On the Reciprocal Sums of Products of Fibonacci Numbers

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Abstract
In this paper we study the reciprocal sums of products of two different Fibonacci numbers. We obtain some identities related to the numbers \( \lfloor (\sum_{k=n}^\infty 1/F_k F_{k+m})^{-1} \rfloor \), \( m \geq 1 \), where \( \lfloor \cdot \rfloor \) indicates the floor function.

1 Introduction
As is well known, the Fibonacci numbers \( F_n \) are generated from the recurrence relation
\[
F_n = F_{n-1} + F_{n-2} \quad (n \geq 2),
\]
with initial condition \( F_0 = 0 \) and \( F_1 = 1 \).

Recently Ohtsuka and Nakamura [7] found interesting properties of the Fibonacci numbers and proved Theorem 1 below.

Theorem 1. For the Fibonacci numbers, the following identities hold:
\[
\left\lfloor \left( \sum_{k=n}^\infty \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} 
F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \text{ is even}; \\
F_n - F_{n-1} - 1, & \text{if } n \geq 3 \text{ and } n \text{ is odd},
\end{cases}
\]
(1)

\[
\left\lfloor \left( \sum_{k=n}^\infty \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} 
F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ and } n \text{ is even}; \\
F_{n-1}F_n, & \text{if } n \geq 3 \text{ and } n \text{ is odd}.
\end{cases}
\]
(2)
Following the paper of Ohtsuka and Nakamura [7], diverse results in the same direction have been reported in the literature [1, 2, 3, 5, 8, 9, 10, 11, 12, 13]. Among them, Liu and Wang [5] considered the product of two reciprocal Fibonacci numbers, and obtained several interesting results. For example, they proved Theorem 2 below for the products of two consecutive Fibonacci numbers.

**Theorem 2.** Let \( m \geq 2 \). Then

\[
\left\lfloor \left( \sum_{k=n}^{mn} \frac{1}{F_k F_{k+1}} \right)^{-1} \right\rfloor = \begin{cases} 
F_{n}^2, & \text{if } n \geq 2 \text{ and } n \text{ is even;} \\
F_{n}^2 - 1, & \text{if } n \geq 3 \text{ and } n \text{ is odd.}
\end{cases}
\] (3)

Motivated by Theorem 2, we study the reciprocal sums of products of two different Fibonacci numbers in this paper. We obtain some identities related to the numbers

\[
\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} \right\rfloor, \quad m \geq 1.
\]

**Remark 3.** The following identity was conjectured by Ohtsuka and proved by Bruckman [6]:

\[
\left( \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} = \sum_{k=1}^{n-1} F_k F_{k+m} - \frac{1}{3} F_{m-2}(-1)^n + O\left( \frac{1}{F_n^2} \right), \quad m \geq 0.
\]

For the case where \( m = 0 \) and \( n \) is large, (2) also can be derived from the above result.

## 2 Main results

We will use Lemma 4 below to prove our main results.

**Lemma 4** (Koshy [4]). For the Fibonacci numbers, we have

\[ F_m F_n - F_{m+k} F_{n-k} = (-1)^{n-k} F_{m-n+k} F_k. \]

Our main results are stated in the following theorem.

**Theorem 5.** For the Fibonacci numbers, (a), (b) and (c) below hold:

(a) Let \( m \geq 1 \). If

\[
\frac{2F_m - F_{m+1}}{3} \notin \mathbb{Z},
\]

then there exist positive integers \( n_0 \) and \( n_1 \) such that

\[
\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} 
F_{n+m-1} F_n + g_m - 1, & \text{if } n \geq n_0 \text{ and } n \text{ is even;} \\
F_{n+m-1} F_n - g_m, & \text{if } n \geq n_1 \text{ and } n \text{ is odd.}
\end{cases}
\] (4)
where

\[ g_m = \left\lfloor \frac{2F_m - F_{m+1}}{3} \right\rfloor + 1. \]

(b) For \( m = 2 \),

\[ \left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} \right\rfloor = F_{n+m-1}F_n, \text{ for } n \geq 1. \]  \hspace{1cm} (5)

