On the Periodicity Problem for Residual $r$-Fubini Sequences

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Abstract

For any positive integer $r$, the $r$-Fubini number with parameter $n$, denoted by $F_{n,r}$, is equal to the number of ways that the elements of a set with $n + r$ elements can be weakly ordered such that the $r$ least elements are in distinct orders. In this article we focus on the sequence of residues of the $r$-Fubini numbers modulo an arbitrary positive integer $s$ and show that this sequence is periodic and then, exhibit how to calculate its period length.

1 Introduction

The Fubini numbers (also known as the ordered Bell numbers) form an integer sequence in which the $n$th term counts the number of weak orderings of a set with $n$ elements. Weak
ordering means that the elements can be ordered, allowing ties. Cayley [2] studied the Fubini numbers as the number of a certain kind of trees with \( n + 1 \) terminal nodes. The Fubini numbers can also be defined as the sum of the Stirling numbers of the second kind, \( \{n\} \), which counts the number of partitions of an \( n \)-element set into \( k \) non-empty subsets. The sequence of residues of the Fubini numbers modulo a positive integer \( s \) was studied by Poonen [6]. He showed that this sequence is periodic and calculated the period length for each positive integer \( s \).

The \( r \)-Stirling numbers of the second kind are defined as an extension to the Stirling numbers of the second kind, in which the first \( r \) elements contained in distinct subsets. Similarly the \( r \)-Fubini numbers, which are denoted by \( F_{n,r} \), are defined as the number of ways which the elements of a set with \( n + r \) elements can be weakly ordered such that the first \( r \) elements are in distinct places. Consider the sequence of remainders of \( F_{n,r} \) modulo an arbitrary number \( s \in \mathbb{N} \) in which \( r \) is fixed, which is denoted by \( A_{r,s} \). One can study the periodicity problem for this sequence. Mezö [4] investigated this problem for \( s = 10 \). In this article \( \omega(A_{r,s}) \), the period of \( A_{r,s} \), is computed for any positive integer \( s \). Based on the fundamental theorem of arithmetic, \( \omega(A_{r,p}) \) is calculated for powers of odd primes \( p^m \). The cases \( s = 2^m \) are studied separately. Therefore if \( s = 2^m p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \) is the prime factorization, then the \( \omega(A_{r,s}) \) is equal to the least common multiple of \( \omega(A_{r,p_i^{m_i}})s \) and \( \omega(A_{r,2^m}) \), for \( i = 1, 2, \ldots, k \).

Section 2 contains the basic definitions and relations. The length of the periods in the case of odd prime powers are computed in the Section 3. The similar results about the 2 powers are stated in the Section 4. The last section contains the final theorem which presents the conclusion of the article.

### 2 Basic concepts

Let \( \{n\} \) be the Stirling number of the second kind with the parameters \( n \) and \( k \) and let \( \{n\}_r \) be the \( r \)-Stirling number of the second kind with parameters \( n \) and \( k \). It is clear that \( n \geq k \geq r \). Fubini numbers are computed as follows [4]:

\[
F_n = \sum_{k=0}^{n} k! \{n\}.
\]

In a similar way we can evaluate the \( r \)-Fubini number \( F_{n,r} \) by

\[
F_{n,r} = \sum_{k=0}^{n} (k + r)! \{n + r\}_r.
\]
There are simple relations and formulae about \( \binom{n}{k} \), which are listed below. One can find a proof of them in [1, 4, 5] and [3, Thm. 4.5.1, p. 158].

\[
\binom{n}{m}_r = \binom{n}{m}_{r-1} - (r-1)\binom{n-1}{m}_{r-1}, 1 \leq r \leq n \tag{1}
\]

\[
\binom{n}{m}_1 = \binom{n}{m} \tag{2}
\]

\[
\binom{n+r}{r}_r = r^n \tag{3}
\]

\[
\binom{n+r}{r+1}_r = (r+1)^n - r^n \tag{4}
\]

\[
\binom{n}{m} = \frac{1}{m!} \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} j^n \tag{5}
\]

\[
\binom{n}{m}_r = \frac{1}{m!} \sum_{j=r}^{m} (-1)^{m-j} \binom{m}{j} j^{n-(r-1)} \binom{(j-1)!(j-r)!}{(j-r)!} \tag{6}
\]

By \( \varphi(n) \) we indicate the number of positive integer numbers less than \( n \) and co-prime to it. It is known as Euler’s totient function. The value of \( \varphi(n) \) can be computed via the following relation [3, Example 4.7.3, p. 167]:

\[
\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).
\]

3 The \( r \)-Fubini residues modulo prime powers

Let \( p \) be a prime number greater than 2 and \( m \) be a positive integer. If \( (F_{n,r}) \) denotes the sequence of \( r \)-Fubini numbers for a fixed positive integer \( r \), we indicate by \( A_{r,q} = (F_{n,r} \mod q) \), for \( n \in \mathbb{N} \), the sequence of residues of the \( r \)-Fubini numbers modulo the positive integer \( q \). In this section we try to compute the period length of the sequence \( A_{r,q} \) when \( q = p^m \). This length is denoted by \( \omega(A_{r,q}) \).

