



ON THE INEQUALITY OF THE DIFFERENCE OF TWO INTEGRAL MEANS AND APPLICATIONS FOR PDFs

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Abstract: A new inequality is presented, which is used to obtain a complement of recently obtained inequality concerning the difference of two integral means. Some applications for pdfs are also given.

Complements of Ostrowski's
Inequality

A.I. Kechriniotis and N.D. Assimakis
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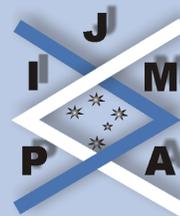
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1. Introduction

In 1938, Ostrowski proved the following inequality [5].

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $|f'(x)| \leq M$ for all $x \in (a, b)$, then,*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) M,$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

In [3] N.S. Barnett, P. Cerone, S.S. Dragomir and A.M. Fink obtained the following inequality for the difference of two integral means:

Theorem 1.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping with the property that $f' \in L_\infty[a, b]$, then for $a \leq c < d \leq b$,*

$$(1.2) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(t) dt \right| \leq \frac{1}{2} (b+c-a-d) \|f'\|_\infty,$$

the constant $\frac{1}{2}$ being the best possible.

For $c = d = x$ this can be seen as a generalization of (1.1).

In recent papers [1], [2], [4], [6] some generalizations of inequality (1.2) are given. Note that estimations of the difference of two integral means are obtained also in the case where $a \leq c < b \leq d$ (see [1], [2]), while in the case where $(a, b) \cap (c, d) = \emptyset$, there is no corresponding result.

In this paper we present a new inequality which is used to obtain some estimations for the difference of two integral means in the case where $(a, b) \cap (c, d) = \emptyset$, which in

limiting cases reduces to a complement of Ostrowski's inequality (1.1). Inequalities for pdfs (Probability density functions) related to some results in [3, p. 245-246] are also given.



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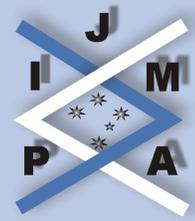
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2. Some Inequalities

The key result of the present paper is the following inequality:

Theorem 2.1. *Let f, g be two continuously differentiable functions on $[a, b]$ and twice differentiable on (a, b) with the properties that,*

$$(2.1) \quad g'' > 0$$

on (a, b) , and that the function $\frac{f''}{g''}$ is bounded on (a, b) . For $a < c \leq d < b$ the following estimation holds,

$$(2.2) \quad \inf_{x \in (a, b)} \frac{f''(x)}{g''(x)} \leq \frac{\frac{f(b)-f(d)}{b-d} - \frac{f(c)-f(a)}{c-a}}{\frac{g(b)-g(d)}{b-d} - \frac{g(c)-g(a)}{c-a}} \leq \sup_{x \in (a, b)} \frac{f''(x)}{g''(x)}.$$

Proof. Let s be any number such that $a < s < c \leq d < b$. Consider the mappings $f_1, g_1 : [d, b] \rightarrow \mathbb{R}$ defined as:

$$(2.3) \quad \begin{aligned} f_1(x) &= f(x) - f(s) - (x-s)f'(s), \\ g_1(x) &= g(x) - g(s) - (x-s)g'(s). \end{aligned}$$

Clearly f_1, g_1 are continuous on $[d, b]$ and differentiable on (d, b) . Further, for any $x \in [d, b]$, by applying the mean value Theorem,

$$g_1'(x) = g'(x) - g'(s) = (x-s)g''(\sigma)$$

for some $\sigma \in (s, x)$, which, combined with (2.1), gives $g_1'(x) \neq 0$, for all $x \in (d, b)$. Hence, we can apply Cauchy's mean value theorem to f_1, g_1 on the interval $[d, b]$ to obtain,

$$\frac{f_1(b) - f_1(d)}{g_1(b) - g_1(d)} = \frac{f_1'(\tau)}{g_1'(\tau)}$$



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for some $\tau \in (d, b)$ which can further be written as,

$$(2.4) \quad \frac{f(b) - f(d) - (b-d)f'(s)}{g(b) - g(d) - (b-d)g'(s)} = \frac{f'(\tau) - f'(s)}{g'(\tau) - g'(s)}.$$

