Spivey’s Bell Number Formula Revisited

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Abstract
This paper introduces an alternative form of the derivation of Spivey’s Bell number formula, which involves the $q$-Boson operators $a$ and $a^\dagger$. Furthermore, a similar formula for the case of the $(q, r)$-Dowling polynomials is obtained, and is shown to produce a generalization of the latter.

1 Introduction

Consider the Stirling numbers of the second kind, denoted by $\{^m_j\}$, which appear as coefficients in the expansion of

$$t^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} (t)_k,$$

where $(t)_k = t(t - 1)(t - 2) \cdots (t - k + 1)$. The Bell numbers, denoted by $B_n$, are defined by

$$B_n = \sum_{j=0}^{n} \left\{ \begin{array}{c} n \\ j \end{array} \right\}$$

and are known to satisfy the recurrence relation

$$B_{n+1} = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) B_k.$$

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In 2008, Spivey [13] obtained a remarkable formula which unifies the defining relation in (2) and the identity (3). The said formula is given by

\[ B_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} j^{n-k} \binom{m}{j} \binom{n}{k} B_k \]  

(4)

and is popularly known as “Spivey’s Bell number formula”. Equation (4) was proved in [13] using a combinatorial approach involving partition of sets. Different proofs and extensions of (4) were later on studied by several authors. For instance, a proof which made use of generating functions was done by Gould and Quaintance [5] which was then generalized by Xu [14] using Hsu and Shuie’s [6] generalized Stirling numbers. Belbachir and Mihoubi [2] presented a proof that involves decomposition of the Bell polynomials into a certain polynomial basis. Mező [12] obtained a generalization of the Spivey’s formula in terms of the \( r \)-Bell polynomials via combinatorial approach. The notion of dual of (4) was also presented in the same paper. On the other hand, the work of Katriel [7] involved the use of the operator \( X \) satisfying

\[ DX - qXD = 1, \]

(5)

where \( D \) is the \( q \)-derivative defined by

\[ Df(x) = \frac{f(qx) - f(x)}{x(q - 1)}. \]

(6)

For the sake of clarity and brevity, this method will be referred to as “Katriel’s proof”.

Now, aside from being implicitly implied in Katriel’s proof, none of the previously-mentioned studies considered establishing \( q \)-analogues. It is, henceforth, the main purpose of this paper to obtain a generalized \( q \)-analogue of Spivey’s Bell number formula.

## 2 Alternative form of “Katriel’s proof”

We direct our attention to the \( q \)-Boson operators \( a \) and \( a^\dagger \) satisfying the commutation relation

\[ [a, a^\dagger]_q = aa^\dagger - qa^\dagger a = 1 \]  

(see [1]). We define the Fock space (or Fock states) by the basis \( \{|s\}; s = 0, 1, 2, \ldots \) so that the relations \( a |s\rangle = \sqrt{s} |s - 1\rangle \) and \( a^\dagger |s\rangle = \sqrt{s + 1} |s + 1\rangle \) form a representation that satisfies (7). The operators \( a^\dagger a \) and \( (a^\dagger)^{k} a^{k} \), when acting on \( |s\rangle \), yield

\[ a^\dagger a |s\rangle = [s]_q |s\rangle \]  

(8)

and

\[ (a^\dagger)^{k} a^{k} |s\rangle = [s]_{q,k} |s\rangle, \]  

(9)
respectively, where \( [s]_q = \frac{q^s - 1}{q - 1} \) and \([s]_{q,k} = [s]_q[s - 1]_q[s - 2]_q \cdots [s - k + 1]_q \). Hence, the \( q \)-Stirling numbers of the second kind \( \{ {n \atop k} \}_q \) \cite{3} can be defined alternatively as

\[
(a^\dagger a)^n = \sum_{k=1}^{n} \left\{ \begin{array}{c}
{n} \\
{k}
\end{array} \right\}_q (a^\dagger)^k a^k.
\] (10)

From (7), it is clear that

\[
[a, (a^\dagger)^k]_q = [a, (a^\dagger)^{k-1}]_q a^\dagger + q^{k-1} (a^\dagger)^{k-1} [a, a^\dagger]_q,
\] (11)

and by induction on \( k \), we have

\[
[a, (a^\dagger)^k]_q = [k]_q (a^\dagger)^{k-1}.
\] (12)

Since \( a \ket{0} = 0 \), then by (12),

\[
a(a^\dagger)^\ell \ket{0} = [a, (a^\dagger)^{\ell-1}]_q \ket{0} = [\ell]_q (a^\dagger)^{\ell-1} \ket{0}.
\]

