ON THE INEQUALITY OF THE DIFFERENCE OF TWO INTEGRAL MEANS AND APPLICATIONS FOR PDFs

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Abstract: A new inequality is presented, which is used to obtain a complement of recently obtained inequality concerning the difference of two integral means. Some applications for pdfs are also given.

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1. Introduction

In 1938, Ostrowski proved the following inequality [5].

Theorem 1.1. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $|f'(x)| \leq M$ *for all* $x \in (a,b)$, *then,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) M, \tag{1.1}$$

for all $x \in [a,b]$. *The constant* $\frac{1}{4}$ *is the best possible.*

In [3] N.S. Barnett, P. Cerone, S.S. Dragomir and A.M. Fink obtained the following inequality for the difference of two integral means:

Theorem 1.2. *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous mapping with the property that* $f' \in L_\infty [a,b]$, *then for* $a \leq c < d \leq b$,

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(t)\,dt \right| \leq \frac{1}{2} (b+c-a-d) \|f'\|_\infty, \tag{1.2}$$

the constant $\frac{1}{2}$ *being the best possible.*

For $c = d = x$ this can be seen as a generalization of (1.1).

In recent papers [1], [2], [4], [6] some generalizations of inequality (1.2) are given. Note that estimations of the difference of two integral means are obtained also in the case where $a \leq c < b \leq d$ (see [1], [2]), while in the case where $(a,b) \cap (c,d) = \varnothing$, there is no corresponding result.

In this paper we present a new inequality which is used to obtain some estimations for the difference of two integral means in the case where $(a,b) \cap (c,d) = \varnothing$, which in

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limiting cases reduces to a complement of Ostrowski's inequality (1.1). Inequalities for pdfs (Probability density functions) related to some results in [3, p. 245-246] are also given.

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2. Some Inequalities

The key result of the present paper is the following inequality:

Theorem 2.1. *Let f, g be two continuously differentiable functions on $[a, b]$ and twice differentiable on (a, b) with the properties that,*

$$g'' > 0 \tag{2.1}$$

on (a, b), and that the function $\frac{f''}{g''}$ is bounded on (a, b). For $a < c \leq d < b$ the following estimation holds,

$$\inf_{x\in(a,b)} \frac{f''(x)}{g''(x)} \leq \frac{\frac{f(b)-f(d)}{b-d} - \frac{f(c)-f(a)}{c-a}}{\frac{g(b)-g(d)}{b-d} - \frac{g(c)-g(a)}{c-a}} \leq \sup_{x\in(a,b)} \frac{f''(x)}{g''(x)}. \tag{2.2}$$

Proof. Let s be any number such that $a < s < c \leq d < b$. Consider the mappings $f_1,\ g_1 : [d, b] \to \mathbb{R}$ defined as:

$$\begin{aligned} f_1(x) &= f(x) - f(s) - (x - s) f'(s), \\ g_1(x) &= g(x) - g(s) - (x - s) g'(s). \end{aligned} \tag{2.3}$$

Clearly f_1, g_1 are continuous on $[d, b]$ and differentiable on (d, b). Further, for any $x \in [d, b]$, by applying the mean value Theorem,

$$g_1'(x) = g'(x) - g'(s) = (x - s) g''(\sigma)$$

for some $\sigma \in (s, x)$, which, combined with (2.1), gives $g_1'(x) \neq 0$, for all $x \in (d, b)$. Hence, we can apply Cauchy's mean value theorem to $f_1,\ g_1$ on the interval $[d, b]$ to obtain,

$$\frac{f_1(b) - f_1(d)}{g_1(b) - g_1(d)} = \frac{f_1'(\tau)}{g_1'(\tau)}$$

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for some $\tau \in (d, b)$ which can further be written as,

$$\frac{f(b) - f(d) - (b-d) f'(s)}{g(b) - g(d) - (b-d) g'(s)} = \frac{f'(\tau) - f'(s)}{g'(\tau) - g'(s)}. \tag{2.4}$$

Applying Cauchy's mean value theorem to f', g' on the interval $[s, \tau]$, we have that for some $\xi \in (s, \tau) \subseteq (a, b)$,

$$\frac{f'(\tau) - f'(s)}{g'(\tau) - g'(s)} = \frac{f''(\xi)}{g''(\xi)}. \tag{2.5}$$

Combining (2.4) and (2.5) we have,

$$m \leq \frac{f(b) - f(d) - (b-d) f'(s)}{g(b) - g(d) - (b-d) g'(s)} \leq M \tag{2.6}$$

for all $s \in (a, c)$, where $m = \inf_{x \in (a,b)} \frac{f''(x)}{g''(x)}$ and $M = \sup_{x \in (a,b)} \frac{f''(x)}{g''(x)}$.