Applying Cauchy's mean value theorem to f' , g' on the interval $[s, \tau]$, we have that for some $\xi \in (s, \tau) \subseteq (a, b)$,

$$(2.5) \quad \frac{f'(\tau) - f'(s)}{g'(\tau) - g'(s)} = \frac{f''(\xi)}{g''(\xi)}.$$

Combining (2.4) and (2.5) we have,

$$(2.6) \quad m \leq \frac{f(b) - f(d) - (b-d)f'(s)}{g(b) - g(d) - (b-d)g'(s)} \leq M$$

for all $s \in (a, c)$, where $m = \inf_{x \in (a,b)} \frac{f''(x)}{g''(x)}$ and $M = \sup_{x \in (a,b)} \frac{f''(x)}{g''(x)}$.

By further application of the mean value Theorem and using the assumption (2.1) we readily get,

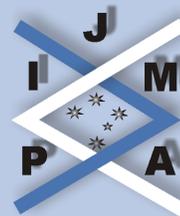
$$(2.7) \quad g(b) - g(d) - (b-d)g'(s) > 0.$$

Multiplying (2.6) by (2.7),

$$(2.8) \quad m(g(b) - g(d) - (b-d)g'(s)) \leq f(b) - f(d) - (b-d)f'(s) \\ \leq M(g(b) - g(d) - (b-d)g'(s)).$$

Integrating the inequalities (2.7) and (2.8) with respect to s from a to c we obtain respectively,

$$(2.9) \quad (c-a)(g(b) - g(d)) - (b-d)(g(c) - g(a)) > 0$$



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and

$$(2.10) \quad \begin{aligned} m((c-a)(g(b)-g(d)) - (b-d)(g(c)-g(a))) \\ \leq (c-a)(f(b)-f(d)) - (b-d)(f(c)-f(a)) \\ \leq M((c-a)(g(b)-g(d)) - (b-d)(g(c)-g(a))). \end{aligned}$$

Finally, dividing (2.10) by (2.9),

$$m \leq \frac{(c-a)(f(b)-f(d)) - (b-d)(f(c)-f(a))}{(c-a)(g(b)-g(d)) - (b-d)(g(c)-g(a))} \leq M$$

as required. ■

Remark 1. It is obvious that Theorem 2.1 holds also in the case where $g'' < 0$ on (a, b) .

Corollary 2.2. Let $a < c \leq d < b$ and F, G be two continuous functions on $[a, b]$ that are differentiable on (a, b) . If $G' > 0$ on (a, b) or $G' < 0$ on (a, b) and $\frac{F'}{G'}$ is bounded (a, b) , then,

$$(2.11) \quad \inf_{x \in (a,b)} \frac{F'(x)}{G'(x)} \leq \frac{\frac{1}{b-d} \int_d^b F(t) dt - \frac{1}{c-a} \int_a^c F(t) dt}{\frac{1}{b-d} \int_d^b G(t) dt - \frac{1}{c-a} \int_a^c G(t) dt} \leq \sup_{x \in (a,b)} \frac{F'(x)}{G'(x)}$$

and

$$(2.12) \quad \begin{aligned} \frac{1}{2}(b+d-a-c) \inf_{x \in (a,b)} F'(x) &\leq \frac{1}{b-d} \int_d^b F(t) dt - \frac{1}{c-a} \int_a^c F(t) dt \\ &\leq \frac{1}{2}(b+d-a-c) \sup_{x \in (a,b)} F'(x). \end{aligned}$$

The constant $\frac{1}{2}$ in (2.12) is the best possible.



