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A Clifford algebra approach to simple Lie algebras of real rank two, I: the A_2 case

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Abstract. By using Clifford algebraic methods, we classify all real semisimple Lie algebras of type A_2. This approach to the classification avoids the need to discuss the real and complex cases separately, and also provides interesting information about the structure of the algebras.

1. Introduction

Since the time of E. Cartan, in the study of Lie algebras, attention has been focused on the involutions of the algebra. It is well known, however, that if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Iwasawa decomposition of $\mathfrak{g}$, then the algebra $\mathfrak{g}$ is uniquely determined by the subalgebra $\mathfrak{a} \oplus \mathfrak{n}$. Thus it should be possible to classify real semisimple Lie algebras by classifying their Iwasawa subalgebras $\mathfrak{a} \oplus \mathfrak{n}$. This is the first paper of a series devoted to this task.

This approach was followed in two papers by M. Cowling, A. Dooley, A. Korányi, and F. Ricci [3, 4], in which the authors classify real rank one Lie algebras and describe them. These papers rest on the observation of Korányi [8] that the nilpotent subalgebra in the Iwasawa decomposition of a real semisimple Lie algebra of rank one is either Abelian or lies in the class of algebras introduced by A. Kaplan in [6] and called generalized Heisenberg algebras.

The following observation holds for general real semisimple Lie algebras, and is the basis of our approach to studying them. The nilpotent subalgebra $\mathfrak{n}$ is the semi-direct product of a generalized Heisenberg algebra $\mathfrak{n}_0$ and a nilpotent subalgebra of the algebra which normalises $\mathfrak{n}_0$ and centralises the centre. This may be seen as a generalization to arbitrary rank of Korányi's original observation. In this paper we use this fact to classify and describe simple real Lie algebras with root system of type A_2, which is the simplest case of a higher rank algebra.

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As far as we know, this approach to the classification has never been tried before. In spite of its difficulty, our technique, which avoids complexification, seems to be more natural than the classical method for dealing with real algebras. This approach also leads us to discover some interesting properties of $\mathfrak{g}$, and clarifies the role played by Clifford algebras in the structure of real semisimple Lie algebras.

We further observe that both complex and noncomplex Lie algebras are treated simultaneously in our method. The real rank two algebras associated to root systems of type A_2 are $\mathfrak{sl}(3,\mathbb{R})$, $\mathfrak{sl}(3,\mathbb{C})$, $\mathfrak{su}^*(6)$, and $\mathfrak{e}_{6(-26)}$, corresponding to the cases where the root spaces have dimensions 1, 2, 4, and 8 (our notation for Lie algebras follows S. Helgason [5, p. 518]).

Here is an outline of the paper. In Section 2, we list some general results on semisimple Lie algebras, focusing attention on restricted roots and on the subalgebra $\mathfrak{m}$. In this section, we quote most standard results without providing the proofs, which may be found in [5].

Section 3 is devoted to generalized Heisenberg Lie algebras. We give the definition and, after some generalities whose proofs may be found in [6], we study the derivations of a generalized Heisenberg algebra in the cases of interest for us.

In Section 4, we choose a set of positive roots $\Delta_+ = \{\alpha, \beta, \alpha+\beta\}$, and prove that the maximal nilpotent subalgebra $\mathfrak{n}$, given by

$$\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta},$$

is a generalized Heisenberg algebra, i.e.,

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z},$$

where $\mathfrak{z} = \mathfrak{g}_{\alpha+\beta} = \mathbb{R}^d$ is the centre and $\mathfrak{v} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$ is a module for the Clifford algebra $\mathcal{C}(d)$. We use this fact to give an elementary proof that all roots in a root system of type A_2 have the same multiplicity, d say, and then we establish that d is in $\{1,2,4,8\}$. Finally, we show that the decomposition of $\mathfrak{v}$ into the orthogonal direct sum $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$ corresponds to the usual decomposition of a $\mathcal{C}(d)$-module into the direct orthogonal sum of two $\mathcal{C}^+(d)$-modules.

Section 5 is devoted to the study of the subalgebra $\mathfrak{m}$ of a real simple Lie algebra with root system A_2. We determine $\mathfrak{m}$ explicitly and describe its action on $\mathfrak{n}$ and $\theta\mathfrak{n}$ for d in $\{1,2,4,8\}$.

In Section 6, we use the results of Sections 4 and 5 to construct a vector space $\mathfrak{g}_d$, endowed with a skew-symmetric product $[\cdot,\cdot]_d$ and an involutive $[\cdot,\cdot]_d$-automorphism θ_d, for all d in $\{1,2,4,8\}$.

In the two last sections we prove that $[\cdot,\cdot]_d$ is a Lie bracket, showing that it satisfies the Jacobi identity. In particular, in Section 7, we show that $(\mathfrak{g}_d,[\cdot,\cdot]_d)$ is isomorphic to the matrix Lie algebra $\mathfrak{sl}(3,\mathbb{F}_d)$, where $\mathbb{F}_d$ is equal to $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, when $d=1$, $d=2$ and $d=4$ respectively. In Section 8, we consider the case where $d=8$. To check the Jacobi identity in $\mathfrak{g}_8$, we prove that for any three elements of $\mathfrak{g}_8$, there are a vector subspace $\mathfrak{g}'$ containing all three elements and a one-to-one linear mapping λ from $\mathfrak{g}'$ onto $\mathfrak{g}_4$ such that

$$\lambda[X,Y]_8 = [\lambda X, \lambda Y]_4.$$

Since at that stage we have already proved that $(\mathfrak{g}_4,[\cdot,\cdot]_4,\theta_4)$ is a Lie algebra, it follows that the Jacobi identity holds in $(\mathfrak{g}_8,[\cdot,\cdot]_8,\theta_8)$.

Finally, up to isomorphism, there is exactly one irreducible generalized Heisenberg algebra with d-dimensional centre, therefore the calculations in Section 5 determine the commutation relations uniquely, and prove that for each d in $\{1, 2, 4, 8\}$ there is exactly one real semisimple Lie algebra of type A_2 for which the dimension of each of the root spaces is d.

2. Preliminaries

In this section, we first recall some familiar facts about semisimple Lie algebras, and then prove some results about the subalgebra $\mathfrak{m}$ in this general context. Let $\mathfrak{g}$ be a real semisimple Lie algebra, not necessarily of real rank 2, and let B be the Killing form of $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$, and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the corresponding Cartan decomposition, where

$$\mathfrak{k} = \{X \in \mathfrak{g} : \theta X = X\} \qquad \text{and} \qquad \mathfrak{p} = \{X \in \mathfrak{g} : \theta X = -X\}.$$

Let c be a positive constant, to be fixed later. For X and Y in $\mathfrak{g}$, let

$$\langle X, Y \rangle = -c\, B(X, \theta Y). \tag{1}$$

Then $\langle \cdot, \cdot \rangle$ is a positive definite inner product. The corresponding norm will be denoted by $\| \cdot \|$. It follows from the invariance of the Killing form that

$$\langle [X, Y], Z \rangle = \langle X, [\theta Y, Z] \rangle \qquad \forall X, Y, Z \in \mathfrak{g}. \tag{2}$$

We will often use (2) without comment in the paper.