By further application of the mean value Theorem and using the assumption (2.1) we readily get,

$$g(b) - g(d) - (b-d) g'(s) > 0. \tag{2.7}$$

Multiplying (2.6) by (2.7),

$$\begin{aligned} m(g(b) - g(d) - (b-d) g'(s)) &\leq f(b) - f(d) - (b-d) f'(s) \\ &\leq M(g(b) - g(d) - (b-d) g'(s)). \end{aligned} \tag{2.8}$$

Integrating the inequalities (2.7) and (2.8) with respect to s from a to c we obtain respectively,

$$(c-a)(g(b) - g(d)) - (b-d)(g(c) - g(a)) > 0 \tag{2.9}$$

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and

$$m\left(\left(c-a\right)\left(g\left(b\right)-g\left(d\right)\right)-\left(b-d\right)\left(g\left(c\right)-g\left(a\right)\right)\right) \\ \leq \left(c-a\right)\left(f\left(b\right)-f\left(d\right)\right)-\left(b-d\right)\left(f\left(c\right)-f\left(a\right)\right) \\ \leq M\left(\left(c-a\right)\left(g\left(b\right)-g\left(d\right)\right)-\left(b-d\right)\left(g\left(c\right)-g\left(a\right)\right)\right). \tag{2.10}$$

Finally, dividing (2.10) by (2.9) ,

$$m \leq \frac{\left(c-a\right)\left(f\left(b\right)-f\left(d\right)\right)-\left(b-d\right)\left(f\left(c\right)-f\left(a\right)\right)}{\left(c-a\right)\left(g\left(b\right)-g\left(d\right)\right)-\left(b-d\right)\left(g\left(c\right)-g\left(a\right)\right)} \leq M$$

as required. ∎

Remark 1. It is obvious that Theorem 2.1 holds also in the case where $g'' < 0$ on (a,b) .

Corollary 2.2. *Let $a < c \leq d < b$ and $F,\ G$ be two continuous functions on $[a,b]$ that are differentiable on (a,b) . If $G' > 0$ on (a,b) or $G' < 0$ on (a,b) and $\frac{F'}{G'}$ is bounded (a,b) , then,*

$$\inf_{x\in(a,b)} \frac{F'(x)}{G'(x)} \leq \frac{\frac{1}{b-d}\int_d^b F(t)\,dt - \frac{1}{c-a}\int_a^c F(t)\,dt}{\frac{1}{b-d}\int_d^b G(t)\,dt - \frac{1}{c-a}\int_a^c G(t)\,dt} \leq \sup_{x\in(a,b)} \frac{F'(x)}{G'(x)} \tag{2.11}$$

and

$$\frac{1}{2}\left(b+d-a-c\right)\inf_{x\in(a,b)} F'(x) \leq \frac{1}{b-d}\int_d^b F(t)\,dt - \frac{1}{c-a}\int_a^c F(t)\,dt \\ \leq \frac{1}{2}\left(b+d-a-c\right)\sup_{x\in(a,b)} F'(x). \tag{2.12}$$

The constant $\frac{1}{2}$ in (2.12) is the best possible.

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Proof. If we apply Theorem 2.1 for the functions,

$$f(x) := \int_a^x F(t)\,dt, \qquad g(x) := \int_a^x G(t)\,dt, \qquad x \in [a,b],$$

then we immediately obtain (2.11). Choosing $G(x) = x$ in (2.11) we get (2.12). ∎

Remark 2. Substituting $d = b$ in (1.2) of Theorem 1.2 we get,

$$\left| \frac{1}{b-a}\int_a^b F(x)\,dx - \frac{1}{b-c}\int_c^b F(x)\,dx \right| \le \frac{1}{2}(c-a)\left\|F'\right\|_\infty . \tag{2.13}$$

Setting $d = c$ in (2.12) of Corollary 2.2 we get,

$$\begin{aligned} \frac{b-a}{2}\inf_{x\in(a,b)} F'(x) &\le \frac{1}{b-c}\int_c^b F(x)\,dx - \frac{1}{c-a}\int_a^c F(x)\,dx \\ &\le \frac{b-a}{2}\sup_{x\in(a,b)} F'(x). \end{aligned} \tag{2.14}$$

