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On Kazhdan's property (T) and Kazhdan constants associated to a Laplacian for SL(3,R)

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Abstract. The first purpose of this paper is to give a very elementary proof of Property (T) for $\mathrm{SL}(3,\mathbf{k})$ over any local field $\mathbf{k}$. Here we use a modification of an argument due to Burger. The second approach to Property (T) is based on spectral properties of a Laplacian in the enveloping algebra. It is shown that for a connected Lie group G Property (T) can be characterized by a spectral property of a Laplacian on the space of smooth K-finite vectors, where K is a compact subgroup of G.

1. Introduction

A locally compact group has *Kazhdan's Property* (T) if the following holds: whenever a (strongly continuous) unitary representation of G has almost invariant vectors, then actually it has a nonzero fixed vector. Recall that a representation $(\pi, \mathcal{H})$ has *almost invariant vectors* if, for any $\varepsilon > 0$ and for any compact set K of G, there exists a unit vector $\xi \in \mathcal{H}$ with $\|\pi(g)\xi - \xi\| < \varepsilon$ for all $g \in K$.

Property (T), discovered in 1967 by D. Kazhdan [11], is a powerful tool, with applications, for instance, in rigidity, geometry, graph theory and operator algebras (see [14, 20, 12, 7]). Most semisimple Lie groups have Property (T). More precisely, all simple Lie groups, except those which are locally isomorphic to $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$, have Property (T). It is an important fact that Property (T) is inherited by lattices. So, for instance, $\mathrm{SL}(n,\mathbb{Z})$ has Property (T) for $n \geq 3$.

The main step in establishing Property (T) for simple Lie groups of $\mathbb{R}$–rank ≥ 2 is the proof for $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{Sp}(2,\mathbb{R})$ of this property. The usual proofs for $\mathrm{SL}(3,\mathbb{R})$ (and for $\mathrm{Sp}(2,\mathbb{R})$) are based on the study - by means of Mackey's theory - of the irreducible unitary representations of a copy of the semi-direct product $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ inside $\mathrm{SL}(3,\mathbb{R})$ (or the semi-direct product of $\mathrm{SL}(2,\mathbb{R})$ by the space of symmetric 2×2–matrices), see [14, 20, 12]. Another argument, due to M. Burger, based on the so-called Furstenberg lemma, appears in [9]. In [10], an alternative proof is given using estimates of independent interest for matrix coefficients of unitary representations.

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The first purpose of this paper is to give a very elementary proof of Property (T) for $\mathrm{SL}(3,\mathbf{k})$ over any local field $\mathbf{k}$. It is a modification of Burger's argument mentioned above. Instead of Furstenberg lemma, it uses the fact that there is no $\mathrm{SL}(2,\mathbf{k})$–invariant mean on the Borel sets in $\mathbf{k}^2 \setminus \{0\}$ (see 2.2 below). In case $\mathbf{k}=\mathbb{R}$, a new proof is given in terms of the Lie algebra for the following well–known but crucial fact (Lemma 2.4): If a vector in a unitary representation of $\mathrm{SL}(2,\mathbb{R})$ is invariant under the upper triangular unipotent matrices then it is invariant under $\mathrm{SL}(2,\mathbb{R})$. In fact, our proof shows that the same result is true for the universal covering group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Our second approach to Property (T) for $\mathrm{SL}(3,\mathbb{R})$ is of a quantitative nature and is based on spectral properties of a *Laplacian* in the enveloping algebra $\mathcal{U}(\mathfrak{sl}(3,\mathbb{R}))$. In [3], Kazhdan's Property (T) for a connected Lie group G is characterized as follows: Let $\mathfrak{g}$ be the Lie algebra of G, let $X_1, X_2, \ldots, X_n$ be a basis of $\mathfrak{g}$, and let $\Delta := -(X_1^2 + X_2^2 + \ldots + X_n^2)$ be the associated Laplacian in $\mathcal{U}(\mathfrak{g})$. Then G has Property (T) if and only if there exists $\varepsilon > 0$ such that $\inf \mathrm{sp}(\overline{d\pi(\Delta)}) \geq \varepsilon$ for any unitary representation $(\pi, \mathcal{H})$ of G without nonzero fixed vector, where $d\pi$ denotes the derived representation of π in the space $\mathcal{H}^\infty$ of C^∞–vectors in $\mathcal{H}$ and sp the spectrum. As this is more convenient for computations, we first show that one may equally consider the smaller space of K–finite vectors for a compact subgroup K of G. Recall that a vector $\xi \in \mathcal{H}$ is K–finite if the linear span of $\pi(K)\xi$ has finite dimension.

Theorem 1.1. *The connected Lie group G has Property (T) if and only if there exists a constant $\varepsilon > 0$ such that*

$$\inf\{\langle d\pi(\Delta)\xi, \xi\rangle,\ \xi \in \mathcal{H}^{\infty,K}, \|\xi\| = 1\} \geq \varepsilon$$

for any unitary representation $(\pi, \mathcal{H})$ of G without nonzero fixed vector, where $\mathcal{H}^{\infty,K}$ is the space of all K–finite C^∞–vectors in $\mathcal{H}$ for a compact subgroup K of G.