Fix a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$. A nonzero linear functional α on $\mathfrak{a}$ is said to be a restricted root of $(\mathfrak{g}, \mathfrak{a})$ if

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H) X\}$$

is nontrivial. The subspace $\mathfrak{g}_\alpha$ is called a root space, and its dimension is called the multiplicity of the root α and denoted by d_α. We will denote the set of roots of $(\mathfrak{g}, \mathfrak{a})$ by Δ, and by $\mathfrak{g}_0$ the commutant of $\mathfrak{a}$ in $\mathfrak{g}$. By θ-invariance,

$$\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{p}) \oplus (\mathfrak{g}_0 \cap \mathfrak{k}) = \mathfrak{a} \oplus \mathfrak{m},$$

say. For α and β in $\Delta \cup \{0\}$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, and in particular, $[\mathfrak{m}, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_\beta$.

For α in Δ, let H_α be the unique vector in $\mathfrak{a}$ such that

$$B(H_\alpha, H) = \alpha(H) \qquad \forall H \in \mathfrak{a}.$$

For α, β in Δ, set

$$(\alpha \mid \beta) = \langle H_\alpha, H_\beta \rangle;$$

this defines an inner product on $\mathfrak{a}^*$ which depends on the constant c in (1). It is well known (see [5, p. 94]) that

$$[X, \theta X] = -\|X\|^2\, H_\alpha \qquad \forall X \in \mathfrak{g}_\alpha. \tag{3}$$

We fix H in $\mathfrak{a}$ such that $\alpha(H) \neq 0$ for all α in Δ, and order the roots by saying that a root α is positive if $\alpha(H) > 0$. Then

$$\Delta = \Delta_+ \cup (-\Delta_+),$$

where Δ_+ is the set of positive roots. Let

$$\mathfrak{n} = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha; \qquad \text{then} \qquad \theta\mathfrak{n} = \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}.$$

It is clear that both $\mathfrak{n}$ and $\theta\mathfrak{n}$ are nilpotent subalgebras. We obtain the decomposition

$$\mathfrak{g} = \theta\mathfrak{n} \oplus \mathfrak{g}_0 \oplus \mathfrak{n} = \theta\mathfrak{n} \oplus (\mathfrak{a} \oplus \mathfrak{m}) \oplus \mathfrak{n}.$$

We now examine the structure of $\mathfrak{m}$ more carefully. We emphasize that the results in this section do not rely on the assumption that $\mathfrak{g}$ is of real rank 2. Much of what follows is "known," but not codified as here (as far as we know).

Proposition 2.1. *Suppose that $X, X' \in \mathfrak{g}_\gamma \setminus \{0\}$ for some root γ and that $\langle X, X' \rangle = 0$. Then*

$$[X, \theta X'] = [X', \theta X] \in \mathfrak{m}.$$

If in addition 2γ is not a root, then $[X, \theta X'] \neq 0$.

Proof. To prove that $[X, \theta X'] \in \mathfrak{m}$, we first observe that $[X, \theta X']$ lies in $\mathfrak{g}_0$. Since the decomposition $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ is orthogonal, $[X, \theta X']$ lies in $\mathfrak{m}$ if and only if $\langle H, [X, \theta X'] \rangle = 0$ for every H in $\mathfrak{a}$. We find that

$$\langle H, [X, \theta X'] \rangle = \langle [H, \theta X], \theta X' \rangle = -\gamma(H) \langle X, X' \rangle = 0.$$

Hence, $[X, \theta X'] \in \mathfrak{m}$. In particular, since θ fixes $\mathfrak{m}$, it follows that

$$[\theta X, X'] = \theta[X, \theta X'] = [X, \theta X'].$$

Finally, if 2γ is not a root, then $[X, \theta X'] \neq 0$, since

$$[[\theta X, X'], X] = [[\theta X, X], X'] = \gamma([\theta X, X]) X',$$

and $\gamma([\theta X, X]) \neq 0$ by (3). ■

For any root γ define

$$\mathfrak{m}_\gamma = \operatorname{span}\{[X, \theta X'] : X, X' \in \mathfrak{g}_\gamma \quad \text{and} \quad \langle X, X' \rangle = 0\}.$$

Since $[\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}] \subseteq \mathbb{R} H_\gamma \oplus \mathfrak{m}_\gamma$, we have $[\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}] \cap \mathfrak{m} = \mathfrak{m}_\gamma$.

Lemma 2.2. *If $\gamma \in \Delta$, then $\mathfrak{m}_\gamma$ is an ideal in $\mathfrak{m}$.*

Proof. Since ad is linear, it suffices to show that $\operatorname{ad}(M)[X, \theta X']$ lies in $\mathfrak{m}_\gamma$ for all M in $\mathfrak{m}$ and all orthogonal vectors X and X' in $\mathfrak{g}_\gamma$. We find that

$$[M, [X, \theta X']] = [X, [M, \theta X']] + [\theta X', [X, M]] = [X, \theta[M, X']] + [[M, X], \theta X'].$$

In this formula, the right hand side lies in $\mathfrak{m}_\gamma \oplus \mathfrak{a}$ and the left hand side lies in $\mathfrak{m}$. So both sides are in $\mathfrak{m} \cap (\mathfrak{a} \oplus \mathfrak{m}_\gamma) = \mathfrak{m}_\gamma$. ■

Proposition 2.3.
$$\mathfrak{m} = \sum_{\gamma \in \Delta_+} \mathfrak{m}_\gamma.$$

Proof. The inclusion $\sum_{\gamma \in \Delta_+} \mathfrak{m}_\gamma \subseteq \mathfrak{m}$ follows from Proposition 2.1.