Moreover,

\[
a^k(a^\dagger)^\ell \ket{0} = \frac{[\ell]_q^k}{[\ell - k]_q!} (a^\dagger)^{\ell-k} \ket{0},
\] (13)

for \( k \leq \ell \) and

\[
a^k(a^\dagger)^\ell \ket{0} = 0,
\] (14)

for \( k > \ell \). Finally,

\[
a^k e_q(xa^\dagger) \ket{0} = x^k e_q(xa^\dagger) \ket{0},
\] (15)

where \( e_q(xa^\dagger) \) is the \( q \)-exponential function defined by

\[
e_q(t) = \sum_{\ell=0}^{\infty} \frac{t^\ell}{[\ell]_q!}.
\] (16)

Applying (15) to (10) yields

\[
(a^\dagger a)^n e_q(ta^\dagger) \ket{0} = B_{n,q}(ta^\dagger) e_q(ta^\dagger) \ket{0},
\] (17)

where \( B_{n,q}(ta^\dagger) \) denotes the \( q \)-Bell polynomials defined by

\[
B_{n,q}(t) = \sum_{k=0}^{n} \left\{ \begin{array}{c}
n \\
k
\end{array} \right\}_q t^k.
\] (18)

Let \( x = ta^\dagger \) so that

\[
(a^\dagger a)^n e_q(x) \ket{0} = B_{n,q}(x) e_q(x) \ket{0}.
\] (19)
Before proceeding, note that by definition,

\[ [a, (a^\dagger) k] q^k = a (a^\dagger)^k - q^k (a^\dagger)^k a. \]  

(20)

By (12),

\[
a(a^\dagger)^k - q^k (a^\dagger)^k a = [k] q (a^\dagger)^{k-1}
\]

\[
a(a^\dagger)^k = q^k (a^\dagger)^k a + [k] q (a^\dagger)^{k-1}.
\]

This can be further expressed as

\[
(a^\dagger)(a^\dagger)^k = (a^\dagger)^k ([k] q + q^k (a^\dagger)a)).
\]

(21)

Now, we have

\[
(a^\dagger a)^{n+m} = (a^\dagger a)^n \sum_{j=0}^{m} \left\{ \begin{array}{c} m \\ j \end{array} \right\}_q (a^\dagger)^j a^j
\]

\[
= \sum_{j=0}^{m} \left\{ \begin{array}{c} m \\ j \end{array} \right\}_q (a^\dagger)^j ([j] q + q^j (a^\dagger)a)^n a^j
\]

\[
= \sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ \begin{array}{c} m \\ j \end{array} \right\}_q \left( \begin{array}{c} n \\ k \end{array} \right)_q [j] q^{n-k} q^{jk} (a^\dagger)^j (a^\dagger a)^k a^j.
\]

Multiplying both sides with \( e_q(x) |0\rangle \) makes the left-hand side

\[
(a^\dagger a)^{n+m} e_q(x) |0\rangle = B_{n+m,q}(x) e_q(x) |0\rangle,
\]

(22)

while the right-hand side becomes

\[
\sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ \begin{array}{c} m \\ j \end{array} \right\}_q \left( \begin{array}{c} n \\ k \end{array} \right)_q [j] q^{n-k} q^{jk} e_q(x) |0\rangle a^j = \sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ \begin{array}{c} m \\ j \end{array} \right\}_q \left( \begin{array}{c} n \\ k \end{array} \right)_q [j] q^{n-k} q^{jk}
\]

\[
B_{k,q}(x) e_q(x) |0\rangle (a^\dagger)^j a^j.
\]

Dividing both sides by \( e_q(x) |0\rangle \) and using (9) gives

\[
B_{n+m,q}(x) = \sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ \begin{array}{c} m \\ j \end{array} \right\}_q \left( \begin{array}{c} n \\ k \end{array} \right)_q [j] q^{n-k} q^{jk} B_{k,q}(x) [x]_{q,j}.
\]

(23)

As \( q \to 1 \), we obtain a polynomial version of Spivey’s Bell number formula which, in return, reduces to (4) when we set \( x = 1 \).