Our main result gives a bound for the constant ε appearing above.

Theorem 1.2. *Let $K := \mathrm{SO}(3,\mathbb{R})$, and let $X_1, X_2, \ldots, X_8$ be the following basis of the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$:*

$$X_1 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_4 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_5 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X_6 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$X_7 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_8 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and let

$$\Delta := -(X_1^2 + X_2^2 + \ldots + X_8^2).$$

Then, for any unitary representation $(\pi, \mathcal{H})$ *of* $\mathrm{SL}(3, \mathbb{R})$ *without nonzero fixed vector,*

$$\inf\{\langle d\pi(\Delta)\xi, \xi\rangle;\ \xi \in \mathcal{H}^{\infty,K},\ \|\xi\| = 1\} \geq \alpha \approx 0.4613 ,$$

where α *is the maximal value of the function* $\frac{2\sin^2\theta}{\pi\theta}$ *on* $\mathbb{R}$.

The reason for the choice of the above basis is that $\Delta = -\mathcal{C} + 2\mathcal{C}_K$, where $\mathcal{C}$ and $\mathcal{C}_K$ are the Casimir operators of $\mathrm{SL}(3, \mathbb{R})$ and $\mathrm{SO}(3, \mathbb{R})$, respectively. The proof of the above estimate is inspired by some ideas due to Howe and Tan [10], Chapter V.3.3.

Kazhdan's constants depending on a generating set may also be defined at the group level (see [9]). Such constants have been studied, for instance, in [1, 4, 5, 6, 8, 15, 17]. The paper is organized as follows. In Section 2, we give the proof of Kazhdan's Property (T) for $\mathrm{SL}(3, \mathbf{k})$. Section 3 is devoted to the proof of Theorem 1.1, and Theorem 1.2 is proved in Section 4.

2. Kazhdan Property (T) for SL(3, k)

The proof depends on the following three lemmas. The first lemma says that the representation $(\pi, \mathcal{H})$ is amenable in the sense of [2], where more general results are proved. We thank S. Popa for the following direct and simple proof.

Lemma 2.1. *Let* G *be a locally compact group, and let* $(\pi, \mathcal{H})$ *be a unitary representation of* G *with almost invariant vectors. Then there is an* $\mathrm{Ad}(G)$*–invariant state* φ *on the* C^**–algebra* $\mathcal{L}(\mathcal{H})$ *of all bounded operators on* $\mathcal{H}$*, that is, a positive linear form* $\varphi : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$ *with* $\varphi(\mathrm{I}) = 1$ *and* $\varphi(\pi(x)T\pi(x)^{-1}) = \varphi(T)$ *for all* $x \in G$, $T \in \mathcal{L}(\mathcal{H})$.

Proof. Let $\{\xi_n\}_n \subseteq \mathcal{H}$, $\|\xi_n\| = 1$, with $\lim_{n\to\infty} \|\pi(x)\xi_n - \xi_n\| = 0$ for all $x \in G$. Define states φ_n on $\mathcal{L}(\mathcal{H})$ by

$$\varphi_n(T) := \langle T\xi_n, \xi_n\rangle, \quad T \in \mathcal{L}(\mathcal{H}) .$$

Since the set of states on $\mathcal{L}(\mathcal{H})$ is a weak–$*$–compact subset of the unit ball of $\mathcal{L}(\mathcal{H})^*$, we may assume (upon passing to a subnet) that there exists a state φ on $\mathcal{L}(\mathcal{H})$ with

$$\lim_{n\to\infty} \varphi_n(T) = \varphi(T) \quad \forall T \in \mathcal{L}(\mathcal{H}) .$$

Then φ is $\mathrm{Ad}(G)$–invariant as, for any $x \in G$, $T \in \mathcal{L}(\mathcal{H})$,

$$\begin{aligned} |\varphi_n(\pi(x)T\pi(x)^{-1}) - \varphi_n(T)| &= |\langle T\pi(x)\xi_n, \pi(x)\xi_n\rangle - \langle T\xi_n, \xi_n\rangle| \\ &\leq 2\|T\|\|\pi(x)\xi_n - \xi_n\| \end{aligned}$$

and hence

$$|\varphi(\pi(x)T\pi(x)^{-1}) - \varphi(T)| = \lim_{n\to\infty} |\varphi_n(\pi(x)T\pi(x)^{-1}) - \varphi_n(T)| = 0 . \qquad \blacksquare$$

Recall that a local field is a locally compact, nondiscrete field (archimedean or nonarchimedean), and that the topology of such a field is defined by an absolute value. In fact, any such field is isomorphic either to $\mathbb{R}$, to $\mathbb{C}$, to a finite extension of the p–adic numbers or to the Laurent series in one variable over a finite field (see [19, Chap. I,A73]).