For the converse, let
$$\mathfrak{m}_+ = \sum_{\gamma \in \Delta_+} \mathfrak{m}_\gamma,$$
and
$$\mathfrak{g}_+ = \theta\mathfrak{n} \oplus (\mathfrak{a} \oplus \mathfrak{m}_+) \oplus \mathfrak{n}.$$
Since each $\mathfrak{m}_\gamma$ is an ideal in $\mathfrak{m}$, $\mathfrak{m}_+$ is an ideal in $\mathfrak{m}$, and $\mathfrak{m}_+ \oplus \mathfrak{a}$ is an ideal in $\mathfrak{m} \oplus \mathfrak{a}$. We prove that $\mathfrak{g}_+$ is an ideal in $\mathfrak{g}$. First, it is a subalgebra, since $\mathfrak{n}$ and $\theta\mathfrak{n}$ are subalgebras and
$$[\mathfrak{n}, \theta\mathfrak{n}] \subseteq \mathfrak{a} \oplus \mathfrak{m}_+ \oplus \mathfrak{n} \oplus \theta\mathfrak{n},$$
and since $\mathfrak{a}$ and $\mathfrak{m}_+$ preserve root spaces under the adjoint action, both being subspaces of $\mathfrak{g}_0$. It remains to show that $\mathfrak{g}_+$ is an ideal. It is already clear that $[\mathfrak{n} \oplus \mathfrak{a} \oplus \theta\mathfrak{n}, \mathfrak{g}_+] \subseteq \mathfrak{g}_+$. Let M be a generic element of $\mathfrak{m}$. Since $\mathrm{ad}(\mathfrak{m})$ preserves root spaces and $\mathfrak{m}_+$ is an ideal in $\mathfrak{m}$, we have by linearity
$$\mathrm{ad}(M)(\mathfrak{n} \oplus \mathfrak{m}_+ \oplus \mathfrak{a} \oplus \theta\mathfrak{n}) \subseteq \mathfrak{n} \oplus \mathfrak{m}_+ \oplus \mathfrak{a} \oplus \theta\mathfrak{n}.$$
Thus $\mathfrak{g}_+$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{g}_+ = \mathfrak{g}$ and so $\mathfrak{m}_+ = \mathfrak{m}$. ∎

Proposition 2.4. *Suppose that γ is a root and that 2γ is not a root. Suppose also that X_1, X_2, X_3, X_4 are pairwise orthogonal vectors in $\mathfrak{g}_\gamma$. Then*
$$[[X_1, \theta X_2], X_2] = \gamma([X_2, \theta X_2])X_1, \qquad [[X_1, \theta X_2], X_1] = -\gamma([X_1, \theta X_1])X_2, \tag{4}$$
$$[[X_1, \theta X_2], X_3] = 0, \tag{5}$$
$$[[X_1, \theta X_2], [X_3, \theta X_2]] = \gamma([\theta X_2, X_2])[X_1, \theta X_3], \tag{6}$$
and
$$[[X_1, \theta X_2], [X_3, \theta X_4]] = 0. \tag{7}$$

Proof. The formulae (4) are easy consequences of the Jacobi identity and of the fact that $2\gamma \notin \Delta$, since
$$[[X_1, \theta X_2], X_2] = [[X_2, \theta X_2], X_1] + [[X_1, X_2], \theta X_2] = \gamma([X_2, \theta X_2])X_1.$$

For (5) we have
$$[[X_1, \theta X_2], X_3] = [[\theta X_1, X_2], X_3] = [[\theta X_1, X_3], X_2] = -[[X_3, \theta X_1], X_2].$$

Permuting the indices twice we get

$$[[X_1, \theta X_2], X_3] = -[[X_3, \theta X_1], X_2] = [[X_2, \theta X_3], X_1] = -[[X_1, \theta X_2], X_3].$$

Hence,

$$[[X_1, \theta X_2], X_3] = 0.$$

Now (6) follows by the Jacobi identity using (4) and (5):

$$\begin{aligned}[][[X_1, \theta X_2], [X_3, \theta X_2]] &= [X_3, [[X_1, \theta X_2], \theta X_2]] + [\theta X_2, [X_3, [X_1, \theta X_2]]] \\ &= \gamma([\theta X_2, X_2])[X_3, \theta X_1].\end{aligned}$$

Similarly we get (7) from the Jacobi identity and (5). ∎

Definition 2.5. Let $\{X_1, \ldots, X_{d_\gamma}\}$ be an orthonormal basis of $\mathfrak{g}_\gamma$. When $i \neq j$, we set

$$X_{ij} = [X_i, \theta X_j].$$

Corollary 2.6. *Suppose that γ is a root and that 2γ is not a root. Then when $i \neq j$ and $k \neq l$, we have*

$$[X_{ij}, X_{kl}] = \gamma([\theta X_1, X_1])\,(\delta_{jl} X_{ik} + \delta_{ik} X_{jl} - \delta_{il} X_{jk} - \delta_{jk} X_{il}). \tag{8}$$

In particular, $\mathfrak{m}_\gamma$ is isomorphic to $\mathfrak{so}(d_\gamma)$ and $\dim \mathfrak{m}_\gamma = \frac{1}{2} d_\gamma (d_\gamma - 1)$.

Proof. By definition

$$\mathfrak{m}_\gamma = \operatorname{span}\{X_{ij} : i < j\}.$$

We know that $X_{ij} \neq 0$ by Proposition 2.1. Since

$$\langle [X_i, \theta X_j], [X_k, \theta X_l] \rangle = -\langle [[X_i, \theta X_j] X_l], X_k \rangle,$$

it follows from Proposition 2.4 that $\langle X_{ij}, X_{kl} \rangle \neq 0$ if and only if $(i,j) = (k,l)$ or $(i,j) = (l,k)$. Therefore, $\{X_{ij} : i < j\}$ is an orthogonal basis of $\mathfrak{m}_\gamma$. From Proposition 2.4 we get (8). These commutation relations show that $\mathfrak{m}_\gamma \cong \mathfrak{so}(d_\gamma)$. Hence, $\dim \mathfrak{m}_\gamma = \frac{1}{2} d_\gamma (d_\gamma - 1)$. ∎

We recall, from the classification of the algebras $\mathfrak{so}(d)$, that $\mathfrak{m}_\gamma$ is simple when $d_\gamma \geq 5$ or when $d_\gamma = 3$, and that when $d_\gamma = 4$,

$$\mathfrak{m}_\gamma = \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are ideals isomorphic to $\mathfrak{so}(3)$, and the decomposition is orthogonal.

Corollary 2.7. *Suppose that γ is a root and that 2γ is not a root. Then $\mathfrak{m}_\gamma$ acts irreducibly on $\mathfrak{g}_\gamma$. Moreover, if $d_\gamma \geq 2$ the action is faithful, that is, if $M \in \mathfrak{m}_\gamma$ and*

$$[M, X] = 0 \qquad \forall X \in \mathfrak{g}_\gamma,$$

then necessarily $M = 0$.

Proof. This holds since $\mathfrak{m}_\gamma$ acts on $\mathfrak{g}_\gamma$ as $\mathfrak{so}(d_\gamma)$ acts on $\mathbb{R}^{d_\gamma}$. ∎

Corollary 2.8. *Suppose that γ and δ are roots and that 2γ is not a root. If there exist two orthogonal vectors X_1 and X_2 in $\mathfrak{g}_\gamma$ such that*

$$[X_1, \theta X_2] \in \mathfrak{m}_\delta,$$

then $\mathfrak{m}_\gamma \subseteq \mathfrak{m}_\delta$.