**Proposition 1.** Let \( p \) be an odd prime and let \( q = p^m, m \in \mathbb{N} \). If \( q \leq r \), then \( \omega(A_{r,q}) = 1 \).

**Proof.** The proof is very simple. Since \( p \leq r \), we can deduce that \( p \mid (k+r)! \), for \( k \geq 0 \), and by the relation \( F_{n,r} = \sum_{k=0}^{n} (k+r)! \binom{n+r}{k+r}_r \), we have \( p \mid F_{n,r} \). Therefore \( \omega(A_{r,p}) = 1 \).

As pointed out in the above proposition, it is sufficient to investigate the period length in the cases of \( q > r \).

**Lemma 2.** Let \( p \) be an odd prime and \( r, m \in \mathbb{N} \) with \( p \geq r + 1 \). Then

\[
p^m - r \geq m.
\]
Proof. For \( m = 1 \) the result is obvious. Suppose the inequality holds for any \( m \geq 2 \). Since \( p(p + m) > 2(p + m) > 2p + m \), we have
\[
p^2 + pm - p \geq p + m. \tag{7}
\]
Since \( p - 1 \geq r \), the induction hypothesis can be reformulated to \( p^n \geq p - 1 + m \). Multiplication by \( p \) results \( p^{n+1} \geq p^2 + pm - p \). By (7) we have \( p^{n+1} \geq p + (m + 1) - 1 \). \qed

**Theorem 3.** Let \( p \) be an odd prime and \( q = p^m \). After the \((m - 1)\)th term the sequence \( A_{r,q} \) has a period with length \( \omega(A_{r,q}) = \varphi(q) \). In other words, \( F_{n+\varphi(q),r} \equiv F_{n,r} \) (mod \( q \)), for \( n \geq m - 1 \).

Proof. If \( n \geq q - r - 1 \) we can write
\[
F_{n+\varphi(q),r} - F_{n,r} = \sum_{k=0}^{n+\varphi(q)} (k + r)! \left\{ \binom{n + \varphi(q) + r}{k + r} \right\} - \sum_{k=0}^{n} (k + r)! \left\{ \binom{n + r}{k + r} \right\}.
\]
\[
\equiv \sum_{k=0}^{q-r-1} (k + r)! \left( \left\{ \binom{n + \varphi(q) + r}{k + r} \right\} - \left\{ \binom{n + r}{k + r} \right\} \right)
\]
\[
\equiv \sum_{k=0}^{q-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} \binom{(j-1)!}{(j-r)!} (j^{\varphi(q)} - 1) \pmod{q}.
\]

If \( j = cp, c \in \mathbb{N} \), then \( j^{n+1} = (cp)^{q-r+h} \), for some \( h \geq 0 \), so from Lemma 2 it follows that \( j^{n+1} \equiv 0 \pmod{q} \). If \( \gcd(j, q) = 1 \), by Euler’s theorem \( j^{\varphi(q) - 1} \equiv 0 \pmod{q} \), so the right hand side of the above congruence relation vanished and we have
\[
F_{n+\varphi(q),r} \equiv F_{n,r} \pmod{q}, \text{ for } n \geq q - r - 1. \tag{8}
\]

If \( m - 1 \leq n < q - r - 1 \) then
\[
F_{n+\varphi(q),r} - F_{n,r} \equiv \sum_{k=0}^{q-r-1} (k + r)! \left( \left\{ \binom{n + \varphi(q) + r}{k + r} \right\} - \left\{ \binom{n + r}{k + r} \right\} \right)
\]
\[
- \sum_{k=n+\varphi(q)+1}^{q-r-1} (k + r)! \left\{ \binom{n + \varphi(q) + r}{k + r} \right\} + \sum_{k=n+1}^{q-r-1} (k + r)! \left\{ \binom{n + r}{k + r} \right\}
\]
\[
\equiv \sum_{k=0}^{q-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} \binom{(j-1)!}{(j-r)!} (j^{\varphi(q)} - 1)
\]
\[
- \sum_{k=n+\varphi(q)+1}^{q-r-1} (k + r)! \left\{ \binom{n + \varphi(q) + r}{k + r} \right\} + \sum_{k=n+1}^{q-r-1} (k + r)! \left\{ \binom{n + r}{k + r} \right\} \pmod{q}.
\]
Since \( n \geq m - 1 \), in the indices where \( j = cp, c \in \mathbb{N} \), we have \( j^{n+1} = (cp)^{m+h} \), for some \( h \geq 0 \), and it is deduced that \( j^{n+1} \equiv 0 \pmod{q} \). When \( \gcd(j, q) = 1 \), again \( j^{r(q)} - 1 \equiv 0 \pmod{q} \) by Euler’s theorem. In the sums \( \sum_{k=0}^{q-r-1} (k + r)!\{n+r\} \) and \( \sum_{k=0}^{q-r-1} (k + r)!\{n+r\} + 2 \), the upper parameter of the \( r \)-Stirling number is less than the lower one, and therefore these two sums are equal to zero. So

\[
F_{n+\varphi(q), r} - F_{n, r} \equiv \sum_{k=0}^{q-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} \binom{j+1}{j} \binom{j}{j-r} \left( j^{\varphi(q)} - 1 \right) \equiv 0 \pmod{q},
\]
and therefore

\[
F_{n+\varphi(q), r} \equiv F_{n, r} \pmod{q} \text{ for } m - 1 \leq n < q - r - 1. \tag{9}
\]