(c) Let \( m \geq 3 \). If

\[ \frac{2F_m - F_{m+1}}{3} \in \mathbb{Z}, \]

then there exist positive integers \( n_2 \) and \( n_3 \) such that

\[ \left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} F_{n+m-1}F_n + \hat{g}_m - 1, & \text{if } n \geq n_2 \text{ and } n \text{ is even;} \\ F_{n+m-1}F_n - \hat{g}_m - 1, & \text{if } n \geq n_3 \text{ and } n \text{ is odd,} \end{cases} \] \hspace{1cm} (6)

where

\[ \hat{g}_m = \frac{2F_m - F_{m+1}}{3}. \]

Proof. (a) To prove (4), consider

\[ X_1 = \frac{1}{F_{n+m-1}F_n + (-1)^n g_m} - \frac{1}{F_{n+m+1}F_{n+2} + (-1)^n g_m} - \frac{1}{F_{n+m}F_{n+m+1}} \]

\[ = \frac{1}{F_{n+m-1}F_n + (-1)^n g_m}\{F_{n+m+1}F_{n+2} + (-1)^n g_m\}F_{n+m+1}F_{n+1}F_{n+m+1}; \]

where, by the identity

\[ F_{n+m+1}F_{n+2} - F_{n+m-1}F_n = F_nF_{n+m} + F_{n+1}F_{n+m+1}, \]

\[ \hat{X}_1 = (F_nF_{n+m} + F_{n+1}F_{n+m+1})\hat{X}_1, \]

with

\[ \hat{X}_1 = F_nF_{n+1}F_{n+m}F_{n+m+1} - F_{n+m-1}F_{n+m+1}F_{n}F_{n+2} \]

\[ - (-1)^ng_m(F_{n+m-1}F_n + F_{n+m+1}F_{n+2}) - g_m^2. \]

From Lemma 4, we have

\[ F_{n+1}F_{n+m} - F_{n+m+1}F_n = (-1)^nF_m, \]
\[ F_{n+m+1}F_n - F_{n+m-1}F_{n+2} = (-1)^n(F_m - F_{m+1}), \]
\[ F_{n+m+1}F_{n-1} - F_{n+m}F_n = (-1)^nF_{m+1}. \]
Then
\[
F_n F_{n+1} F_{n+m} F_{n+m+1} - F_{n+m-1} F_{n+m+1} F_n F_{n+2} = F_{n+m+1} F_n \left\{ F_{n+m+1} F_n + (-1)^n F_m \right\}
\]
\[
- F_{n+m+1} F_n \left\{ F_{n+m+1} F_n + (-1)^n (F_m + 1 - F_m) \right\}
\]
\[
= (-1)^n F_{n+m+1} F_n (2F_m - F_{m+1}),
\]
and
\[
F_{n+m-1} F_n + F_{n+m+1} F_{n+2} = 3F_{n+m+1} F_n + F_{n+m+1} F_{n-1} - F_{n+m} F_n
\]
\[
= 3F_{n+m+1} F_n + (-1)^n F_m+1.
\]
Hence
\[
\tilde{X}_1 = (-1)^n F_{n+m+1} F_n (2F_m - F_{m+1} - 3g_m) - g_m F_{m+1} - g_m^2.
\]
Assume that \( n \) is even. Since \( g_m > 0 \) and \( 2F_m - F_{m+1} - 3g_m < 0 \), then \( X_1 < 0 \) and
\[
\frac{1}{F_{n+m-1} F_n + g_m} - \frac{1}{F_{n+m+1} F_{n+2} + g_m} < \frac{1}{F_{n+m+1} F_n} + \frac{1}{F_{n+1} F_{n+m+1}}.
\]
Repetedly applying the above inequality, we have
\[
\frac{1}{F_{n+m-1} F_n + g_m} < \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}}, \text{ if } n \geq 2 \text{ and } n \text{ is even.} \tag{7}
\]
Similarly, if \( n \) is odd, then there exists a positive integer \( m_1 \) such that, for \( n \geq m_1 \), \( X_1 > 0 \) and
\[
\frac{1}{F_{n+m} F_n} + \frac{1}{F_{n+1} F_{n+m+1}} < \frac{1}{F_{n+m-1} F_n - g_m} - \frac{1}{F_{n+m+1} F_{n+2} - g_m},
\]
from which we obtain
\[
\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} < \frac{1}{F_{n+m-1} F_n - g_m}, \text{ if } n \geq m_1 \text{ and } n \text{ is odd.} \tag{8}
\]
Next, consider
\[
X_2 = \frac{1}{F_{n+m-1} F_n + (-1)^n g_m - 1} - \frac{1}{F_{n+m} F_{n+1} + (-1)^{n+1} g_m - 1} - \frac{1}{F_{n} F_{n+m}}
\]
\[
= \frac{\tilde{X}_2}{\{F_{n+m-1} F_n + (-1)^n g_m - 1\}\{F_{n+m} F_{n+1} + (-1)^{n+1} g_m - 1\} F_n F_{n+m}},
\]
where
\[
\dot{X}_2 = F_n F_{n+m}^2 F_{n+1} - F_{n+m} F_{n+m-1} F_n F_{n+1} - F_n^2 F_{n+m-1} F_{n+m} \\
-(-1)^n g_m (2F_n F_{n+m} - F_{n+m-1} F_n + F_{n+m} F_{n+1}) \\
+ F_{n+m-1} F_n + F_{n+m} F_{n+1} + g_m^2 - 1.
\]