Proof. The statement is vacuous if $d_\gamma = 1$ and obvious if $d_\gamma = 2$. Assume therefore that $d_\gamma \geq 3$. Extend $\{X_1, X_2\}$ to an orthogonal basis $\{X_1, X_2, \dots, X_{d_\gamma}\}$ of $\mathfrak{g}_\gamma$. Then the set $\{[X_i, \theta X_j] : i \neq j\}$ spans $\mathfrak{m}_\gamma$. If $i \geq 3$, since $[X_1, \theta X_2] \in \mathfrak{m}_\delta$, it follows from (6) that

$$-\gamma([\theta X_2, X_2])[X_1, \theta X_i] = [[X_1, \theta X_2], [X_2, \theta X_i]] \in \mathfrak{m}_\delta.$$

The same argument shows that $[X_i, \theta X_j] \in \mathfrak{m}_\delta$ whenever $1 \leq i < j \leq d_\gamma$, proving the statement. ∎

Proposition 2.9. *Let γ and δ be roots, and assume that $2\gamma \notin \Delta$. If the action of $\mathfrak{m}_\delta$ on $\mathfrak{g}_\gamma$ is trivial then $\mathfrak{m}_\gamma \cap \mathfrak{m}_\delta = \{0\}$. Suppose further that $d_\gamma \geq 3$. If $\mathfrak{m}_\gamma \cap \mathfrak{m}_\delta = \{0\}$, then the action of $\mathfrak{m}_\delta$ on $\mathfrak{g}_\gamma$ is trivial.*

Proof. Take M in $\mathfrak{m}_\gamma \cap \mathfrak{m}_\delta$. Since $M \in \mathfrak{m}_\delta$, we have

$$[M, Y] = 0 \qquad \forall Y \in \mathfrak{g}_\gamma.$$

However, M lies in $\mathfrak{m}_\gamma$ and the action of $\mathfrak{m}_\gamma$ on $\mathfrak{g}_\gamma$ is faithful by Corollary 2.7. Hence, $M = 0$.

Assume that $d_\gamma \geq 3$ and $\mathfrak{m}_\gamma \cap \mathfrak{m}_\delta = \{0\}$. As in Lemma 2.2, the intersection $\mathfrak{m}_\gamma \cap \mathfrak{m}_\delta$ is an ideal in both $\mathfrak{m}_\gamma$ and $\mathfrak{m}_\delta$. Since $[\mathfrak{m}_\gamma, \mathfrak{m}_\delta]$ is a subset of $\mathfrak{m}_\gamma \cap \mathfrak{m}_\delta$, it is trivial. In other words, for all M in $\mathfrak{m}_\delta$ the operator $\mathrm{ad}(M)$ lies in the commutant of $\mathfrak{m}_\gamma$. But $\mathfrak{m}_\gamma \cong \mathfrak{so}(d_\gamma)$ and any skew-symmetric operator on $\mathbb{R}^{d_\gamma}$ which commutes with $\mathfrak{so}(d_\gamma)$ is zero when $d_\gamma \geq 3$. ∎

Note that in the statement of Proposition 2.9 we suppose that $d_\gamma \geq 3$. It is clear from a study of $\mathfrak{sl}(3, \mathbb{C})$ that this hypothesis is necessary.

3. Generalized Heisenberg Lie algebras

In this section, we introduce generalized Heisenberg Lie algebras, and next examine their derivations. We then examine carefully four examples of these algebras, and finally show that the Iwasawa $\mathfrak{n}$ summands of simple Lie algebras of type A_2 are generalized Heisenberg Lie algebras.

The following definition of generalized Heisenberg (or H-type) Lie algebra is due to Kaplan [6].

Definition 3.1. Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra, endowed with an inner product $\langle\cdot,\cdot\rangle$. Denote by $\mathfrak{v}$ the orthogonal complement of the centre $\mathfrak{z}$ of $\mathfrak{n}$, and for each $Z \in \mathfrak{z}$ define a map $J_Z : \mathfrak{v} \to \mathfrak{v}$ by

$$\langle J_Z X, Y\rangle = \langle [X,Y], Z\rangle \qquad \forall X, Y \in \mathfrak{v}.$$

We say that $\mathfrak{n}$ is a *generalized Heisenberg algebra* if J_Z is orthogonal whenever $\|Z\| = 1$, or equivalently if

$$\|J_Z X\| = \|Z\|\,\|X\| \qquad \forall Z \in \mathfrak{z} \quad \forall X \in \mathfrak{v}.$$

Since the Lie bracket is skew-symmetric, we have

$$\langle J_Z X, Y\rangle = -\langle X, J_Z Y\rangle\ ,$$

for all X and Y in $\mathfrak{v}$. Hence,

$$J_Z^2 = -\|Z\|^2 I.$$

By polarization, we obtain

$$\langle J_Z X, J_{Z'} X\rangle = -\langle Z, Z'\rangle \|X\|^2\, I\ ,$$

and

$$J_Z J_{Z'} + J_{Z'} J_Z = -2\langle Z, Z'\rangle I.$$

In particular the last relation shows that, if d denotes the dimension of $\mathfrak{z}$, then $\mathfrak{v}$ is a module for the Clifford algebra with d generators $\mathcal{C}(d)$ (for the definition and properties of Clifford algebras, see, for instance, [9] or [10]).

Take Z in $\mathfrak{z}$. Let $\hat{J}_Z$ be the extension to $\mathfrak{n}$ of J_Z which acts on the centre as minus the reflection in the hyperplane orthogonal to Z. Then $\hat{J}_Z$ is an automorphism of the Lie algebra $\mathfrak{n}$.

Fix $d \geq 1$. We recall that there exists an H-type algebra with a d-dimensional centre. We say that two H-type algebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are *isomorphic* if there is an orthogonal Lie isomorphism between $\mathfrak{n}_1$ and $\mathfrak{n}_2$. We say that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is *irreducible* if there is no $\mathfrak{v}' \subset \mathfrak{v}$ such that $\mathfrak{v}' \oplus \mathfrak{z}$ is H-type.

Proposition 3.2. *Up to isomorphism there is exactly one irreducible H-type algebra with centre of dimension d.*

We will denote the irreducible generalized Heisenberg algebra with d-dimensional centre by $\mathfrak{n}_d = \mathfrak{v}_d \oplus \mathfrak{z}_d$.

We now discuss the derivations of a generalized Heisenberg algebra, following C. Riehm [11]. Let $\mathcal{D}(\mathfrak{n})$ be the space of *skew-symmetric* derivations of $\mathfrak{n}$. If $D \in \mathcal{D}(\mathfrak{n})$, then D, being a derivation, maps $\mathfrak{z}$ into itself, and since D is skew-symmetric it also maps $\mathfrak{v}$ into itself.