Lemma 2.2. *Let* $\mathbf{k}$ *be a local field . Let* $\mathrm{SL}(2,\mathbf{k})$ *act on* $\mathbf{k}^2$ *in the natural way. Then the Dirac measure at* $\{0\}$ *is the only finitely additive* $\mathrm{SL}(2,\mathbf{k})$*–invariant probability measure on the Borel sets of* $\mathbf{k}^2$*. Equivalently, there is no finitely additive,* $\mathrm{SL}(2,\mathbf{k})$*–invariant probability measure on the Borel sets of* $\mathbf{k}^2 \setminus \{0\}$.

Proof. Let

$$\mu : \mathcal{B}(\mathbf{k}^2) \to \mathbb{R}^+$$

be a finitely additive, $\mathrm{SL}(2,\mathbf{k})$–invariant probability measure on the Borel sets $\mathcal{B}(\mathbf{k}^2)$ of $\mathbf{k}^2$. Let $|\cdot|$ be an absolute value on $\mathbf{k}$. Let

$$\Omega := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{k}^2 \setminus \{0\};\ |y| \geq |x| \right\} .$$

Take a sequence $\{\lambda_n\}_n \subseteq \mathbf{k}$ with $|\lambda_{n+1}| > |\lambda_n| + 2$ for all $n \in \mathbb{N}$, and let

$$g_n := \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbf{k}) .$$

Then

$$\Omega_n := g_n \Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{k}^2 \setminus \{0\};\ \frac{|x|}{|\lambda_n| + 1} \leq |y| \leq \frac{|x|}{|\lambda_n| - 1} \right\} .$$

Indeed, for $\begin{pmatrix} x \\ y \end{pmatrix} \in \Omega$,

$$g_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \lambda_n y \\ y \end{pmatrix}$$

and

$$\begin{aligned} |x + \lambda_n y| &\geq |\lambda_n y| - |x| \geq (|\lambda_n| - 1)|y| , \\ |x + \lambda_n y| &\leq |x| + |\lambda_n y| \leq (|\lambda_n| + 1)|y| . \end{aligned}$$

Clearly the sets Ω_n are pairwise disjoint, as

$$\frac{1}{|\lambda_n| - 1} < \frac{1}{|\lambda_m| + 1} \quad \text{for } n > m .$$

Hence $\sum_{i=1}^n \mu(\Omega_i) \leq \mu(\mathbf{k}^2 \setminus \{0\}) \leq 1$ for all $n \in \mathbb{N}$. Since $\mu(\Omega_i) = \mu(g_i \Omega) = \mu(\Omega)$, this shows that $\mu(\Omega) = 0$. Now, let

$$\Omega' := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{k}^2 \setminus \{0\};\ |x| \geq |y| \right\} .$$

Then $\mu(\Omega') = \mu(\Omega) = 0$. Since $\Omega \cup \Omega' = \mathbf{k}^2 \setminus \{0\}$, $\mu(\mathbf{k}^2 \setminus \{0\}) = 0$. So, μ is the Dirac measure at 0. ∎

Remark 2.3. As the proof shows, the above lemma applies to other groups than $\mathrm{SL}(2,\mathbf{k})$, for instance to $\mathrm{SL}(2,\mathbb{Z})$ when $\operatorname{char}\mathbf{k}=0$. The conclusion of the lemma is certainly known to several people. A. Valette showed us a proof of the lemma, in the case $k=\mathbb{R}$, using only two matrices from $\mathrm{SL}(2,\mathbb{Z})$.

The last ingredient is the following well-known lemma (see [9, 14, 20, 12]) for which we give a new proof based on consideration of the Lie algebra, in the case where $\mathbf{k}=\mathbb{R}$.

Lemma 2.4. *Let $(\pi,\mathcal{H})$ be a unitary representation of $\mathrm{SL}(2,\mathbf{k})$. Assume that*

$$N := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbf{k} \right\}$$

has a nonzero fixed vector $\xi \in \mathcal{H}$. Then ξ is fixed by $\mathrm{SL}(2,\mathbf{k})$.

Proof. ($\mathbf{k}=\mathbb{R}$) Let

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard basis of $\mathfrak{sl}(2,\mathbb{R})$ with usual commutator relations

$$[X,Y]=H, \quad [H,X]=2X, \quad [H,Y]=-2Y\ .$$

The space $\mathcal{H}^N$ of the N–invariant vectors is invariant under the subgroup A generated by H. Hence, $\mathcal{H}^N$ contains a dense subspace $\mathcal{D}$ of C^∞–vectors under the action of AN. Clearly, $X=0$ on $\mathcal{D}$ and $\mathcal{D}$ is H–invariant. We first show that $H=0$ on $\mathcal{D}$. We write $W\xi$ instead of $d\pi(W)\xi$ for $W\in\mathfrak{sl}(2,\mathbb{R}), \xi\in\mathcal{H}$, whenever this makes sense. Recall that each W is skew-symmetric on $\mathcal{H}^\infty$. In fact, it is well-known that iW is essentially selfadjoint on $\mathcal{H}^\infty$ for any $W\in\mathfrak{sl}(2,\mathbb{R})$. Consider