Combining results (8) and (9) gives \( F_{n+\varphi(q), r} \equiv F_{n, r} \pmod{q} \), for \( n \geq m - 1 \). \( \square \)

### 4 The \( r \)-Fubini residues modulo powers of 2

As in many other computations in number theory, the case of \( p = 2 \) has its own difficulties that require special attention. In the case of powers of 2, initially we calculate the residues of 2-Fubini numbers and then use the results in the case of the \( r \)-Fubini numbers. We classify the sequence of remainders of 2-Fubini numbers modulo \( 2^m, m \geq 7 \), in Theorem 6 and then, work on remainders of the \( r \)-Fubini numbers modulo \( 2^m, m \geq 7 \) in Theorem 9. The special cases will be proved in Theorems 4, 7 and 8. The trivial cases in which \( 2^m \leq r \) with period length 1 are omitted.

**Theorem 4.** If \( 3 \leq m \leq 6 \), then after the \( (m - 1) \)th term the sequence \( A_{2,2^m} \) has a period with length \( \omega(A_{2,2^m}) = 2 \).

**Proof.** By using the formula \( F_{n,2} = \sum_{k=0}^{n} (k + 2)!\{n+2\} \), we prove that \( F_{n+2,2} - F_{n,2} \equiv 0 \pmod{64} \). Then \( F_{n+2,2} - F_{n,2} \equiv 0 \pmod{2^m} \) for \( 3 \leq m \leq 5 \).

\[
F_{n+2,2} - F_{n,2} = \sum_{k=0}^{n+2} (k + 2)!\left\{\binom{n+4}{k+2}\right\} - \sum_{k=0}^{n} (k + 2)!\left\{\binom{n+2}{k+2}\right\}
\]

\[
\equiv \sum_{k=0}^{5} (k + 2)!\left(\binom{n+4}{k+2} - \binom{n+2}{k+2}\right)
\]

\[
\equiv \sum_{k=0}^{5} \sum_{j=2}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} \binom{j+1}{j} (j^2 - 1)(j - 1) \pmod{64}.
\]

In the case \( m = 6 \) then \( n \geq 5 \), so if \( j \) is even, then \( j^{n+1} = (2c)^{6+h} \), for some \( h \geq 0 \) and therefore \( 64 \mid j^{n+1} \). For odd \( j \) we have \( \gcd(j, 64) = 1 \), so by Euler’s theorem we have

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\(j^{32} \equiv 1 \pmod{64}\), and therefore \(j^{n+1+32} \equiv j^{n+1} \pmod{64}\). This implies that

\[
F_{n+2,2} - F_{n,2} \equiv \sum_{k=0}^{\lfloor (k+1)/2 \rfloor} \sum_{l=1}^{5} (-1)^{k+2-(2l+1)} \binom{k + 2}{2l + 1} (2l + 1)^{n+1} ((2l + 1)^2 - 1) \times 2l
\]

\[
\equiv 16 \sum_{k=0}^{\lfloor (k+1)/2 \rfloor} (-1)^{k+1} \sum_{l=1}^{5} \binom{k + 2}{2l + 1} (2l + 1)^{n+1} \left(\frac{l(l+1)}{2}\right) l \pmod{64}.
\]

Enumerating the last summation for \(2 \leq n \leq 33\) shows that it is divisible by 64 and because of periodicity of remainders of \(j^{n+1} \pmod{64}\), the result follows.

Analogous to Lemma 2, it can be easily deduced by induction, showing that for each positive integer \(m > 1\) we have

\[
2^m - 2 \geq m. \tag{10}
\]

This can be shown by using the relation \(2^{m+1} \geq 2m + 4 > m + 3\), for \(m > 1\). The following lemma provides a simple but essential relation used in the next theorem. Its proof is provided in Appendix A.

**Lemma 5.** For \(m \geq 7\) and \(5 \leq i \leq 2^{m-6}\) we have \(2^{m-6} - i \mid 2^{i-5} \cdot \binom{2^{m-6}-1}{i} \).

**Theorem 6.** If \(m \geq 7\), after the \((m-1)th\) term, the sequence \(A_{2,2^m}\) has a period with length \(\omega(A_{2,2^m}) = 2^{m-6}\).