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Proof. If we apply Theorem 2.1 for the functions,

$$f(x) := \int_a^x F(t) dt, \quad g(x) := \int_a^x G(t) dt, \quad x \in [a, b],$$

then we immediately obtain (2.11). Choosing $G(x) = x$ in (2.11) we get (2.12). ■

Remark 2. Substituting $d = b$ in (1.2) of Theorem 1.2 we get,

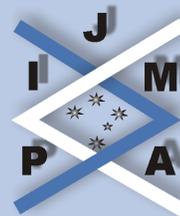
$$(2.13) \quad \left| \frac{1}{b-a} \int_a^b F(x) dx - \frac{1}{b-c} \int_c^b F(x) dx \right| \leq \frac{1}{2} (c-a) \|F'\|_\infty.$$

Setting $d = c$ in (2.12) of Corollary 2.2 we get,

$$(2.14) \quad \begin{aligned} \frac{b-a}{2} \inf_{x \in (a,b)} F'(x) &\leq \frac{1}{b-c} \int_c^b F(x) dx - \frac{1}{c-a} \int_a^c F(x) dx \\ &\leq \frac{b-a}{2} \sup_{x \in (a,b)} F'(x). \end{aligned}$$

Now,

$$\begin{aligned} \frac{1}{b-c} \int_c^b F(x) dx - \frac{1}{c-a} \int_a^c F(x) dx \\ &= \frac{1}{c-a} \left(\frac{c-a}{b-c} \int_c^b F(x) dx - \int_a^c F(x) dx \right) \\ &= \frac{1}{c-a} \left(\frac{c-a}{b-c} \int_c^b F(x) dx - \int_a^b F(x) dx + \int_c^a F(x) dx \right) \end{aligned}$$



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$$\begin{aligned}
 &= \frac{1}{c-a} \left(\frac{b-a}{b-c} \int_c^b F(x) dx - \int_a^b F(x) dx \right) \\
 &= \frac{b-a}{c-a} \left(\frac{1}{b-c} \int_c^b F(x) dx - \frac{1}{b-a} \int_a^b F(x) dx \right).
 \end{aligned}$$

Using this in (2.14) we derive the inequality,

$$\frac{c-a}{2} \inf_{x \in (a,b)} F'(x) \leq \frac{1}{b-c} \int_c^b F(x) dx - \frac{1}{b-a} \int_a^b F(x) dx \leq \frac{c-a}{2} \sup_{x \in (a,b)} F'(x).$$

From this we clearly get again inequality (2.13). Consequently, inequality (2.12) can be seen as a complement of (1.2).

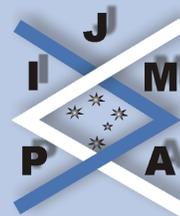
Corollary 2.3. *Let F, G be two continuous functions on an interval $I \subset \mathbb{R}$ and differentiable on the interior $\overset{\circ}{I}$ of I with the properties $G' > 0$ on $\overset{\circ}{I}$ or $G' < 0$ on $\overset{\circ}{I}$ and $\frac{F'}{G'}$ bounded on $\overset{\circ}{I}$. Let a, b be any numbers in $\overset{\circ}{I}$ such that $a < b$, then for all $x \in I - (a, b)$, that is, $x \in I$ but $x \notin (a, b)$, we have the estimation:*

$$(2.15) \quad \inf_{t \in (\{a, b, x\})} \frac{F'(t)}{G'(t)} \leq \frac{\frac{1}{b-a} \int_a^b F(t) dt - F(x)}{\frac{1}{b-a} \int_a^b G(t) dt - G(x)} \leq \sup_{t \in (\{a, b, x\})} \frac{F'(t)}{G'(t)},$$

where $(\{a, b, x\}) := (\min \{a, x\}, \max \{x, b\})$.