$$\mathcal{C} := H^2+2(XY+YX) = H^2+4XY-2H\ ,$$

the Casimir operator of $\mathfrak{sl}(2,\mathbb{R})$. We have, for any $\xi\in\mathcal{D}, \eta\in\mathcal{H}^\infty$:

$$\begin{aligned} \langle \xi, \mathcal{C}\eta\rangle &= \langle \xi, H^2\eta\rangle + 4\langle \xi, XY\eta\rangle - 2\langle \xi, H\eta\rangle \\ &= \langle \xi, (H^2-2H)\eta\rangle = \langle \left(H^2+2H\right)\xi, \eta\rangle\ . \end{aligned}$$

Thus, $\mathcal{D}$ is contained in the domain of $\mathcal{C}^*$ and $\mathcal{C}^*=H^2+2H$ on $\mathcal{D}$.
As is well-known (see [18, p.269, Ex.(3)], for instance), $\mathcal{C}$ is essentially selfadjoint on $\mathcal{H}^\infty$. Whence $\mathcal{C}\big|_{\mathcal{D}}=\mathcal{C}^*\big|_{\mathcal{D}}$ or $(H^2+2H)^*=H^2-2H$ on $\mathcal{D}$. Thus $H=0$ on $\mathcal{D}$. Now, fix $\xi\in\mathcal{D}$. Then , for any $\eta\in\mathcal{H}^\infty$,

$$\langle \xi, YH\eta\rangle = 2\langle \xi, Y\eta\rangle\ ,$$

as $[H,Y]=-2Y$ and $H\xi=0$.
On the other hand, the range of $H-2\mathrm{I}$ as an operator on $\mathcal{H}^\infty$ is dense in $\mathcal{H}$. Indeed, this follows from the fact that iH is essentially selfadjoint on $\mathcal{H}$.
Thus,

$$\langle \xi, Y\eta\rangle = 0 \quad \text{for all } \eta\in\mathcal{D}'\ ,$$

where $\mathcal{D}' = (H - 2\mathrm{I})\mathcal{H}^\infty$. This shows that ξ is in the domain of $Y^* = -\overline{Y}$ and that $\overline{Y}\xi = 0$. Hence, by Stone's theorem,

$$\exp(tY)\xi \;=\; \exp(t\overline{Y})\xi \;=\; \xi$$

for all $t \in \mathbb{R}$. Thus ξ is fixed by the subgroup $\overline{N}$, generated by Y . Hence, any $\xi \in \mathcal{D}$ is fixed by $\mathrm{SL}(2,\mathbb{R})$. By density, this is true for any N–fixed vector in $\mathcal{H}$.

■

Remark 2.5. The above proof is somewhat involved because it is not a priori clear whether the space of N–fixed vectors contains any nonzero C^∞–vector. The arguments above become much shorter in the case of an N–invariant C^∞–vector ξ as the reader may wish to verify.
On the other hand, because it relies on Lie algebra considerations, the proof works for any covering group of $\mathrm{SL}(2,\mathbb{R})$ (where N has to be taken as the one–parameter subgroup generated by X).

Theorem 2.6. [11] $\mathrm{SL}(3,\mathbf{k})$ *has Kazhdan's Property (T).*

Proof. Let $(\pi,\mathcal{H})$ be a unitary representation of $\mathrm{SL}(3,\mathbf{k})$ with almost invariant vectors. Let

$$H \;:=\; \left\{ \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbf{k}), \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{k}^2 \right\} \simeq \mathrm{SL}(2,\mathbf{k}) \ltimes \mathbf{k}^2,$$

and let

$$V \;:=\; \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{k}^2 \right\} \simeq \mathbf{k}^2 \,.$$

Let

$$P : \mathcal{B}(\hat{V}) \to \mathcal{L}(\mathcal{H}), \quad E \mapsto P(E)$$

be the projection valued measure associated with the unitary representation $\pi|_V$ of the abelian group V , see e. g. , [13]. Clearly

$$\pi(g)P(E)\pi(g)^{-1} \;=\; P(g\cdot E) \quad \forall g \in \mathrm{SL}(2,\mathbf{k}), E \in \mathcal{B}(\hat{V}) \,. \tag{1}$$