Let $\{Z_1, \ldots, Z_d\}$ be an orthonormal basis of $\mathfrak{z}$. We will set $J_i = J_{Z_i}$. For each pair (i,k) with $i \neq k$ define a linear endomorphism D^0_{ik} of $\mathfrak{n}_d$ by

$$D^0_{ik} V = J_i J_k V \qquad \forall V \in \mathfrak{v},$$

and

$$D^0_{ik} Z_l = 2\delta_{il} Z_k - 2\delta_{kl} Z_i \qquad \forall 1 \leq l \leq d.$$

Then set

$$\mathcal{D}_{\mathfrak{z}}(\mathfrak{n}) = \operatorname{span}\{D^0_{ik} : i \neq k\}.$$

It is easy to verify that $\mathcal{D}_{\mathfrak{z}}(\mathfrak{n})$ is a subspace of $\mathcal{D}(\mathfrak{n})$. We see from the definition that the restrictions of the elements of $\mathcal{D}_{\mathfrak{z}}(\mathfrak{n})$ to $\mathfrak{z}$ span the algebra of all linear skew-symmetric maps of $\mathfrak{z}$ into itself. Now, since the D^0_{ik}'s are derivations, an easy computation shows that they satisfy the commutation relations of $\mathfrak{so}(d)$. Hence, $\mathcal{D}_{\mathfrak{z}}(\mathfrak{n})$ is a Lie algebra isomorphic to $\mathfrak{so}(d)$.

Let $\mathcal{D}_0(\mathfrak{n})$ denote the space of the skew-symmetric derivations of $\mathfrak{n}$ which act trivially on $\mathfrak{z}$.

Proposition 3.3. *Take D in $\mathcal{D}(\mathfrak{n})$. For all Z in $\mathfrak{z}$ we have*

$$D|_{\mathfrak{v}} J_Z = J_{DZ} + J_Z D|_{\mathfrak{v}}.$$

Furthermore, $\mathcal{D}(\mathfrak{n}) = \mathcal{D}_{\mathfrak{z}}(\mathfrak{n}) \oplus \mathcal{D}_0(\mathfrak{n})$, and $\mathcal{D}_{\mathfrak{z}}$ and $\mathcal{D}_0$ commute.

Proof. Since D is skew-symmetric and a derivation,

$$\begin{aligned}
\langle DJ_Z X, Y\rangle &= -\langle J_Z X, DY\rangle \\
&= -\langle Z, [X, DY]\rangle \\
&= -\langle Z, D[X,Y]\rangle + \langle Z, [DX, Y]\rangle \\
&= \langle DZ, [X,Y]\rangle + \langle J_Z DX, Y\rangle \\
&= \langle J_{DZ} X, Y\rangle + \langle J_Z DX, Y\rangle,
\end{aligned}$$

for all Z in $\mathfrak{z}$ and all X and Y in $\mathfrak{v}$, which proves the first result.

Let $D|_{\mathfrak{z}}$ be the restriction of a derivation D to $\mathfrak{z}$. Since $D|_{\mathfrak{z}}$ is a skew-symmetric linear map of $\mathfrak{z}$ into itself, it is an element of $\mathfrak{so}(d)$. Therefore,

$$D|_{\mathfrak{z}} = \sum_{i \neq k} c_{ik} D^0_{ik}|_{\mathfrak{z}},$$

where the coefficients $c_{ik} = -c_{ki}$ in $\mathbb{R}$ are uniquely determined by $D|_{\mathfrak{z}}$. Let

$$\tilde{D} = D - \sum_{i \neq k} c_{ik} D^0_{ik}. \tag{9}$$

Then $\tilde{D}$ is a skew-symmetric derivation of $\mathfrak{n}$ acting trivially on $\mathfrak{z}$. Further the decomposition (9) is unique and $\mathcal{D}_{\mathfrak{z}}$ and $\mathcal{D}_0$ commute, since

$$\mathcal{D}_{\mathfrak{z}}(\mathfrak{n}) \cap \mathcal{D}_0(\mathfrak{n}) = \{0\},$$

as required. ■

Now we are interested in studying the generalized Heisenberg algebras with centres of dimension of 1, 2, 4, and 8. For the reader's convenience we recall that $\mathcal{C}(1) \cong \mathbb{C}$, $\mathcal{C}(2) \cong \mathbb{H}$, $\mathcal{C}(4) \cong \mathbb{H}(2)$, and $\mathcal{C}(8) \cong \mathbb{R}(16)$, where $\mathbb{F}(n)$ denotes the space of $n \times n$ matrices with entries in the field $\mathbb{F}$.

It is well known (see, for instance, [9] or [10]) that any Clifford algebra $\mathcal{C}(d)$ splits into the direct sum of its even and odd parts, denoted by $\mathcal{C}^+(d)$ and

$\mathcal{C}^-(d)$ respectively, and that $\mathcal{C}^+(d)$ is a Clifford algebra isomorphic to $\mathcal{C}(d-1)$. Corresponding to this decomposition of $\mathcal{C}(d)$, when d in $\{1,2,4,8\}$, the $\mathcal{C}(d)$-modules $\mathfrak{v}_d$ split into the sum $\mathfrak{v}_{d1} \oplus \mathfrak{v}_{d2}$, where $\mathfrak{v}_{d1}$ and $\mathfrak{v}_{d2}$ are irreducible $\mathcal{C}^+(d)$-modules. The algebras $\mathcal{C}^+(1)$ and $\mathcal{C}^+(2)$, which are isomorphic to $\mathbb{R}$ and $\mathbb{C}$ have, up to equivalence, only one irreducible module: the irreducible module for $\mathcal{C}^+(1)$ is isomorphic to $\mathbb{R}$ and that for $\mathcal{C}^+(2)$ is isomorphic to $\mathbb{C}$. However, $\mathcal{C}^+(4) \cong \mathbb{H} \oplus \mathbb{H}$ and $\mathcal{C}^+(8) \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$ have two inequivalent irreducible modules, $\mathfrak{v}_{d1}$ and $\mathfrak{v}_{d2}$ say. As real vector spaces, the irreducible $\mathcal{C}^+(4)$-modules are of dimension 4, while the irreducible modules of $\mathcal{C}^+(8)$ are of dimension 8. These modules may be distinguished by looking at the action of $\epsilon = J_1 J_2 \dots J_d$, which acts as the identity on $\mathfrak{v}_{d1}$ and as minus the identity on $\mathfrak{v}_{d2}$.

Corollary 3.4. *Suppose that d in $\{1,2,4,8\}$ and that $D \in \mathcal{D}(\mathfrak{n})$. If $D|_{\mathfrak{z}} = 0$ and $D\,\mathfrak{v}_{d1} \subseteq \mathfrak{v}_{d1}$, then the restrictions $D|_{\mathfrak{v}_{d1}}$ and $D|_{\mathfrak{v}_{d2}}$ of D are related by the formula*

$$D|_{\mathfrak{v}_{d1}} = -J_Z D|_{\mathfrak{v}_{d2}} J_Z \tag{10}$$

for all Z in $\mathfrak{z}$ of norm one.

