A Variation on Mills-Like Prime-Representing Functions

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Abstract

Mills showed that there exists a constant $A$ such that $\lfloor A^3n \rfloor$ is prime for every positive integer $n$. Kuipers and Ansari generalized this result to $\lfloor A^cn \rfloor$ where $c \in \mathbb{R}$ and $c \geq 2.106$. The main contribution of this paper is a proof that the function $\lceil Be^n \rceil$ is also a prime-representing function, where $\lceil X \rceil$ denotes the ceiling or least integer function. Moreover, the first 10 primes in the sequence generated in the case $c = 3$ are calculated. Lastly, the value of $B$ is approximated to the first 5500 digits and is shown to begin with 1.2405547052. . . .

1 Introduction

Mills [6] showed in 1947 that there exists a constant $A$ such that $\lfloor A^3n \rfloor$ is prime for all positive integers $n$. Kuipers [5] and Ansari [1] generalized this result to all $\lfloor A^cn \rfloor$ where $c \in \mathbb{R}, c \geq 2.106$, i.e., there exist infinitely many $A$’s such that the above expression yields a prime for all positive integers $n$. Caldwell and Cheng [2] calculated the minimum constant $A$ for the case $c = 3$ up to the first 6850 digits ($A051021$), and found it to be approximately equal to 1.3063778838. . . . This process involved computing the first 10 primes $b_i$ in the sequence generated by the function ($A051254$), with $b_{10}$ having 6854 decimal digits.

The main contribution of this paper is a proof that the function $\lceil Be^n \rceil$ satisfies the same criteria, where $\lceil X \rceil$ denotes the ceiling function (the least integer greater than or equal to $X$). In other words, there exists a constant $B$ such that for all positive integers $n$, the expression $\lceil Be^n \rceil$ yields a prime for $c \geq 3, c \in \mathbb{N}$. Moreover, the sequence of primes generated by such
functions is monotonically increasing. Lastly, analogously to [2] the case \( c = 3 \) is studied in more detail and the value of \( B \) is approximated up to the first 5500 decimal digits by calculating the first 10 primes \( b_i \) of the sequence.

In contrast to Mills’ formula and given that here the floor function is replaced by a ceiling function, the process of generating the prime number sequence \( P_0, P_1, P_2, \ldots \) involves taking the greatest prime smaller than \( P_n^c \) at each step instead of smallest prime greater than \( P_n^c \), in order to find \( P_{n+1} \). As a consequence, the sequence of primes generated by \( \lceil B^c \rceil \) is different than the one generated by \( \lfloor A^c \rfloor \) for the same value of \( c \) and the same starting prime (apart from the first element of course).

## 2 The prime-representing function

This paper begins with a proof of the case \( c = 3 \) and will proceed to a generalization of the function to all \( c \geq 3, c \in \mathbb{N} \).

By using Ingham’s result [4] on the difference of consecutive primes:

\[
p_{n+1} - p_n < K P_n^{5/8},
\]

and analogously to Mills’ reasoning [6], we construct an infinite sequence of primes \( P_0, P_1, P_2, \ldots \) such that \( \forall n \in \mathbb{N} : (P_n - 1)^3 + 1 < P_{n+1} < P_n^3 \) using the following lemma.

**Lemma 1.** \( \forall N > K^8 + 1 \in \mathbb{N} : \exists p \in \mathbb{P} : (N - 1)^3 + 1 < p < N^3 \), where \( \mathbb{P} \) denotes the set of prime numbers.

**Proof.** Let \( p_n \) be the greatest prime smaller than \((N-1)^3\).

\[
\begin{align*}
(N - 1)^3 &< p_{n+1} \\
&< p_n + K P_n^{5/8} \\
&< (N - 1)^3 + K ((N - 1)^3)^{5/8} & \text{(since } p_n < (N - 1)^3) \\
&< (N - 1)^3 + (N - 1)^2 & \text{(since } N > K^8 + 1) \\
&< N^3 - 2N^2 + N \\
&< N^3.
\end{align*}
\]

Note that since \((N-1)^3 < p_{n+1}, (N-1)^3 + 1 < p_{n+1} \) since \((N-1)^3+1 = N(N^2 - 3N + 3)\) is not prime.

Given the above we can construct an infinite sequence of primes \( P_0, P_1, P_2, \ldots \) such that for every positive integer \( n \), we have: \((P_n - 1)^3 + 1 < P_{n+1} < P_n^3\).