From Lemma 4, we have
\[
F_{n+m-1} F_n - F_{n+m-2} F_{n+1} = (-1)^{n+1} F_{m-2} = (-1)^n (F_{m+1} - 2F_m).
\]

Then
\[
F_n F_{n+m} F_{n+1} F_{n+m} - F_{n+m} F_{n+m-1} F_{n+1} - F_n^2 F_{n+m-1} F_{n+m} \\
= F_n F_{n+m} (F_{n+1} F_{n+m-2} - F_n F_{n+m-1}) \\
= (-1)^n F_n F_{n+m} (2F_{m+2} - F_{m+1}),
\]

and
\[
2F_n F_{n+m} + F_{n+m} F_{n+1} - F_{n+m-1} F_n \\
= 3F_n F_{n+m} + F_{n+m} F_{n-1} - F_{n+m-1} F_n \\
= 3F_n F_{n+m} + (-1)^n (2F_{m+2} - F_{m+3}).
\]

Hence
\[
\dot{X}_2 = (-1)^n F_n F_{n+m} (2F_m - F_{m+1} - 3g_m) + F_{n+m-1} F_n + F_{n+m} F_{n+1} \\
- g_m (2F_{m+2} - F_{m+3}) + g_m^2 - 1.
\]

Suppose that \( n \) is even. Since
\[
-2 \leq 2F_m - F_{m+1} - 3g_m \leq -1,
\]

then
\[
F_n F_{n+m} (2F_m - F_{m+1} - 3g_m) + (F_{n+m-1} F_n + F_{n+m} F_{n+1}) \\
\geq -2F_n F_{n+m} + F_n F_{n+m-1} + F_{n+m} F_{n+1} \\
= (F_{n-1} - F_n) (F_{n+m-1} + F_{n-m-2}) + F_n F_{n+m-1} \\
= F_{n-1} F_{n+m-1} - F_{n-2} F_{n+m-2} \\
> 0,
\]

and there exists a positive integer \( m_2 \) such that, for \( n \geq m_2 \), \( X_2 > 0 \) and
\[
\frac{1}{F_n F_{n+m}} < \frac{1}{F_{n+m-1} F_n + (-1)^n g_m - 1} - \frac{1}{F_{n+m} F_{n+1} + (-1)^{n+1} g_m - 1}.
\]
Repeatedly applying the above inequality, we have
\[
\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} < \frac{1}{F_{n+m-1} F_n + g_m - 1}, \quad \text{if } n \geq m_2 \text{ and } n \text{ is even.} \quad (9)
\]

On the other hand,
\[
X_3 = \frac{1}{F_{n+m-1} F_n + (-1)^n g_m + 1} - \frac{1}{F_{n+m} F_{n+1} + (-1)^{n+1} g_m + 1} - \frac{1}{F_n F_{n+m}}
\]
\[
= \frac{1}{\{F_{n+m-1} F_n + (-1)^n g_m + 1\}\{F_{n+m} F_{n+1} + (-1)^{n+1} g_m + 1\} F_n F_{n+m}},
\]
where
\[
\hat{X}_3 = \hat{X}_2 - 2(F_{n+m-1} F_n + F_{n+m} F_{n+1})
\]
\[
= (-1)^n F_n F_{n+m}(2F_m - F_{m+1} - 3g_m) - F_{n+m-1} F_n - F_{n+1} F_{n+1}
\]
\[
- g_m(2F_{m+2} - F_{m+3}) + g^2 - 1.
\]