Proof. Since D acts trivially on $\mathfrak{z}$, we get from Proposition 3.3

$$D|_{\mathfrak{v}_d} J_Z = J_Z D|_{\mathfrak{v}_d} \qquad \forall Z \in \mathfrak{z},$$

which, for Z with norm one, implies

$$D|_{\mathfrak{v}_{d2}} = -J_Z D|_{\mathfrak{v}_{d1}} J_Z,$$

as required. ∎

We denote by $\mathcal{D}_0^+(\mathfrak{n}_d)$ the subspace of $\mathcal{D}_0(\mathfrak{n}_d)$ which consists of derivations preserving the $\mathcal{C}^+(d)$-modules $\mathfrak{v}_{d1}$ and $\mathfrak{v}_{d2}$. In general $\mathcal{D}_0^+(\mathfrak{n}_d)$ is a proper subspace of $\mathcal{D}_0(\mathfrak{n}_d)$. We observe however that for d in $\{4,8\}$, when $\mathfrak{v}_d$ is the sum of two inequivalent modules, then necessarily $\mathcal{D}_0^+(\mathfrak{n}_d) = \mathcal{D}_0(\mathfrak{n}_d)$ since any D in $\mathcal{D}_0^+(\mathfrak{n}_d)$ which commutes with J_Z also commutes with ϵ.

Proposition 3.5. *The dimensions of the spaces $\mathcal{D}_0^+(\mathfrak{n}_d)$ are equal to 0, 1, 3, and 0 when d is equal to 1, 2, 4, and 8 respectively.*

Proof. As we observed $\mathfrak{v}_{d1}$ and $\mathfrak{v}_{d2}$ are irreducible $\mathcal{C}^+(d)$-modules. Moreover, any derivation in $\mathcal{D}_0^+(\mathfrak{n}_d)$ commutes with the action of $\mathcal{C}^+(d)$. Now for $d=1$, since $\mathcal{C}^+(1) \cong \mathbb{R}$, the commutant is $\mathbb{R}$. For $d=2$, since $\mathcal{C}^+(2) \cong \mathbb{C}$, the commutant is $\mathbb{C}$. For $d=4$, since $\mathcal{C}^+(4) \cong \mathbb{H} \oplus \mathbb{H}$, the commutant is $\mathbb{H}$. For $d=8$, since $\mathcal{C}^+(8) \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$, the commutant is $\mathbb{R}$. ∎

We conclude this section by showing that generalized Heisenberg algebras occur naturally in real simple Lie algebras. This provides the link between the Clifford algebraic approach just presented and semisimple Lie algebras.

Lemma 3.6. *Suppose that $\alpha, \beta, \alpha+\beta \in \Delta$ and that $2\alpha+\beta, \alpha+2\beta \notin \Delta$. Then*

$$(\alpha \mid \alpha+\beta) = (\beta \mid \alpha+\beta) = \frac{1}{2}(\alpha+\beta \mid \alpha+\beta). \tag{11}$$

Proof. Recall that if γ, δ are distinct roots, then (see [1, Chapter VI, 1.3])

$$(\gamma \mid \delta) < 0 \Rightarrow \gamma+\delta \in \Delta, \tag{12}$$

moreover, if $(\gamma \mid \delta) = 0$, then

$$\gamma+\delta \in \Delta \Longleftrightarrow \gamma-\delta \in \Delta. \tag{13}$$

We also recall that

$$2\frac{(\gamma \mid \delta)}{(\gamma \mid \gamma)} \in \{0, \pm 1, \pm 2, \pm 3\}. \tag{14}$$

We claim that

$$(\alpha \mid \alpha+\beta) > 0 \qquad \text{and} \qquad (\beta \mid \alpha+\beta) > 0. \tag{15}$$

Indeed, if $(\alpha \mid \alpha+\beta) < 0$, then $2\alpha+\beta \in \Delta$ by (12), contradicting the hypothesis, and $(\alpha \mid \alpha+\beta) \neq 0$ by (13) since $\beta \in \Delta$. Thus $(\alpha \mid \alpha+\beta) > 0$; exchanging the roles of α and β shows that $(\beta \mid \alpha+\beta) > 0$. Since

$$\frac{(\alpha \mid \alpha+\beta)}{(\alpha+\beta \mid \alpha+\beta)} + \frac{(\beta \mid \alpha+\beta)}{(\alpha+\beta \mid \alpha+\beta)} = 1,$$

(14) and (15) imply (11). ■

Set

$$\mathfrak{n} = (\mathfrak{g}_\alpha + \mathfrak{g}_\beta) \oplus \mathfrak{g}_{\alpha+\beta}, \qquad \mathfrak{z} = \mathfrak{g}_{\alpha+\beta} \qquad \text{and} \qquad \mathfrak{v} = \mathfrak{g}_\alpha + \mathfrak{g}_\beta,$$

so that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$.

Definition 3.7. For Z in $\mathfrak{z}$, we define an operator J_Z on $\mathfrak{v}$ by

$$J_Z X = [Z, \theta X] \qquad \forall X \in \mathfrak{v}.$$

Theorem 3.8. *Fix the constant c in (1) such that*

$$(\alpha+\beta \mid \alpha+\beta) = 2.$$

Then $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, endowed with the inner product $\langle \cdot, \cdot \rangle$, is a generalized Heisenberg algebra with centre $\mathfrak{z}$. In particular, for all Z in $\mathfrak{z}$ and X and Y in $\mathfrak{v}$

$$\langle J_Z X, Y \rangle = -\langle X, J_Z Y \rangle, \tag{16}$$
$$[X, J_Z X] = \|X\|^2 Z, \tag{17}$$

and

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle. \tag{18}$$

Proof. We first prove that J_Z is skew-symmetric. Take X and Y in $\mathfrak{v}$. From the definition of J_Z and (2) it follows that

$$\begin{aligned}\langle J_Z X, Y\rangle &= \langle [Z, \theta X], Y\rangle = \langle Z, [X, Y]\rangle = -\langle Z, [Y, X]\rangle \\ &= -\langle [Z, \theta Y], X\rangle = -\langle J_Z Y, X\rangle = -\langle X, J_Z Y\rangle,\end{aligned}$$

i.e., (16) holds.

Now we prove that $J_Z^2 X = -\|Z\|^2 X$ for all X in $\mathfrak{v}$. Let Z be in $\mathfrak{z}$ and suppose that X is in $\mathfrak{g}_\alpha$. From the definition of J_Z we get

$$J_Z^2 X = [Z, \theta[Z, \theta X]].$$

Hence, from the Jacobi identity

$$J_Z^2 X = [Z, [\theta Z, X]] = [[Z, \theta Z], X] = -\alpha([\theta Z, Z])X = -\|Z\|^2 X, \tag{19}$$

by (3.8). The case where $X \in \mathfrak{g}_\beta$ is similar.