Proof. Let u, w, y, z be any numbers in I such that $u < w \leq y < z$. According to Corollary 2.2 we then have the inequality,

$$(2.16) \quad \inf_{t \in (u, z)} \frac{F'(t)}{G'(t)} \leq \frac{\frac{1}{z-y} \int_y^z F(t) dt - \frac{1}{w-u} \int_u^w F(t) dt}{\frac{1}{z-y} \int_y^z G(t) dt - \frac{1}{w-u} \int_u^w G(t) dt} \leq \sup_{t \in (u, z)} \frac{F'(t)}{G'(t)}.$$



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We distinguish two cases:

If $x < a$, then by choosing $y = a$, $z = b$ and $u = w = x$ in (2.12) and assuming that $\frac{1}{w-u} \int_u^w F(t) dt = F(x)$ and $\frac{1}{w-u} \int_u^w G(t) dt = G(x)$ as limiting cases, (2.16) reduces to,

$$\inf_{t \in (x,b)} \frac{F'(t)}{G'(t)} \leq \frac{\frac{1}{b-a} \int_a^b F(t) dt - F(x)}{\frac{1}{b-a} \int_a^b G(t) dt - G(x)} \leq \sup_{t \in (x,b)} \frac{F'(t)}{G'(t)}.$$

Hence (2.15) holds for all $x < a$.

If $x > b$, then by choosing $u = a$, $w = b$ and $y = z = x$, in (2.16), similarly to the above, we can prove that for all $x > b$ the inequality (2.15) holds. ■

Corollary 2.4. Let F be a continuous function on an interval $I \subset \mathbb{R}$. If $F' \in L_\infty \overset{\circ}{I}$, then for all $a, b \in \overset{\circ}{I}$ with $b > a$ and all $x \in I - (a, b)$ we have:

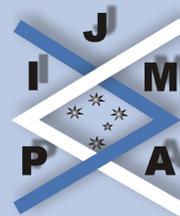
$$(2.17) \quad \left| F(x) - \frac{1}{b-a} \int_a^b F(t) dt \right| \leq \frac{|b+a-2x|}{2} \|F'\|_\infty, (\min\{a,x\}, \max\{b,x\}).$$

The inequality (2.17) is sharp.

Proof. Applying (2.15) for $G(x) = x$ we readily get (2.17). Choosing $F(x) = x$ in (2.17) we see that the equality holds, so the constant $\frac{1}{2}$ is the best possible. ■

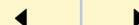
(2.17) is now used to obtain an extension of Ostrowski's inequality (1.1).

Proposition 2.5. Let F be as in Corollary 2.3, then for all $a, b \in I$ with $b > a$ and



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for all $x \in I$,

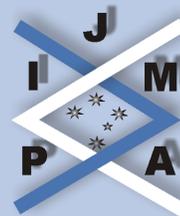
$$(2.18) \quad \left| F(x) - \frac{1}{b-a} \int_a^b F(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|F'\|_{\infty, (\min\{a,x\}, \max\{b,x\})}.$$

Proof. Clearly, the restriction of inequality (2.18) on $[a, b]$ is Ostrowski's inequality (1.1). Moreover, a simple calculation yields

$$\frac{|b+a-2x|}{2} \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)$$

for all $x \in \mathbb{R}$.

Combining this latter inequality with (2.17) we conclude that (2.18) holds also for $x \in I - (a, b)$ and so (2.18) is valid for all $x \in I$. ■



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3. Applications for PDFs

We now use inequality (2.2) in Theorem 2.1 to obtain improvements of some results in [3, p. 245-246].

Assume that $f : [a, b] \rightarrow \mathbb{R}_+$ is a probability density function (pdf) of a certain random variable X , that is $\int_a^b f(x) dx = 1$, and

$$\Pr(X \leq x) = \int_a^x f(t) dt, \quad x \in [a, b]$$

is its cumulative distribution function. Working similarly to [3, p. 245-246] we can state the following:

Proposition 3.1. *With the previous assumptions for f , we have that for all $x \in [a, b]$,*

$$(3.1) \quad \frac{1}{2} (b-x)(x-a) \inf_{x \in (a,b)} f'(x) \leq \frac{x-a}{b-a} - \Pr(X \leq x) \\ \leq \frac{1}{2} (b-x)(x-a) \sup_{x \in (a,b)} f'(x),$$

provided that $f \in C[a, b]$ and f is differentiable and bounded on (a, b) .