Suppose that \(n\) is odd. As shown above, we have
\[
-F_n F_{n+m}(2F_m - F_{m+1} - 3g_m) - F_{n+m-1} F_n - F_{n+m} F_{n+1} < F_{n-2} F_{n+m-2} - F_{n-1} F_{n+m-1}.
\]
Hence there exists a positive integer \(m_3\) such that, for \(n \geq m_3\), \(X_3 < 0\) and
\[
\frac{1}{F_{n+m-1} F_n + (-1)^n g_m + 1} - \frac{1}{F_{n+m} F_{n+1} + (-1)^{n+1} g_m + 1} < \frac{1}{F_n F_{n+m}},
\]
from which we have
\[
\frac{1}{F_{n+m-1} F_n - g_m + 1} < \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}}, \quad \text{if } n \geq m_3 \text{ and } n \text{ is odd.} \quad (10)
\]

Then, (4) follows from (7), (8), (9) and (10).

(b) Since \(F_{n+2} F_{n+3} - F_n F_{n+1} = F_n F_{n+2} + F_{n+1} F_{n+3}\), we have
\[
\frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+2} F_{n+3}} = \frac{1}{F_{n+2} F_{n+3}} - \frac{1}{F_n F_{n+1}} = \frac{F_{n+2} F_{n+3} - F_n F_{n+1} - (F_n F_{n+2} + F_{n+1} F_{n+3})}{F_n F_{n+1} F_{n+2} F_{n+3}} = 0,
\]
i.e.,
\[
\frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+2} F_{n+3}} = \frac{1}{F_{n+1} F_{n+2}} + \frac{1}{F_{n+1} F_{n+3}}.
\]
Repeatedly applying the above equality, we obtain (5).
(c) Let \( m \geq 3 \) and assume that
\[
\hat{g}_m = \frac{2F_m - F_{m+1}}{3} \in \mathbb{Z}.
\]
We recall the proof of (a). Replacing \( g_m \) by \( \hat{g}_m \) in \( \tilde{X}_1 \), we have
\[
\tilde{X}_1 = -\hat{g}_m F_{m+1} - \hat{g}_m^2 < 0.
\]
Then \( X_1 < 0 \) if \( n \geq 2 \) and \( n \) is even or if \( n \geq m_4 \) and \( n \) is odd for some positive integer \( m_4 \), and we have
\[
\frac{1}{F_{n+m-1}F_n + (-1)^n \hat{g}_m} < \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}}, \quad \text{if } n \geq 2 \text{ (} n \text{ is even) or if } n \geq m_4 \text{ (} n \text{ is odd).} \quad (11)
\]
Similarly there exist positive integers \( m_5 \) and \( m_6 \) such that \( X_2 > 0 \) if \( n \geq m_5 \) and \( n \) is even, or if \( n \geq m_6 \) and \( n \) is odd, from which we have
\[
\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} < \frac{1}{F_{n+m-1}F_n + (-1)^n \hat{g}_m - 1}, \quad \text{if } n \geq m_5 \text{ (} n \text{ is even) or if } n \geq m_6 \text{ (} n \text{ is odd).} \quad (12)
\]
Then, (6) follows from (11) and (12).

Remark 6. From Theorem 5, we have
\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+1}} \right)^{-1} \right] = \begin{cases} F_n^2, & \text{if } n \geq 2 \text{ and } n \text{ is even;} \\
F_n^2 - 1, & \text{if } n \geq 1 \text{ and } n \text{ is odd,} \end{cases}
\]
\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+6}} \right)^{-1} \right] = \begin{cases} F_{n+5}F_n, & \text{if } n \geq 2 \text{ and } n \text{ is even;} \\
F_{n+5}F_n - 2, & \text{if } n \geq 1 \text{ and } n \text{ is odd,} \end{cases}
\]

etc.

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References


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