It is important to emphasize that this is not a new proof, but an alternative form of Katzriél’s proof, since the operators \(a, a^\dagger\) and the operators \(X, D\) generate isomorphic algebras.
3 A generalization of Spivey’s Bell number formula

The main result of this paper is the following identity:

$$D_{m,r,q}(n + \ell, x) = \sum_{j=0}^{\ell} \sum_{k=0}^{n} m^j W_{m,r,q}(\ell, j) \binom{n}{k} (m[j]_q + r)^{n-k} q^{jk} D_{m,0,q}(k, x)_q, \quad (24)$$

Here, $D_{m,r,q}(n, x)$ is a $(q, r)$-Dowling polynomial defined previously by the author and Katriel [9] as

$$D_{m,r,q}(n, x) = \sum_{k=0}^{n} W_{m,r,q}(n, k) x^k, \quad (25)$$

where $W_{m,r,q}(n, k)$ is the $(q, r)$-Whitney numbers of the second kind. Several properties of $D_{m,r,q}(n, x)$ can be seen in [8, 9].

To derive (24), we first multiply both sides of (21) by $m$ and then add $r(a^\dagger)^k$ to yield

$$(ma^\dagger + r)(a^\dagger)^k = (a^\dagger)^k (m[k]_q + r + mq^k a^\dagger a). \quad (26)$$

Also, multiplying both sides of the defining relation in [9, Equation 16] by $e_q(ta^\dagger) |0\rangle$ and applying (15) yields

$$\sum_{k=0}^{n} W_{m,r,q}(n, k) (a^\dagger)^k a^k e_q(ta^\dagger) |0\rangle = D_{m,r,q}(n, mta^\dagger) e_q(ta^\dagger) |0\rangle.$$ 

Now, by (26),

$$(ma^\dagger + r)^{n+\ell} = \sum_{j=0}^{\ell} m^j W_{m,r,q}(\ell, j) (ma^\dagger + r)^n (a^\dagger)^j a^j$$

$$= \sum_{j=0}^{\ell} m^j W_{m,r,q}(\ell, j) (a^\dagger)^j (m[j]_q + r + mq^j a^\dagger a)^n a^j$$

$$= \sum_{j=0}^{\ell} \sum_{k=0}^{n} m^{j+k} W_{m,r,q}(\ell, j) \binom{n}{k} (a^\dagger)^j (m[j]_q + r)^{n-k} q^{kj} (a^\dagger a)^k a^j.$$ 

Applying this expression to the operator identity $e_q(ta^\dagger) |0\rangle$, combining with the previous equation, using (9), (19) and $W_{m,0,q}(k, i) = m^{k-i} \binom{k}{i} i!_q$ (see [9, Equation 18]), and then dividing both sides of the resulting identity by $e_q(ta^\dagger) |0\rangle$ completes the derivation.
4 Remarks

Since $W_{1,0,q}(\ell, j) = \binom{\ell}{j}_q$, then by setting $x = 1$, $m = 1$ and $r = 0$, we have

$$D_{1,0,q}(n + \ell, 1) = \sum_{j=0}^{\ell} \sum_{k=0}^{n} \binom{\ell}{j}_q \binom{n}{k}_q [j]_q^{n-k} q^{jk} B_{k,q}.$$  \quad (27)

where $B_{k,q} := B_{k,q}(1)$. This is a $q$-analogue of (4) which was first obtained by Katriel [7]. On the other hand, setting $x = 1$ and then taking the limit of (24) as $q \to 1$ provides a generalization of Spivey’s Bell number formula in terms of the $r$-Whitney numbers of the second kind, denoted by $W_{m,r}(\ell, j)$, and the $r$-Dowling numbers, denoted by $D_{m,r}(n)$, (see [4, 11]), given by

$$D_{m,r}(n + \ell) = \sum_{j=0}^{\ell} \sum_{k=0}^{n} m^j W_{m,r}(\ell, j) \binom{n}{k}_q (m j + r)^{n-k} D_{m,0}(k). \quad (28)$$

In a recent paper, Mansour et al. [10] obtained the following generalization of Spivey’s Bell number formula:

$$D_{p,q}(a + b; x) = \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{\ell=0}^{j} (mq^i)^{j-\ell} x^{i+\ell} \binom{b}{j}_q ([r]_p + m[i]_q)^{b-j} W_{p,q}(a, i) S_q(j, \ell). \quad (29)$$

Here, $D_{p,q}(n; x)$ and $W_{p,q}(n, k)$ denote the $(p, q)$-analogues of the $r$-Dowling polynomials and the $r$-Whitney numbers of the second kind, respectively. The $(p, q)$-analogues are natural generalizations of $q$-analogues. However, since the manner by which the numbers $W_{m,r,q}(n, k)$ were defined in [9] differs from the work of Mansour et al. [10], the main result of this paper is not generalized by (29).

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