Proof. Apply Theorem 2.1 for $f(x) = \Pr(X \leq x)$, $g(x) = x^2$, $c = d = x$. ■

Proposition 3.2. *Let f be as above, then,*

$$(3.2) \quad \frac{1}{12} (x-a)^2 (3b-a-2x) \inf_{x \in (a,b)} f'(x) \\ \leq \frac{(x-a)^2}{2(b-a)} - x \Pr(X \leq x) + E_x(X)$$

$$\leq \frac{1}{12} (x - a)^2 (3b - a - 2x) \sup_{x \in (a,b)} f'(x),$$

for all $x \in [a, b]$, where

$$E_x(X) := \int_a^x t \Pr(X \leq t) dt, \quad x \in [a, b].$$

Proof. Integrating (3.1) from a to x and using, in the resulting estimation, the following identity,

$$\begin{aligned} (3.3) \quad \int_a^x \Pr(X \leq x) dx &= x \Pr(X \leq x) - \int_a^x x (\Pr(X \leq x))' dx \\ &= x \Pr(X \leq x) - E_x(X) \end{aligned}$$

we easily get the desired result. ■

Remark 3. Setting $x = b$ in (3.2) we get,

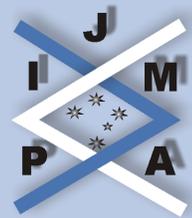
$$\frac{1}{12} (b - a)^3 \inf_{x \in (a,b)} f'(x) \leq E(X) - \frac{a + b}{2} \leq \frac{1}{12} (b - a)^3 \sup_{x \in (a,b)} f'(x).$$

Proposition 3.3. Let $f, \Pr(X \leq x)$ be as above. If $f \in L_\infty[a, b]$, then we have,

$$\begin{aligned} \frac{1}{2} (b - x)(x - a) \inf_{x \in [a,b]} f(x) &\leq \frac{x - a}{b - a} (b - E(X)) - x \Pr(X \leq x) + E_x(X) \\ &\leq \frac{1}{2} (b - x)(x - a) \sup_{x \in [a,b]} f(x) \end{aligned}$$

for all $x \in [a, b]$.

Proof. Apply Theorem 2.1 for $f(x) := \int_a^x \Pr(X \leq t) dt$, $g(x) := x^2$, $x \in [a, b]$, and identity (3.3). ■



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References

- [1] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ AND I. PERIĆ, Estimates of the difference between two weighted integral means via weighted Montgomery identity, *Math. Inequal. Appl.*, **7**(3) (2004), 315–336.
- [2] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ AND A. VUKELIĆ, The extension of Montgomery identity via Fink identity with applications, *J. Inequal. Appl.*, **2005**(1), 67–79.
- [3] N.S. BARNETT, P. CERONE, S.S. DRAGOMIR AND A. M. FINK, Comparing two integral means for absolutely continuous mappings whose derivatives are in $L_\infty [a, b]$ and applications, *Comput. Math. Appl.*, **44**(1-2) (2002), 241–251.
- [4] P. CERONE AND S.S. DRAGOMIR, Differences between means with bounds from a Riemann-Stieltjes integral, *Comp. and Math. Appl.*, **46** (2003), 445–453.
- [5] A. OSTROWSKI, Über die Absolutabweichung einer differenzierbaren funktion von ihren integralmittelwert, *Comment. Math. Helv.*, **10** (1938), 226–227 (German).
- [6] J. PEČARIĆ, I. PERIĆ AND A. VUKELIĆ, Estimations of the difference between two integral means via Euler-type identities, *Math. Inequal. Appl.*, **7**(3) (2004), 365–378.



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