Now,

$$\begin{aligned} &\frac{1}{b-c}\int_c^b F(x)\,dx - \frac{1}{c-a}\int_a^c F(x)\,dx \\ &\qquad = \frac{1}{c-a}\left(\frac{c-a}{b-c}\int_c^b F(x)\,dx - \int_a^c F(x)\,dx\right) \\ &\qquad = \frac{1}{c-a}\left(\frac{c-a}{b-c}\int_c^b F(x)\,dx - \int_a^b F(x)\,dx + \int_c^b F(x)\,dx\right) \end{aligned}$$

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$$= \frac{1}{c-a}\left(\frac{b-a}{b-c}\int_c^b F(x)\,dx - \int_a^b F(x)\,dx\right)$$
$$= \frac{b-a}{c-a}\left(\frac{1}{b-c}\int_c^b F(x)\,dx - \frac{1}{b-a}\int_a^b F(x)\,dx\right).$$

Using this in (2.14) we derive the inequality,

$$\frac{c-a}{2}\inf_{x\in(a,b)} F'(x) \le \frac{1}{b-c}\int_c^b F(x)\,dx - \frac{1}{b-a}\int_a^b F(x)\,dx \le \frac{c-a}{2}\sup_{x\in(a,b)} F'(x).$$

From this we clearly get again inequality (2.13). Consequently, inequality (2.12) can be seen as a complement of (1.2).

Corollary 2.3. *Let F, G be two continuous functions on an interval $I \subset \mathbb{R}$ and differentiable on the interior $\overset{\circ}{I}$ of I with the properties $G' > 0$ on $\overset{\circ}{I}$ or $G' < 0$ on $\overset{\circ}{I}$ and $\frac{F'}{G'}$ bounded on $\overset{\circ}{I}$. Let a, b be any numbers in $\overset{\circ}{I}$ such that $a < b$, then for all $x \in I - (a,b)$, that is, $x \in I$ but $x \notin (a,b)$, we have the estimation:*

$$\inf_{t\in(\{a,b,x\})} \frac{F'(t)}{G'(t)} \le \frac{\frac{1}{b-a}\int_a^b F(t)\,dt - F(x)}{\frac{1}{b-a}\int_a^b G(t)\,dt - G(x)} \le \sup_{t\in(\{a,b,x\})} \frac{F'(t)}{G'(t)}, \tag{2.15}$$

where $(\{a,b,x\}) := (\min\{a,x,\}, \max\{x,b\})$.

Proof. Let u, w, y, z be any numbers in I such that $u < w \le y < z$. According to Corollary 2.2 we then have the inequality,

$$\inf_{t\in(u,z)} \frac{F'(t)}{G'(t)} \le \frac{\frac{1}{z-y}\int_y^z F(t)\,dt - \frac{1}{w-u}\int_u^w F(t)\,dt}{\frac{1}{z-y}\int_y^z G(t)\,dt - \frac{1}{w-u}\int_u^w G(t)\,dt} \le \sup_{t\in(u,z)} \frac{F'(t)}{G'(t)}. \tag{2.16}$$

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We distinguish two cases:

If $x < a$, then by choosing $y = a,\ z = b$ and $u = w = x$ in (2.12) and assuming that $\frac{1}{w-u}\int_u^w F(t)\,dt = F(x)$ and $\frac{1}{w-u}\int_u^w G(t)\,dt = G(x)$ as limiting cases, (2.16) reduces to,

$$\inf_{t\in(x,b)} \frac{F'(t)}{G'(t)} \le \frac{\frac{1}{b-a}\int_a^b F(t)\,dt - F(x)}{\frac{1}{b-a}\int_a^b G(t)\,dt - G(x)} \le \sup_{t\in(x,b)} \frac{F'(t)}{G'(t)}.$$

Hence (2.15) holds for all $x < a$.

If $x > b$, then by choosing $u = a,\ w = b$ and $y = z = x$, in (2.16), similarly to the above, we can prove that for all $x > b$ the inequality (2.15) holds. ∎

Corollary 2.4. *Let F be a continuous function on an interval $I \subset \mathbb{R}$. If $F' \in L_\infty \overset{\circ}{I}$, then for all $a, b \in \overset{\circ}{I}$ with $b > a$ and all $x \in I - (a, b)$ we have:*

$$\left| F(x) - \frac{1}{b-a}\int_a^b F(t)\,dt \right| \le \frac{|b+a-2x|}{2} \|F'\|_{\infty,\ (\min\{a,x\},\max\{b,x\})}. \tag{2.17}$$

The inequality (2.17) is sharp.