Here

$$g\cdot\gamma\begin{pmatrix} x \\ y \end{pmatrix} \;:=\; \gamma\left(g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right), \quad g \in \mathrm{SL}(2,\mathbf{k}), \gamma \in \hat{V}$$

is the dual action.
By Lemma 2.1, there exists an $\mathrm{Ad}(\mathrm{SL}(3,\mathbf{k}))$–invariant state φ on $\mathcal{L}(\mathcal{H})$. Define

$$m(E) \;:=\; \varphi(P(E)) \quad \forall E \in \mathcal{B}(\mathbf{k}^2) \,.$$

Then, m is a finitely additive probability measure on $\mathcal{B}(\hat{V})$. Moreover, m is $\mathrm{SL}(2,\mathbf{k})$–invariant, by (1). Now $\hat{\mathbf{k}}$ identifies with $\mathbf{k}$ in such a way that the action of $\mathrm{SL}(2,\mathbf{k})$ corresponds to the transpose of the action of $\mathrm{SL}(2,\mathbf{k})$ on $\mathbf{k}^2$ (see [19]). In this way, m becomes an invariant finitely additive probability measure on the

Borel sets of $\mathbf{k}^2$. So, by Lemma 2.2, m is the Dirac measure at 0. In particular, $P(\{0\}) \neq 0$. This shows that $\pi|_V$ has a nonzero invariant vector ξ. By Lemma 2.4, ξ is invariant under the following two copies of $\mathrm{SL}(2,\mathbf{k})$

$$\begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix};$$

which generate together $\mathrm{SL}(3,\mathbf{k})$. This concludes the proof. ∎

3. Kazhdan Constants and K-Finite Vectors

In this section, we prove Theorem 1.1. Let $(\pi,\mathcal{H})$ be a (strongly continuous) unitary representation of a connected Lie group G and denote by $d\pi$ the derived representation of the Lie algebra $\mathfrak{g}$. Let $(X_1, X_2, \dots, X_n)$ be a basis of $\mathfrak{g}$ and $\Delta := -\sum_{j=1}^n X_i^2 \in \mathcal{U}(\mathfrak{g})$ be the associated Laplacian. Then $d\pi(\Delta)$ is defined on $\mathcal{H}^\infty$, positive and essentially selfadjoint. Let $\overline{d\pi(\Delta)}$ be its closure.

In [3] the operators $d\pi(\Delta)$ and $\overline{d\pi(\Delta)}$ are used in order to decide whether or not $(\pi,\mathcal{H})$ contains weakly the trivial representation. The main result of that paper is

Theorem 3.1. *Let $(\pi,\mathcal{H})$ be a unitary representation of a connected Lie group G . Then the following are equivalent:*

(i) $(\pi,\mathcal{H})$ *contains weakly the trivial representation.*

(ii) 0 *is an approximative eigenvalue of* $d\pi(\Delta)$.

(iii) 0 *is a spectral value of* $\overline{d\pi(\Delta)}$.

We are going to see that one may also consider the restriction of the Laplacian to smaller subspaces where the computations become easier. Recall that $\xi \in \mathcal{H}$ is called *analytic for* π if the mapping

$$g \mapsto \langle \pi(g)\xi, \eta \rangle$$

is analytic on G for all $\eta \in \mathcal{H}$. The space of analytic vectors is denoted by $\mathcal{H}^\omega$. Finally, let T be an operator in $\mathcal{H}$. A vector $\xi \in \mathcal{H}$ is called *analytic for* T if the series

$$\sum_{j=1}^{\infty} \frac{\|T^j\xi\|}{j!} t^j$$

has positive convergence radius. By the spectral theorem, the analytic vectors of a selfadjoint operator are dense in $\mathcal{H}$. The following result of R. Goodman relates these notions of analyticity.

Define

$$B_\pi := \left(\mathrm{I} + \overline{d\pi(\Delta)}\right)^{\frac{1}{2}} .$$

B_π is a selfadjoint, positive operator, $B_\pi \geq \mathrm{I}$. Then ξ is an analytic vector for B_π if and only if ξ is an analytic vector for π and this is the case if and only if $\|d\pi(\Delta)^m\xi\| \leq (2m)!M^m\|\xi\|$, for all $m \in \mathbb{N}_0$ and a suitable $M > 0$. For all these results, see [18, Chapter 4.4].

Proposition 3.2. *With the above notations, let $\mathcal{D}$ be a subspace of $\mathcal{H}$. Assume that $\mathcal{D} \cap \mathcal{H}^\omega$ is dense in $\mathcal{H}$. Then the following are equivalent:*

(i) *$(\pi, \mathcal{H})$ contains weakly the trivial representation.*

(ii) $\inf\{\langle d\pi(\Delta)\xi, \xi\rangle;\ \xi \in \mathcal{D}, \|\xi\| = 1\} = 0$.