We now define the following two functions:
\( \forall n \in \mathbb{Z}^+ : u_n = (P_n - 1)^{3^{-n}}, \)
\( \forall n \in \mathbb{Z}^+ : v_n = P_n^{3^{-n}}. \)

The following statements can immediately be deduced:

- \( u_n < v_n, \)
- \( u_{n+1} = (P_{n+1} - 1)^{3^{-n-1}} > ((P_n - 1)^{3} + 1)^{3^{-n-1}} = (P_n - 1)^{3-n} = u_n, \)
- \( v_{n+1} = P_{n+1}^{3^{-n-1}} < (P_n^{3})^{3^{-n-1}} = P_n^{3-n} = v_n. \)

It follows that \( u_n \) forms a bounded and monotone increasing sequence.

**Theorem 2.** There exists a positive real constant \( B \) such that \( \lceil B^{3^n} \rceil \) is a prime-representing function for all positive integers \( n \).

**Proof.** Since \( u_n \) is bounded and strictly monotone, there exists a number \( B := \lim_{n \to \infty} u_n \).

From the above deduced properties of \( u_n \) and \( v_n \), we have

\[
\begin{align*}
&u_n < B < v_n, \\
&(P_n - 1)^{3^{-n}} < B < P_n^{3^{-n}}, \\
&P_n - 1 < B^{3^n} < P_n.
\end{align*}
\]

**Theorem 3.** There exists a positive real constant \( B \) such that \( \lceil B^{c^n} \rceil \) is a prime-representing function for \( c \geq 3, c \in \mathbb{N} \) and all positive integers \( n \).

**Proof.** We can use the generalizations to Mills’ function as shown by Kuipers [5] and Dudley [3] in order to show that \( \lceil B^{c^n} \rceil \) is also a prime-representing function for \( c \geq 3, c \in \mathbb{N} \). This proof is short as it is essentially identical to the one presented above, with the following modifications.

As shown by Kuipers [5] for Mills’ function, we first define \( a = 3c - 4, b = 3c - 1. \) Therefore \( a/b \geq 5/8. \) This means that in Ingham’s equation there exists a constant \( K' \) such that

\[ p_{n+1} - p_n < K' p_n^{a/b}. \]

Lemma 1 can then be modified by taking \( N > K'^b + 1, \) defining \( p_n \) as the greatest prime smaller than \( (N - 1)^c \) and noticing that \( ca + 1 = b(c - 1). \) Analogously to the proof in Lemma 1, we quickly obtain the bounds \( (N - 1)^c + 1 < p < N^c. \) This means that we can construct a sequence of primes \( P_0, P_1, P_2, \ldots \) such that for every positive integer \( n, \)

\[
(P_n - 1)^{c} + 1 < P_{n+1} < P_n^c.
\]

This is then concluded with a similar reasoning as in the proof of Theorem 2.
3 Numerical calculation of $B$

In this section, a numerical approximation of $B$ is presented for the case $c = 3$. Mills [6] suggested using the lower bound $K = 8$ for the first prime in the classical Mills function $\lfloor A^{3^n} \rfloor$, where $K$ is the constant defined in Ingham’s paper [4]. Other authors, including Caldwell and Cheng [2], decided to begin with the prime 2 and then choose the least possible prime at each step. In this case, since the ceiling function replaces the floor function, we choose the greatest possible prime smaller than $P_n^3$ as the next element $P_{n+1}$.

If $p_i$ denotes the $i^{th}$ prime in the sequence, we obtain

- $p_1 = 2$
- $p_2 = 7$
- $p_3 = 337$
- $p_4 = 38272739$
- $p_5 = 56062005704198360319209$
- $p_6 = 17619999581432728735667120910458586439705503907211069\backslash$
  \quad 6028654438846269$
- $p_7 = 54703823381492990628407924713718713957740513297193414\backslash$
  \quad 21259587335767096542227048457036456872683352033529421007878\backslash$
  \quad 29141860830768725102854526098825035518110731140339908096068\backslash$
  \quad 8125590506176016285837338837682469$

The primes $p_8$, $p_9$ and $p_{10}$ are far too large to show in this paper — for instance $p_{10}$ has 5528 decimal digits. The primes $p_i$ for $i \leq 8$ were verified using a deterministic primality test in Wolfram Mathematica 11 with the ProvablePrimeQ function in the PrimalityProving package, while $p_9$ and $p_{10}$ were certified prime by the Primo software [7]. The certification of $p_{10}$ took 14 hours and 23 minutes on an Intel i7-4770 CPU and 4GB RAM. The prime certificates for $p_9$ and $p_{10}$ as well as the primes themselves can be found alongside this paper as auxiliary files.

The value of $B$ was calculated up to its first 5500 decimal digits. The first 600 are presented below:
References


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