**Proof.** In the case of \(n \geq 2^m - 3\), from (10) we can deduce that \(n \geq 2^m - 3 \geq m - 1\). So we have

\[
F_{n+2^{m-6},2} - F_{n,2} \equiv \sum_{k=0}^{n+2^{m-6}} (k + 2)! \left\{ \binom{n + 2^{m-6} + 2}{k + 2} \right\} - \sum_{k=0}^{n} (k + 2)! \left\{ \binom{n + 2}{k + 2} \right\}
\]

\[
\equiv \sum_{k=0}^{2^m-3} (k + 2)! \left\{ \binom{n + 2^{m-6} + 2}{k + 2} \right\} - \sum_{k=0}^{n} (k + 2)! \left\{ \binom{n + 2}{k + 2} \right\}
\]

\[
\equiv \sum_{k=0}^{2^m-3} \sum_{j=2}^{k+2} (-1)^{k+2-j} \binom{k + 2}{j} j^{n+1} (2^{m-6} - 1)(j - 1) \pmod{2^m}.
\]

When \(j\) is even, then \(j^{n+1} = (2c)^{2^{m-2}+h}\), for some \(h \geq 0\). So by (10), \(2^m \mid j^{n+1}\). For odd \(j\) we have

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\[ F_{n+2^{m-6}, 2} - F_{n, 2} = \sum_{k=0}^{2^{m-3}} \sum_{l=1}^{[k+1]/2} (-1)^{k+2-(2l+1)} \binom{k+2}{2l+1} (2l+1)^{n+1}((2l+1)2^{m-6} - 1) \times 2l \]

\[ = 2^{m-4} \sum_{k=0}^{2^{m-3}} (-1)^{k+1} \sum_{l=1}^{[k+1]/2} \binom{k+2}{2l+1} (2l+1)^{n+1} \left( \frac{(2l+1)2^{m-6} - 1}{2^{m-5}} \right) l \]

\[ = 2^{m-4} \sum_{k=0}^{2^{m-3}} (-1)^{k+1} \sum_{l=1}^{[k+1]/2} \binom{k+2}{2l+1} (2l+1)^{n+1} \sum_{i=1}^{2^{m-6}} l^{2i-1} \left( \frac{(2^{m-6} - 1)!}{i!(2^{m-6} - i)!} \right) \times l \pmod{2^m}. \]

The last expression contains \( m - 4 \) factors of 2, so it is sufficient to prove that the last summation is divisible by 16. This summation is denoted by \( S \). Simplify the summation \( \sum_{i=1}^{2^{m-6}} l^{i2i-1} \frac{(2^{m-6} - 1)!}{i!(2^{m-6} - i)!} \) and using Lemma 5 gives

\[ \sum_{i=1}^{2^{m-6}} l^{i2i-1} \frac{(2^{m-6} - 1)!}{i!(2^{m-6} - i)!} \equiv \sum_{i=1}^{4} l^{i2i-1} \frac{(2^{m-6} - 1)!}{i!(2^{m-6} - i)!} \equiv l + l^2(2^{m-6} - 1) \]

\[ + \frac{l^3 \times 2(2^{m-6} - 1)(2^{m-6} - 2)}{3} + l^4(2^{m-6} - 1)(2^{m-6} - 2)(2^{m-6} - 3) \pmod{16}. \]

Assume \( m \geq 10 \) (the case \( 7 \leq m \leq 9 \) is studied at the end of the proof). So \( 16 \mid 2^{m-6} \). Let \( 3a = 2(2^{m-6} - 1)(2^{m-6} - 2) \) and \( 3b = (2^{m-6} - 1)(2^{m-6} - 2)(2^{m-6} - 3) \). Then \( 3a \equiv 4 \pmod{16} \) and \( 3b \equiv -6 \pmod{16} \). Therefore \( a \equiv -4 \pmod{16} \) and \( b \equiv -2 \pmod{16} \). So the proof continues as follows:

\[ S \equiv \sum_{k=0}^{2^{m-3}} (-1)^{k+1} \sum_{l=1}^{[k+1]/2} \binom{k+2}{2l+1} (2l+1)^{n+1}(l - l^2 - 4l^3 - 2l^4) \pmod{16} \]

\[ S \equiv \sum_{k=0}^{2^{m-3}} (-1)^{k+1} \binom{k+2}{2l+1} (2l+1)^{n+1} \left( \frac{l(l+1)}{2} \right) (-2l^2 - 2l + 1)l \pmod{8}. \]

Let \( P(l) \) and \( A(k, r, n) \) be the remainder of \( \frac{1}{2}(2l+1)^{n+1}(l(l+1))(-2l^2 - 2l + 1)l \) and \( \sum_{l=-\infty}^{k+1} \frac{k+2}{2l+r} P(l) \) divided by 8, respectively. By Pascal’s identity, we have \( \binom{k+2}{2l+r} = \binom{k+1}{2l+r-1} \) and therefore

\[ \sum_{l=-\infty}^{\infty} \binom{k+2}{2l+r} P(l) = \sum_{l=-\infty}^{\infty} \binom{k+1}{2l+r} P(l) + \sum_{l=-\infty}^{\infty} \binom{k+1}{2l+r-1} P(l), \]

so

\[ A(k, r, n) = A(k - 1, r, n) + A(k - 1, r - 1, n). \]
We can write

\[ A(k, r + 32, n) \equiv \sum_{l=-\infty}^{\infty} \left( \frac{k + 2}{2l + r + 32} \right) P(l) \pmod{8}. \]

The sequence \((P(l))_{l=-\infty}^{\infty}\) has period 16, so \(P(l + 16) = P(l)\). Set \(l' = l + 16\), then

\[ A(k, r + 32, n) \equiv \sum_{l'=-\infty}^{\infty} \left( \frac{k + 2}{2l' + r} \right) P(l') \equiv A(k, r, n) \pmod{8}. \]  

(12)