It is clear from the structure of the root system that $\mathfrak{g}_{\alpha+\beta} \subseteq \mathfrak{z}$. We prove the converse. It follows from Lemma 3.6 and formula (3.8) that

$$\begin{aligned}[X, J_Z X] &= [X, [Z, \theta X]] = [Z, [X, \theta X]] = (\alpha + \beta)([\theta X, X])Z \\ &= \frac{1}{2}(\alpha + \beta \mid \alpha + \beta)\|X\|^2 Z = \|X\|^2 Z \qquad \forall X \in \mathfrak{v}.\end{aligned}$$

This proves that for all nonzero X in $\mathfrak{v}$ the vector $J_Z X$ is such that $[X, J_Z X] \neq 0$, whence $\mathfrak{v} \cap \mathfrak{z} = \{0\}$. Therefore, $\mathfrak{z}$ coincides with $\mathfrak{g}_{\alpha+\beta}$. ∎

4. The structure of $\mathfrak{n}$

From now on, we consider real semisimple Lie algebras with root system A_2. This means that

$$\Delta_+ = \{\alpha, \beta, \alpha + \beta\}, \tag{20}$$

and that all roots have the same length, hence

$$(\alpha \mid \beta) = -\frac{1}{2}(\alpha \mid \alpha). \tag{21}$$

It is clear that

$$\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta}.$$

As before, we set

$$\mathfrak{z} = \mathfrak{g}_{\alpha+\beta} \qquad \text{and} \qquad \mathfrak{v} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta.$$

If we choose the constant c in (1) so that for all γ in Δ,

$$\langle H_\gamma, H_\gamma\rangle = 2,$$

that is

$$\gamma([\theta X, X]) = 2\|X\|^2, \tag{22}$$

for all X in $\mathfrak{g}_\gamma$, then $\mathfrak{n}$, endowed with the inner product $\langle\cdot,\cdot\rangle$, is a generalized Heisenberg algebra, by Theorem 3.8. From (21) and (22) it follows that

$$\alpha([\theta Z, Z]) = \beta([\theta Z, Z]) = \|Z\|^2, \tag{23}$$

for all Z in $\mathfrak{g}_{\alpha+\beta}$, and that

$$\alpha([\theta Y, Y]) = -\|Y\|^2 \qquad \text{and} \qquad \beta([\theta X, X]) = -\|X\|^2, \tag{24}$$

for all Y in $\mathfrak{g}_\beta$ and all X in $\mathfrak{g}_\alpha$.

Lemma 4.1. *For all nonzero vectors X in $\mathfrak{g}_\alpha$ the map $Z \mapsto J_Z X$ from $\mathfrak{g}_{\alpha+\beta}$ into $\mathfrak{g}_\beta$ is one-to-one and onto. Its inverse is $\|X\|^{-2}\,\mathrm{ad}_X$. Similarly, for all nonzero Y in $\mathfrak{g}_\beta$ the map $Z \mapsto J_Z Y$ from $\mathfrak{g}_{\alpha+\beta}$ into $\mathfrak{g}_\alpha$ is one-to-one and onto and its inverse is $\|Y\|^{-2}\,\mathrm{ad}_Y$. In particular, $d_\alpha = d_\beta = d_{\alpha+\beta}$.*

Proof. If $J_Z X = 0$ and $X \neq 0$, it follows that $Z = 0$ from (17). Therefore, $Z \mapsto J_Z X$ is one-to-one. This map is also onto because given Y in $\mathfrak{g}_\beta$, using the Jacobi identity and the fact that $\alpha - \beta \notin \Delta$, we find that

$$J_{[X,Y]}X = [[X,Y],\theta X] = [[X,\theta X],Y] = \beta([X,\theta X])Y = \|X\|^2\,Y.$$

The inverse of this map is $\|X\|^{-2}\,\mathrm{ad}_X$ by (17).

By symmetry the same is true for $Z \mapsto J_Z Y$. ■

From the lemma we immediately get the following corollaries.

Corollary 4.2. *The following relations hold:*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}, \qquad [\mathfrak{g}_{\alpha+\beta}, \mathfrak{g}_{-\beta}] = \mathfrak{g}_\alpha \quad \textit{and} \quad [\mathfrak{g}_{\alpha+\beta}, \mathfrak{g}_{-\alpha}] = \mathfrak{g}_\beta. \tag{25}$$

Actually, as we show in [2], formulae (25) hold in all simple real Lie algebras.

Corollary 4.3. *Let $\{Z_1, Z_2, \ldots, Z_d\}$ be an orthonormal basis of $\mathfrak{z}$ and let X be a unit vector in $\mathfrak{g}_\alpha$. Then $\{X_k \equiv J_1 J_k X : 1 \le k \le d\}$ and $\{Y_k \equiv J_k X : 1 \le k \le d\}$ are orthonormal bases of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ respectively. Furthermore,*

$$\{X_{ik} \equiv [J_1 J_i X, \theta J_1 J_k X] : i < k\} \qquad \textit{and} \qquad \{Y_{ik} \equiv [J_i X, \theta J_k X] : i < k\}$$

are orthogonal bases of $\mathfrak{m}_\alpha$ and $\mathfrak{m}_\beta$ respectively.

Theorem 4.4. *The root spaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_\beta$, and $\mathfrak{g}_{\alpha+\beta}$ all have the same dimension, which can be either 1, 2, 4, or 8. Furthermore, $\mathfrak{v}$ is an irreducible $\mathcal{C}(d)$-module.*

Proof. All the roots have the same multiplicity by Lemma 4.1. In other words, the number of generators ($\dim\mathfrak{z} = \dim\mathfrak{g}_{\alpha+\beta}$) of the Clifford algebra involved in the generalized Heisenberg algebra is half the dimension of the Clifford module $\mathfrak{v}$ (which is $\dim\mathfrak{g}_\alpha + \dim\mathfrak{g}_\beta$). There are few Clifford modules with this property. These are the irreducible modules for the Clifford algebras $\mathcal{C}(1) = \mathbb{C}$, $\mathcal{C}(2) = \mathbb{H}$, $\mathcal{C}(4) = \mathbb{C}(4)$, and $\mathcal{C}(8) = \mathbb{R}(16)$, whose real irreducible modules have dimension $2, 4, 8, 16$ respectively (see [9, p. 28]). ■

Proposition 4.5. *The root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are irreducible $\mathcal{C}^+(d_\alpha)$-modules. If $d_\alpha \in \{4, 8\}$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are inequivalent $\mathcal{C}^+(d_\alpha)$-modules.*

Proof. The first part follows from the discussion preceding Corollary 3.4.

