. 2, art. 36, 2009

Title Page

Contents



Page 23 of 23

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756