Proof. By 3.1 it is enough to prove that (i) implies (ii).
Since $\mathcal{D} \cap \mathcal{H}^\omega$ is dense in $\mathcal{H}$, Goodman's result implies that $\mathcal{D}$ contains a dense set of vectors analytic for $B_\pi|_\mathcal{D}$, where B_π is defined as above. Since $B_\pi|_\mathcal{D}$ is symmetric, Nelson's theorem (see e.g., [16, X.39]) shows that $B_\pi|_\mathcal{D}$ is essentially selfadjoint, whence $\overline{B_\pi|_\mathcal{D}} = B_\pi$. By functional calculus, there is a sequence $\{\xi_n\}_n$ of unit vectors in the domain of B_π, satisfying $\|B_\pi(\xi_n) - \xi_n\| < \frac{1}{2n}$. Let $\{\eta_n\}_n$ be sequence in $\mathcal{D}$ with $\|B_\pi\xi_n - B_\pi\eta_n\| + \|\xi_n - \eta_n\| < \frac{1}{2n}$. Then $\|B_\pi(\eta_n) - \eta_n\| < \frac{1}{n}$. Let $\psi_n := \frac{\eta_n}{\|\eta_n\|} \in \mathcal{D}$. Then $\lim_{n\to\infty}\langle (B_\pi - \mathrm{I})\psi_n, \psi_n\rangle = 0$, by Cauchy–Schwarz inequality and, hence,

$$\lim_{n\to\infty} \langle d\pi(\Delta)\psi_n, \psi_n\rangle = 0 = \lim_{n\to\infty} \langle (B_\pi - \mathrm{I})^2\psi_n, \psi_n\rangle + 2\lim_{n\to\infty}\langle (B_\pi - \mathrm{I})\psi_n, \psi_n\rangle = 0$$

as desired. ∎

An often useful choice of $\mathcal{D}$ is the following: Let K be a compact subgroup of G. By the Peter–Weyl theorem, we may decompose $(\pi|_K, \mathcal{H})$ in K-isotypic components $\mathcal{H}(\sigma)$

$$\mathcal{H} = \sum_{\sigma\in\hat{K}} \mathcal{H}(\sigma) .$$

Then the subspaces (algebraic sum)

$$\mathcal{H}^{\infty,K} := \sum_{\sigma\in\hat{K}} (\mathcal{H}(\sigma) \cap \mathcal{H}^\infty) \quad \text{and} \quad \mathcal{H}^{\omega,K} := \sum_{\sigma\in\hat{K}} (\mathcal{H}(\sigma) \cap \mathcal{H}^\omega)$$

are dense in $\mathcal{H}$ [18, 4.4.3.1,4.4.5.16].

Corollary 3.3. *With the above notations, the following are equivalent:*

(i) *$(\pi, \mathcal{H})$ contains weakly the trivial representation.*

(ii) $\kappa_K(d\pi(\Delta), G) := \inf\{\langle d\pi(\Delta)\xi, \xi\rangle,\ \xi \in \mathcal{H}_K^\infty, \|\xi\| = 1\} = 0$.

We define the infinitesimal Kazhdan constant $\kappa_K(\Delta, G)$ to be the least upper bound of all$\{\kappa_K(d\pi(\Delta), G)$ where π ranges through all unitary representations which do not contain the trivial representation$\}$.

Corollary 3.4. *Let G be a connected Lie group. Then G has property (T) if and only if there exists $\varepsilon > 0$ such that*

$$\kappa_K(\Delta, G) > \varepsilon .$$

Proof. Using 3.2, this is similar to the proof of [3, 3.10]. ∎

4. Kazhdan Constants for $\mathrm{SL}(\mathbf{3}, \mathbb{R})$

We now give a bound for the above Kazhdan's constant in the case of $\mathrm{SL}(3, \mathbb{R})$. Consider the subgroup

$$H := \left\{ \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),\, x, y \in \mathbb{R} \right\} \simeq \mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2 .$$

Observe that if $(\pi, \mathcal{H})$ is a unitary representation of $\mathrm{SL}(3, \mathbb{R})$ and if ξ is fixed under the action of the subgroup

$$\mathbb{R}^2 \simeq V := \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y \in \mathbb{R} \right\},$$

then ξ is a fixed vector under $\mathrm{SL}(3, \mathbb{R})$ (see Lemma 2.5 above, and the proof of 2.6).
So, we may assume that $(\pi, \mathcal{H})$ is a unitary representation of $\mathrm{SL}(3, \mathbb{R})$ without V–fixed vectors. As in the proof of 2.6, there exists a projection valued measure P on the Borel sets of $\hat{V} \simeq V$ such that

$$\pi|_V = \int_{\hat{V}} \gamma dP(\gamma) \quad \text{and} \quad P(\{0\}) = 0 ,$$

and we have for all Borel sets $E \subseteq \hat{V}$:

$$P(g \cdot E) = \pi(g) P(E) \pi(g)^{-1}$$

(compare with (1)) for all $g \in \mathrm{SL}(2, \mathbb{R})$. Next, choose as a basis of $\mathfrak{sl}(2, \mathbb{R})$:

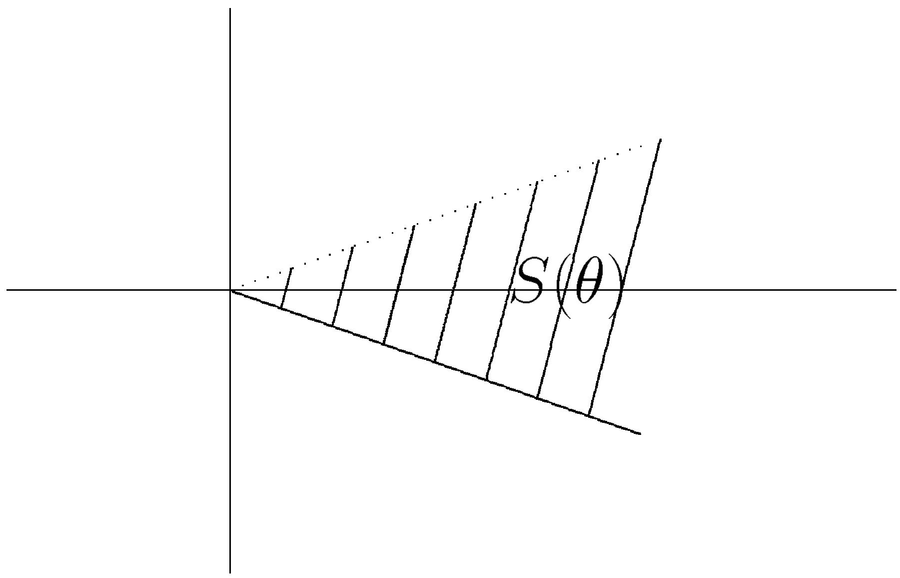
$$K := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

If we embed $\mathrm{SL}(2, \mathbb{R})$ in $\mathrm{SL}(3, \mathbb{R})$ as above, K, A, B correspond to X_1, X_2, and X_3 in Theorem 1.2.
We have the commutator relations

$$[K, A] = -2B,\ [K, B] = 2A,\ [A, B] = 2K .$$

For an angle $0 \leq \theta \leq \pi$, let $S(\theta)$ be the sector

$$S(\theta) := \left\{ v \in \hat{V},\ \arg v \in \left[-\frac{\theta}{2}, \frac{\theta}{2} \right[\right\} .$$



The first crucial fact is that, for any unit K–eigenvector v, we have

$$\|P(S(\theta))v\|^2 = \frac{\theta}{2\pi} . \tag{2}$$

This is an easy computation (see [10, p.223]). Indeed, since V has no nonzero fixed vector, $P(\mathbb{R}^2 \setminus \{0\}) = \mathrm{I}$. For an arc E on the unit circle, set $\mu(E) = \|P(S(E))v\|^2$, where $S(E)$ is the sector defined by E. The usual properties of a spectral measure show that μ is a measure on the circle. Further, μ is rotation invariant by the above relation (1). Hence, μ is the normalized arc lengh on the unit circle proving (2).

Set $a_t := \exp(tA)$, and let θ_t be the angle corresponding to the sector $a_t \cdot S(\theta)$. One easily computes that

$$\theta_t = 2\mathrm{arctan}(e^{2t} \tan \frac{\theta}{2}) . \tag{3}$$

For a C^∞– vector v , we have

$$\|P(S(\theta_t))v\|^2 = \|\pi(a_t)P(S(\theta))\pi(a_{-t})v\|^2 = \|P(S(\theta))\pi(a_{-t})v\|^2 ,$$

by (1), and hence, differentiating at $t = 0$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\|P(S(\theta_t))v\|^2 &= \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\|P(S(\theta))\pi(a_{-t})v\|^2 \\
&= \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\langle P(S(\theta))\pi(a_{-t})v, \pi(a_{-t})v\rangle \\
&= -\langle d\pi(A)v, P(S(\theta))v\rangle - \langle P(S(\theta))v, d\pi(A)v\rangle .
\end{aligned}$$

Let v be a C^∞– vector which is a unit K–eigenvector. Then, (2) and (3) show that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\|P(S(\theta_t))v\|^2 &= \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\mathrm{arctan}(e^{2t}\tan\frac{\theta}{2}) = \frac{1}{\pi}\frac{2\tan\frac{\theta}{2}}{1+(\tan\frac{\theta}{2})^2} \\
&= \frac{1}{\pi}\sin\theta .
\end{aligned}$$

Hence,

$$-\langle d\pi(A)v, P(S(\theta))v\rangle - \langle P(S(\theta))v, d\pi(A)v\rangle \;=\; \frac{1}{\pi}\sin\theta\;.$$

Using Cauchy-Schwarz inequality, we find

$$2\|d\pi(A)v\|\sqrt{\frac{\theta}{2\pi}} \;=\; 2\|d\pi(A)v\|\|P(S(\theta))v\| \;\geq\; \frac{1}{\pi}\sin\theta\;,$$

and

$$\|d\pi(A)v\| \;\geq\; \frac{1}{\sqrt{2\pi}}\frac{\sin\theta}{\sqrt{\theta}}\;,$$

for all smooth unit K–eigenvectors v .
Observe that the same inequality holds for B , since B is conjugate to A under $\mathrm{SO}(2,\mathbb{R})$.