Since \(\gcd(2l + 1, 16) = 1\), Euler’s theorem implies \((2l + 1)^8 \equiv 1 \pmod{16}\) and therefore \((2l + 1)^{n+1+8} \equiv (2l + 1)^{n+1} \pmod{16}\). The quantity \(A(6, r, n)\) vanishes for \(1 \leq r \leq 32\) and \(9 \leq n \leq 24\), by enumeration, then by (11) and (12), we deduce that

\[ A(k, r, n) = 0, \text{ for } k \geq 6. \]  

(13)

Therefore

\[ A(k, 1, n) \equiv \sum_{l=-\infty}^{\infty} \left( \frac{k + 2}{2l + 1} \right) (2l + 1)^{n+1} \left( \frac{l(l + 1)}{2} \right) (-2l^2 - 2l + 1)l \]

\[ \equiv \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \left( \frac{k + 2}{2l + 1} \right) (2l + 1)^{n+1} \left( \frac{l(l + 1)}{2} \right) (-2l^2 - 2l + 1)l \equiv 0 \pmod{8}, \]

for \(k \geq 6\). If \(1 \leq k \leq 5\), \(9 \leq n \leq 24\) and \(1 \leq r \leq 32\) we have \(\sum_{k=1}^{5} (-1)^{k+1} A(k, r, n) \equiv 0 \pmod{8}\). The period length of \(A(k, r, n)\) with respect to \(r\) and \(n\) implies that

\[ \sum_{k=1}^{5} (-1)^{k+1} A(k, 1, n) \equiv 0 \pmod{8}, \text{ for } n \geq 9. \]

Combining this with (13) we have

\[ S \equiv \sum_{k=1}^{2^{m-3}} (-1)^{k+1} A(k, 1, n) \equiv 0 \pmod{8}, \text{ for } n \geq 0. \]
So the result follows in the case of \( n \geq 2^m - 3 \). If \( m - 1 \leq n < 2^m - 3 \) we can write

\[
F_{n+2^m-6,2} - F_{n,2} = \sum_{k=0}^{n+2^m-6} (k+2)! \left\{ \frac{n+2^m-6+2}{k+2} \right\} - \sum_{k=0}^{n} (k+2)! \left\{ \frac{n+2}{k+2} \right\}
\]

\[
= \sum_{k=0}^{2^m-3} (k+2)! \left( \left\{ \frac{n+2^m-6+2}{k+2} \right\} - \left\{ \frac{n+2}{k+2} \right\} \right)
\]

\[
- \sum_{k=n+2^m-6+1}^{2^m-3} (k+2)! \left\{ \frac{n+2^m-6+2}{k+2} \right\} + \sum_{k=n+1}^{2^m-3} (k+2)! \left\{ \frac{n+2}{k+2} \right\}
\]

\[
\equiv \sum_{k=0}^{2^m-3} \sum_{j=1}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1} (j^{2^m-6} - 1)(j - 1) \pmod{2^m}.
\]

When \( j \) is even, then \( j^{n+1} = (2e)^{m+h} \), for some \( h \geq 0 \), so \( 2^m \mid j^{n+1} \). Since \( m \geq 10 \), for odd \( j \) we have

\[
\sum_{k=0}^{2^m-3} \sum_{j=1}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1} (j^{2^m-6} - 1)(j - 1)
\]

\[
\equiv 2^{m-4} \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\frac{(k+1)/2}{2l+1}} \binom{k+2}{2l+1} (2l+1)^{n+1}(l - l^2 - 4l^3 - 2l^4)l \pmod{2^m}.
\]

The last summation is exactly the \( S \) and the proof will be similar as above. Combine with the previous case we have the following congruence relation

\[
F_{n+2^m-6,2} \equiv F_{n,2} \pmod{2^m}, \text{ for } m \geq 10. \tag{14}
\]

In the case where \( 7 \leq m \leq 9 \), the remainder value of the sum

\[
\sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\frac{(k+1)/2}{2l+1}} \binom{k+2}{2l+1} (2l+1)^{n+1} \left( \sum_{i=1}^{4} l^i 2^{i-1} \frac{(2^m-6-1)!}{i!(2^m-6-i)!} \right) l
\]

modulo 16 is computed for \( m - 1 \leq n \leq m + 14 \). Divisibility of all these values by 16 implies that the recent sum is divisible by 16, and therefore

\[
F_{n+2^m-6,2} \equiv F_{n,2} \pmod{2^m}, \text{ for } 7 \leq m \leq 9. \tag{15}
\]

Summing up the congruence relations (14) and (15) gives

\[
\omega(A_{2,2^m}) = 2^{m-6}, \text{ for } m \geq 7.
\]

\[\square\]
Theorem 7. For \( m = 1 \) and \( m = 2 \), the sequence \( A_{r,2^m} \) is periodic from the first term and the period length is \( \omega(A_{r,2^m}) = 1 \).