Proof. Applying (2.15) for $G(x) = x$ we readily get (2.17). Choosing $F(x) = x$ in (2.17) we see that the equality holds, so the constant $\frac{1}{2}$ is the best possible. ∎

(2.17) is now used to obtain an extension of Ostrowski's inequality (1.1).

Proposition 2.5. *Let F be as in Corollary 2.3, then for all $a, b \in I$ with $b > a$ and*

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for all $x \in I$,

$$\left| F(x) - \frac{1}{b-a} \int_a^b F(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| F' \right\|_{\infty, (\min\{a,x\}, \max\{b,x\})} . \tag{2.18}$$

Proof. Clearly, the restriction of inequality (2.18) on $[a, b]$ is Ostrowski's inequality (1.1) . Moreover, a simple calculation yields

$$\frac{|b + a - 2x|}{2} \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)$$

for all $x \in \mathbb{R}$.

Combining this latter inequality with (2.17) we conclude that (2.18) holds also for $x \in I - (a, b)$ and so (2.18) is valid for all $x \in I$. ∎

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3. Applications for PDFs

We now use inequality (2.2) in Theorem 2.1 to obtain improvements of some results in [3, p. 245-246].

Assume that $f : [a,b] \to \mathbb{R}_+$ is a probability density function (pdf) of a certain random variable X, that is $\int_a^b f(x)\,dx = 1$, and

$$\Pr(X \le x) = \int_a^x f(t)\,dt, \qquad x \in [a,b]$$

is its cumulative distribution function. Working similarly to [3, p. 245-246] we can state the following:

Proposition 3.1. *With the previous assumptions for f, we have that for all $x \in [a,b]$,*

$$\text{(3.1)} \qquad \frac{1}{2}(b-x)(x-a)\inf_{x\in(a,b)} f'(x) \le \frac{x-a}{b-a} - \Pr(X \le x) \le \frac{1}{2}(b-x)(x-a)\sup_{x\in(a,b)} f'(x),$$

provided that $f \in C[a,b]$ and f is differentiable and bounded on (a,b).

Proof. Apply Theorem 2.1 for $f(x) = \Pr(X \le x)$, $g(x) = x^2$, $c = d = x$. ∎

Proposition 3.2. *Let f be as above, then,*

$$\text{(3.2)} \qquad \frac{1}{12}(x-a)^2(3b-a-2x)\inf_{x\in(a,b)} f'(x) \le \frac{(x-a)^2}{2(b-a)} - x\Pr(X \le x) + E_x(X)$$

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$$\leq \frac{1}{12}(x-a)^2(3b-a-2x)\sup_{x\in(a,b)} f'(x),$$

for all $x \in [a,b]$, *where*

$$E_x(X) := \int_a^x t \Pr(X \leq t)\,dt, \qquad x \in [a,b].$$

Proof. Integrating (3.1) from a to x and using, in the resulting estimation, the following identity,

$$\text{(3.3)} \qquad \int_a^x \Pr(X \leq x)\,dx = x\Pr(X \leq x) - \int_a^x x(\Pr(X \leq x))'\,dx = x\Pr(X \leq x) - E_x(X)$$

we easily get the desired result. ∎

Remark 3. Setting $x = b$ in (3.2) we get,

$$\frac{1}{12}(b-a)^3 \inf_{x\in(a,b)} f'(x) \leq E(X) - \frac{a+b}{2} \leq \frac{1}{12}(b-a)^3 \sup_{x\in(a,b)} f'(x).$$

Proposition 3.3. *Let* f, $\Pr(X \leq x)$ *be as above. If* $f \in L_\infty[a,b]$, *then we have,*

$$\frac{1}{2}(b-x)(x-a)\inf_{x\in[a,b]} f(x) \leq \frac{x-a}{b-a}(b-E(X)) - x\Pr(X \leq x) + E_x(X) \leq \frac{1}{2}(b-x)(x-a)\sup_{x\in[a,b]} f(x)$$

for all $x \in [a,b]$.

Proof. Apply Theorem 2.1 for $f(x) := \int_a^x \Pr(X \leq t)\,dt$, $g(x) := x^2$, $x \in [a,b]$, and identity (3.3). ∎

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