Now, we consider a second copy of $\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^2$ in $\mathrm{SL}(3,\mathbb{R})$, namely

$$H' := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & a & b \\ y & c & d \end{pmatrix},\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),\, x,y\in\mathbb{R} \right\}.$$

The same computation as above shows that, for $j\in\{2,3\}$ (respectively $j\in\{5,6\}$) :

$$\|d\pi(X_j)v\| \;\geq\; \frac{1}{\sqrt{2\pi}}\frac{\sin\theta}{\sqrt{\theta}}\;, \tag{4}$$

if v is a smooth unit X_1–eigenvector (respectively, if v is a smooth unit X_4–eigenvector).

Consider the Cartan decomposition $\mathfrak{sl}(3,\mathbb{R}) = \mathfrak{so}(3,\mathbb{R})\oplus\mathfrak{p}$, where $\mathfrak{p}$ are the symmetric matrices in $\mathfrak{sl}(3,\mathbb{R})$. Then X_1, X_4, X_7 is an orthonormal basis of $\mathfrak{so}(3,\mathbb{R})$ and X_2, X_3, X_5, X_6, X_8 is one of $\mathfrak{p}$, with respect to $\frac{1}{6}K(X,Y)$, where $K(X,Y)$ is the Killing form on $\mathrm{SL}(3,\mathbb{R})$. Hence, the Casimir operator $\mathcal{C}$ on $\mathrm{SL}(3,\mathbb{R})$ is

$$\mathcal{C} = (X_2^2 + X_3^2 + X_5^2 + X_6^2 + X_8^2) \,-\, (X_1^2 + X_4^2 + X_7^2),$$

the Casimir operator on $\mathrm{SO}(3,\mathbb{R})$ is $-(X_1^2+X_4^2+X_7^2)$, and

$$\Delta \;=\; -2(X_1^2 + X_4^2 + X_7^2) \,-\, \mathcal{C}\;, \tag{5}$$

Now, let w be a smooth $\mathrm{SO}(3,\mathbb{R})$–finite unit vector. Then $w = \sum_{j=1}^k v_j$, where the v_j belong to the isotypic components of pairwise inequivalent representations (in particular, they are pairwise orthogonal). By (5), $d\pi(\Delta)$ preserves the isotypic components. Hence,

$$\begin{aligned} \langle \Delta w, w\rangle \;&=\; \sum_{i,j=1}^k \langle d\pi(\Delta)v_i, v_j\rangle \;=\; \sum_{j=1}^k \langle d\pi(\Delta)v_j, v_j\rangle \\ &\geq\; \sum_{j=1}^k \left(\langle d\pi(\Delta_1)v_j, v_j\rangle + \langle d\pi(\Delta_2)v_j, v_j\rangle\right)\;, \end{aligned}$$

where $\Delta_1 := -\sum_{\nu=1}^{3} X_\nu^2$ and $\Delta_2 := -\sum_{\nu=4}^{6} X_\nu^2$. Moreover, observe that $\Delta_1 = -\Box_1 - 2X_1^2$ and $\Delta_2 = -\Box_2 - 2X_4^2$, where $\Box_i$, $i = 1,2$, are the respective Casimir operators in the copies of $\mathfrak{sl}(2,\mathbb{R})$. Hence, if $v_j = \sum_{\ell=1}^{r_j} u_\ell^j$ is the orthogonal decomposition of the X_1–finite vector v_j in X_1–eigenvectors, equation (4) yields

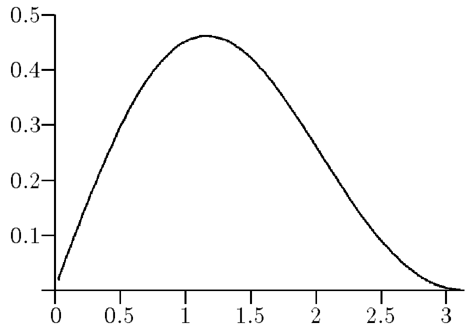
$$\langle d\pi(\Delta_1)v_j, v_j\rangle \geq \sum_{\ell=1}^{r_j} 2\left(\frac{1}{\sqrt{2\pi}}\frac{\sin\theta}{\sqrt{\theta}}\right)^2 \|u_\ell^j\|^2 = \frac{\sin^2\theta}{\pi\theta}\|v_j\|^2$$

Decomposing v_j into X_4–eigenvectors yields the same inequality, with Δ_1 replaced by Δ_2. Thus,

$$\langle \Delta w, w\rangle \geq \frac{2\sin^2\theta}{\pi\theta} \tag{6}$$

for any smooth $\mathrm{SO}(3,\mathbb{R})$–finite unit vector w .

Numerical computations show that the function $f(\theta) := \frac{2\sin^2\theta}{\pi\theta}$ assumes its maximal value of ≈ 0.4613 at $\theta \approx 1.1656$.



Thus

$$\langle \Delta w, w\rangle \geq 0.4613 ,$$

for all unit $\mathrm{SO}(3,\mathbb{R})$–finite vectors w .

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