Proof. The proof of this theorem is divided into three cases. For \( r = 2 \) we have

\[
F_{n+1,2} - F_{n,2} = \sum_{k=0}^{n+1} (k+2)! \binom{n+3}{k+2} - \sum_{k=0}^{n} (k+2)! \binom{n+2}{k+2}
\equiv 2 \left( \binom{n+3}{2} - \binom{n+2}{2} \right) + 6 \left( \binom{n+3}{3} - \binom{n+2}{3} \right) \pmod{4}
\equiv 2 \left( 2^{n+1} - 2^n \right) + 6 \left( 3^{n+1} - 2^{n+1} - 3^n - 2^n \right)
\equiv 2^{n+1} + 6(2 \times 3^n - 2^n) = 4(2^{n-1} + 3^{n+1} - 3 \times 2^{n-1})
\equiv 0 \pmod{4}.
\]

So we can deduce that \( \omega(A_{2,4}) = 1 \) and obviously \( \omega(A_{2,2}) = 1 \).

For \( r = 3 \) we can write

\[
F_{n+1,3} - F_{n,3} = \sum_{k=0}^{n+1} (k+3)! \binom{n+4}{k+3} - \sum_{k=0}^{n} (k+3)! \binom{n+3}{k+3}
\equiv 6 \left( \binom{n+4}{3} - \binom{n+3}{3} \right) \pmod{4}
\equiv 6(3^{n+1} - 3^n) = 6 \times 2 \times 3^n = 4 \times 3^{n+1} \equiv 0 \pmod{4}.
\]

Therefore we have \( \omega(A_{3,4}) = 1 \) and \( \omega(A_{3,2}) = 1 \).

Finally if \( r \geq 4 \), let \( r = 4 + h \), for some \( h \geq 0 \), then

\[
F_{n+1,r} - F_{n,r} = \sum_{k=0}^{n+1} (k+r)! \binom{n+1+r}{k+r} - \sum_{k=0}^{n} (k+r)! \binom{n+r}{k+r}.
\]

Since \( 4 \mid (k+r)! \), for all \( k \geq 0 \), we can write \( F_{n+1,r} - F_{n,r} \equiv 0 \pmod{4} \). Therefore \( \omega(A_{r,4}) = 1 \) and \( \omega(A_{r,2}) = 1 \).

Theorem 8. If \( 3 \leq m \leq 6 \), after the \( (m - 1) \)th term, the sequence \( A_{r,2^m} \) has a period with length \( \omega(A_{r,2^m}) = 2 \).

Proof. The proof of this theorem is similar to the proof of Theorem 4. It is enough to prove the theorem for \( m = 6 \); then the result follows for \( m = 3, 4 \) and 5. Since \( n \geq m - 1 \), then
for $m = 6$ we have $n \geq 5$. For $3 \leq r \leq 7$ we have

$$F_{n+2,r} - F_{n,r} = \sum_{k=0}^{n+2} (k+r)! \left\{ \binom{n+2+r}{k+r} - \binom{n}{k+r} \binom{n+r}{r} \right\} - \sum_{k=0}^{n} (k+r)! \left\{ \binom{n+r}{k+r} \binom{n+r}{r} \right\}$$

$$\equiv \sum_{k=0}^{7-r} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^2 - 1) \binom{(j-1)!}{(j-r)!} (j-1)^{n+1} \binom{(j-2)!}{(j-r)!} \pmod{64}.$$  

When $j$ is even, then $j^{n+1} = (2c)^{6+h}$, for some $h \geq 0$, and so $64 | j^{n+1}$. For odd $j$ we have $\gcd(j,64) = 1$ and Euler's theorem gives $j^{32} \equiv 1 \pmod{64}$. Therefore $j^{n+1+32} \equiv j^{n+1} \pmod{64}$, and we can write

$$\sum_{k=0}^{7-r} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^2 - 1) \binom{(j-1)!}{(j-r)!} \equiv 7 - r \sum_{k=0}^{7-r} \sum_{l=\lfloor r/2 \rfloor}^{\lfloor (k+r-1)/2 \rfloor} (-1)^{k+r-2l+1} \binom{k+r}{2l+1} (2l+1)^{n+1} \frac{(2l+1)^2 - 1}{2l} \binom{2l+1-1}{2l+1-r} \pmod{64}.$$  

By computation we see that the recent summation is divisible by 4, for $2 \leq n \leq 33$. So the proof for $3 \leq r \leq 7$ is completed.

If $r \geq 8$, since $64 | 8!$, then $64 | (k+r)!$, and

$$F_{n+2,r} - F_{n,r} = \sum_{k=0}^{n+2} (k+r)! \left\{ \binom{n+2+r}{k+r} - \binom{n}{k+r} \binom{n+r}{r} \right\} - \sum_{k=0}^{n} (k+r)! \left\{ \binom{n+r}{k+r} \binom{n+r}{r} \right\} \equiv 0 \pmod{64},$$

so $\omega(A_{r,26}) = 2$, for $r \geq 8$, and the proof is completed.

**Theorem 9.** If $m \geq 7$, after the $(m-1)\text{th}$ term, the sequence $A_{r,2m}$ has a period with length $\omega(A_{r,2m}) = 2^{m-6}$.

**Proof.** The proof of this theorem is similar to the proof of Theorem 6. In the case of
\[ n \geq 2^m - r - 1 \text{ and } r \geq 8 \text{ we have } \]
\[ F_{n+2^{m-6}, r} - F_{n, r} = \sum_{k=0}^{n+2^{m-6}} (k + r)! \left\{ \frac{n + 2^{m-6} + r}{k + r} \right\}_r - \sum_{k=0}^{n} (k + r)! \left\{ \frac{n + r}{k + r} \right\}_r, \]
\[ = \sum_{k=0}^{2^m - r - 1} (k + r)! \left( \left\{ \frac{n + 2^{m-6} + r}{k + r} \right\}_r - \left\{ \frac{n + r}{k + r} \right\}_r \right) \]
\[ = \sum_{k=0}^{2^m - r - 1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k + r}{j} j^{n+1} (j^{2^{m-6}} - 1) \frac{(j - 1)!}{(j - r)!} \pmod{2^m}. \]

In the case of \( 2^m > r > 2^m - m \), since \( m \geq 7 \) this implies that \( r > 2^m - m \geq 2^{m-1} \), so
\[ 2^m \mid (2^{m-1})! \mid (k + r)! \text{, for each } k \geq 0. \]

Therefore both summations in the above first equation are zero modulo \( 2^m \) and in this case \( \omega(A_{r, 2^m}) = 2^{m-6} \). When \( r \leq 2^m - m \), if \( j \) is even then \( j^{n+1} = (2c)^{2^m - r + h} \), for some \( h \geq 0 \). So \( 2^m \mid j^{n+1} \). For odd \( j \) we have \((j, 2^m-5) = 1\), and \( 2^m \mid j^{2^{m-6}} - 1 \) by Euler’s theorem. Since \( r \geq 8 \) we can write \((j-1)! \pmod{2^m} = \frac{(j-8)!}{(j-2)!} \sum_{i=1}^{7} (j-i) \). Therefore \( 32 \mid \frac{(j-1)!}{(j-2)!} \) and
\[ 2^m \mid (j^{2^{m-6}} - 1) \left( \frac{(j-1)!}{(j-2)!} \right). \]

In the case of \( m - 1 \leq n < 2^m - r - 1 \) and \( r \geq 8 \) we have
\[ F_{n+2^{m-6}, r} - F_{n, r} = \sum_{k=0}^{n+2^{m-6}} (k + r)! \left\{ \frac{n + 2^{m-6} + r}{k + r} \right\}_r - \sum_{k=0}^{n} (k + r)! \left\{ \frac{n + r}{k + r} \right\}_r, \]
\[ = \sum_{k=0}^{2^m - r - 1} (k + r)! \left( \left\{ \frac{n + 2^{m-6} + r}{k + r} \right\}_r - \left\{ \frac{n + r}{k + r} \right\}_r \right) \]
\[ - \sum_{k=n+2^{m-6}+1}^{2^m - r - 1} (k + r)! \left\{ \frac{n + 2^{m-6} + r}{k + r} \right\}_r + \sum_{k=n+1}^{2^m - r - 1} (k + r)! \left\{ \frac{n + r}{k + r} \right\}_r \pmod{2^m} \]
\[ = \sum_{k=0}^{2^m - r - 1} (k + r)! \left( \left\{ \frac{n + 2^{m-6} + r}{k + r} \right\}_r - \left\{ \frac{n + r}{k + r} \right\}_r \right) + 0, \]
and the proof proceeds as in the previous case. In the case of \( 3 \leq r \leq 7 \) one can deduce similarly to the proof of Theorem 6 that
\[ F_{n+2^{m-6}, r} - F_{n, r} \equiv \sum_{k=0}^{2^m - r - 1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k + r}{j} j^{n+1} (j^{2^{m-6}} - 1) \frac{(j - 1)!}{(j - r)!} \pmod{2^m}. \]

Exactly the same as Theorem 6, the terms with even \( j \) vanish and only the terms with odd
$j$ remain. So we have
\[ F_{n+2^m-6,r} - F_{n,r} = \sum_{k=0}^{2^m-1} \sum_{l=0}^{[k+r-1]/2} (-1)^{k+r-(2l+1)}\left(\frac{k+r}{2l+1}\right)(2l+1)^{n+1}(2l + 1)^{2^m-1} \]
\[ \times \left(\frac{(2l+1) - (r)}{(2l+1) - r}\right) \]
\[ \equiv 2^{m-5} \sum_{k=0}^{2^m-1} (-1)^{k+r+1} \sum_{l=0}^{[k+r+(1)/2]} \left(\frac{k+r}{2l+1}\right)(2l+1)^{n+1} \]
\[ \times \left(\sum_{i=1}^{2^{m-6}} i^2^{i-1} \frac{(2^{m-6} - i)!}{i!(2^{m-6} - i)!}\right) \left(\frac{(2l)!}{(2l - r + 1)!}\right) \pmod{2^m}. \]

Since $\gcd(2l + 1, 16) = 1$, Euler’s theorem shows that $(2l + 1)^{n+1+8} \equiv (2l + 1)^{n+1} \pmod{16}$. If $m \geq 10$, we have
\[ F_{n+2^m-6,r} - F_{n,r} = \sum_{k=0}^{2^m-1} (-1)^{k+r+1} \sum_{l=0}^{[k+r+(1)/2]} \left(\frac{k+r}{2l+1}\right)(2l+1)^{n+1} \left(\frac{l l+1}{2}\right) \]
\[ \times (-2l^2 - 2l + 1) \left(\frac{(2l)!}{(2l - r + 1)!}\right) \pmod{2^m}. \]

Therefore it is sufficient to compute the above summation (without factor $2^{m-4}$) for $3 \leq r \leq 7$ and $9 \leq n \leq 16$ to show that it is divisible by 16.

For $7 \leq m \leq 9$ we evaluate the sum
\[ \sum_{k=0}^{2^m-1} (-1)^{k+r+1} \sum_{l=0}^{[k+r+(1)/2]} \left(\frac{k+r}{2l+1}\right)(2l+1)^{n+1} \sum_{i=1}^{2^{m-6}} i^2^{i-1} \frac{(2^{m-6} - i)!}{i!(2^{m-6} - i)!}\left(\frac{(2l)!}{(2l - r + 1)!}\right) \]
for $m-1 \leq n \leq m+6$ to show that it is divisible by 32. Then it follows that $\omega(A_{r,2^m}) = 2^{m-6}$, for all $m \geq 7$.

\[ \square \]

5 The conclusion

We now state the final theorem, which shows how to compute $\omega(A_{r,s})$ for any $s \in \mathbb{N}$.

**Theorem 10.** Let $s \in \mathbb{N}$ and $s > 1$ with the prime factorization $s = 2^m p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ and let $D = \{p_i^{m_i} | p_i^{m_i} > r, 1 \leq i \leq k\}$. Define $E = \{m_i - 1 | p_i^{m_i} \in D\}$, $F = \{\varphi(p_i^{m_i}) | p_i^{m_i} \in D\}$ and $a = \max(E \cup \{m - 1\})$ and let $b$ be the least common multiple (lcm) of the elements of $F$. Then
\[ \omega(A_{r,s}) = \begin{cases} b, & \text{if } 0 \leq m \leq 2 \text{ or } 2^m \leq r; \\ \lcm(2, b), & \text{if } 3 \leq m \leq 6 \text{ and } 2^m > r; \\ \lcm(2^{m-6}, b), & \text{if } m \geq 7 \text{ and } 2^m > r, \end{cases} \] (16)
and periodicity of the sequence $A_{r,s}$ is seen after the $a$-th term.

Proof. Let $l$ be the right hand side of (16). For each $d \in D \cup \{2^m\}$, $\omega(A_{r,d}) \mid l$ and for each $p_j^{m_j} \not\in D$ such that $1 \leq j \leq k$, we have $1 = \omega(A_{r,p_j^{m_j}}) \mid l$, so

$$F_{n+l,r} \equiv F_{n,r} \pmod{2^m}$$

$$F_{n+l,r} \equiv F_{n,r} \pmod{p_i^{m_i}}, \text{ for } i = 1, 2, \ldots, k.$$  

Since $\gcd(2^m, p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k}) = 1$, the multiplication of all above congruence relations gives the required result. \qed

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A Proof of Lemma 5

After simplifying the lemma’s relation we have

$$2^{i-5}(\binom{2^m-6}{i} - 1) \cdot \frac{2^{i-5}(2^m-6-1)(2^m-6-2) \cdots (2^m-6-i+1)}{i!}.$$  

It is sufficient to show that the right hand side of (17) is integer. We know that $\binom{2^m-6}{i} \in \mathbb{N}$, i.e.,

$$i! \mid 2^{m-6}(2^m-6-1) \cdots (2^m-6-i+1).$$

If $O_i$ denotes the product of the odd factors of $i!$, since $(O_i, 2^{m-6}) = 1$, then $O_i \mid (2^{m-6} - 1) \cdots (2^{m-6} - i + 1)$. So in (17) we only need to prove that

$$\nu_2(2^{i-5}(2^m-6-1)(2^m-6-2) \cdots (2^m-6-i+1)) \geq \nu_2(i!),$$

where by $\nu_2(x)$ we mean that $2^{\nu_2(x)} \mid x$, but $2^{\nu_2(x)+1} \nmid x$. Let $A = \nu_2((2^m-6-1)(2^m-6-2) \cdots (2^m-6-i+1))$ and $B = \nu_2(i!)$. Let $e$ be the unique integer such that $2^e \leq i < 2^{e+1}$. So

$$A = \sum_{k=1}^{e} \left\lfloor \frac{i-1}{2^k} \right\rfloor, \quad B = \sum_{k=1}^{e} \left\lfloor \frac{i}{2^k} \right\rfloor.$$  

If we show that

$$B - A \leq e$$  

(19)
then the lemma is concluded if it is proved that

$$i + A \geq B + 5.$$  \hspace{1cm} (20)

It can easily be shown that $B = \nu_2(i!)$ and $A = \nu_2((i - 1)!)$, so $B - A = \nu_2(i)$. Since $2^e \leq i < 2^{e+1}$, therefore $\nu_2(i) \leq e$ and (19) follows. For $e = 2$, integer possibilities for inequality (20) are as follows:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

For $e \geq 3$ one can deduce by simple induction that

$$2^e \geq e + 5,$$

so $i \geq 2^e \geq e + 5$. Add $B - e$ to these inequalities and use (19) demonstrates (20) for $i \geq 8$.

References


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