Now assume that $d_\alpha \in \{4, 8\}$. Since $\epsilon = J_1 J_2 \dots J_{d_\alpha}$ commutes with $J_j J_k$ for all j, k in $\{1, \dots, d_\alpha\}$, and since $\epsilon^2 = 1$, it follows from the irreducibility of the $\mathcal{C}^+(d_\alpha)$-modules $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ that ϵ is $\pm I$ on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$. Since ϵ anti-commutes with J_1, it follows that if $\epsilon = I$ on $\mathfrak{g}_\alpha$, then $\epsilon = -I$ on $\mathfrak{g}_\beta$ and vice versa. ∎

5. The structure of $\mathfrak{m}$

Lemma 5.1.

$$\mathfrak{m} = \mathfrak{m}_\alpha + \mathfrak{m}_\beta = \mathfrak{m}_\alpha + \mathfrak{m}_{\alpha+\beta} = \mathfrak{m}_\beta + \mathfrak{m}_{\alpha+\beta}.$$

Proof. Proposition 2.3 implies that $\mathfrak{m} = \mathfrak{m}_\alpha + \mathfrak{m}_\beta + \mathfrak{m}_{\alpha+\beta}$. Thus, to prove that $\mathfrak{m} = \mathfrak{m}_\alpha + \mathfrak{m}_\beta$, it suffices to show that

$$\mathfrak{m}_{\gamma+\delta} \subseteq \mathfrak{m}_\gamma + \mathfrak{m}_\delta.$$

Suppose that M is in $\mathfrak{m}_{\gamma+\delta}$. By Proposition 2.3, we may write

$$M = \sum_{i=1}^{m} [Z_i, \theta Z_i'],$$

where Z_i and Z_i' are orthogonal vectors in $\mathfrak{g}_{\gamma+\delta}$ for each index i. It is sufficient to prove that if $Z, Z' \in \mathfrak{g}_{\gamma+\delta}$ and $\langle Z, Z' \rangle = 0$, then $[Z, \theta Z'] \in \mathfrak{m}_\gamma + \mathfrak{m}_\delta$. Since $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] = \mathfrak{g}_{\gamma+\delta}$ by Corollary 4.2, there are vectors X_j and Y_j in $\mathfrak{g}_\gamma$ and $\mathfrak{g}_\delta$ respectively such that

$$Z' = \sum_{1 \le j \le n} [X_j, Y_j],$$

where n is a positive integer. By the Jacobi identity,

$$\begin{aligned}[Z, \theta Z'] &= \sum_{1 \le j \le n} [Z, \theta[X_j, Y_j]] \\ &= \sum_{1 \le j \le n} [\theta X_j, \theta[\theta Z, Y_j]] + \sum_{1 \le j \le n} [[Z, \theta X_j], \theta Y_j] \in (\mathfrak{m}_\gamma + \mathfrak{m}_\delta) \oplus \mathfrak{a}.\end{aligned}$$

But the left hand side lies in $\mathfrak{m}$, hence $[Z, \theta Z']$ belongs to $\mathfrak{m}_\gamma + \mathfrak{m}_\delta$.

The other equalities are proved similarly. ∎

Lemma 5.2. *Let $\{Z_1, Z_2\}$ be an orthonormal pair in $\mathfrak{z}$, and X be a unit vector in $\mathfrak{v}$. Then*

$$[Z_1, \theta Z_2] = [X, \theta J_1 J_2 X] + [J_1 X, \theta J_2 X]. \tag{26}$$

Proof. Since $Z_2 = [X, J_2 X]$ by (17), the Jacobi identity implies (26). ∎

Observe that if $d_\alpha = 1$, then $\mathfrak{m} = \{0\}$. For d in $\{2,4,8\}$, we consider the decomposition

$$\mathfrak{m} = \mathfrak{m}_{\alpha+\beta} \oplus \mathfrak{m}_{\alpha+\beta}^{\perp}.$$

The elements of $\mathfrak{m}$ are clearly derivations of the generalized Heisenberg algebra $\mathfrak{n}$. If Z_1, Z_2 are in $\mathfrak{g}_{\alpha+\beta}$ and V is in $\mathfrak{v}$, then

$$[[Z_1, \theta Z_2], V] = [[V, \theta Z_2], Z_1] = [Z_1, \theta[Z_2, \theta V]] = J_1 J_2 V. \tag{27}$$

Now, since $\mathfrak{m}_{\alpha+\beta} \cong \mathfrak{so}(d_\alpha)$ and since the space of restrictions to $\mathfrak{g}_{\alpha+\beta}$ of the elements of $\mathcal{D}_{\mathfrak{z}}(\mathfrak{n})$ is isomorphic to $\mathfrak{so}(d_\alpha)$, we see that

$$\mathfrak{m}_{\alpha+\beta} = \mathcal{D}_{\mathfrak{z}}(\mathfrak{n}).$$

We will see later that $\mathfrak{m}_{\alpha+\beta}^{\perp} = \mathcal{D}_0^+(\mathfrak{n})$.

Proposition 5.3. *Suppose that $d_\alpha \geq 2$. If $\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha = \{0\}$, then $d_\alpha = 2$. If $d_\alpha = 2$, then $\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha = \mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\beta = \mathfrak{m}_\beta \cap \mathfrak{m}_\alpha = \{0\}$ and $\mathfrak{m}_\alpha \cong \mathfrak{m}_\beta \cong \mathfrak{so}(2)$; moreover,*

$$\mathfrak{m}_{\alpha+\beta} = \mathbb{R}\{[\theta X, J_1 J_2 X] + [J_1 X, \theta J_2 X]\},$$
$$\mathfrak{m}_{\alpha+\beta}^{\perp} = \mathbb{R}\{[\theta X, J_1 J_2 X] - [J_1 X, \theta J_2 X]\},$$

and

$$\mathcal{D}_0^+(\mathfrak{n}_2) = \mathfrak{m}_{\alpha+\beta}^{\perp}.$$

Proof. Assume that $\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha = \{0\}$. If d_α were greater than or equal to 3, then the action of $\mathfrak{m}_{\alpha+\beta}$ on $\mathfrak{g}_\alpha$ would be trivial by Proposition 2.9, yielding a contradiction, since $(\operatorname{ad}[Z_1, \theta Z_2])X = J_1 J_2 X$. Hence, $d_\alpha = 2$.

For the converse, assume that $d_\alpha = 2$, and suppose by contradiction that $\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha \neq \{0\}$. Then we would have

$$[Z_1, \theta Z_2] = c\,[X, \theta J_1 J_2 X], \tag{28}$$

with c in $\mathbb{R}$. Applying (28) to Z_1, we deduce from the Jacobi identity that

$$2Z_2 = c\,[[X, \theta J_1 J_2 X], Z_1] = c\,[[Z_1, \theta J_1 J_2 X], X] = c\,[J_1^2 J_2 X, X] = cZ_2,$$

whence $c = 2$. On the other hand, (28) applied to X gives

$$J_1 J_2 X = 2[[\theta X, J_1 J_2 X], X] = 2\alpha([\theta X, X]) J_1 J_2 X,$$

which implies that $(\alpha \mid \alpha) = 1/2$, yielding a contradiction. Thus, the sum of $\mathfrak{m}_\alpha$ and $\mathfrak{m}_\beta$ is direct, and we conclude from Lemma 5.1 that

$$\mathfrak{m} = \mathfrak{m}_\alpha \oplus \mathfrak{m}_\beta.$$

From Corollary 2.6, it follows that $\mathfrak{m}_\alpha \simeq \mathfrak{m}_\beta \simeq \mathfrak{m}_{\alpha+\beta} \simeq \mathfrak{so}(2)$.

From (26), we obtain immediately that

$$\mathfrak{m}_{\alpha+\beta} = \operatorname{span}\{[X, \theta J_1 J_2 X] + [J_1 X, \theta J_2 X]\},$$

which implies that

$$\mathfrak{m}_{\alpha+\beta}^{\perp} = \mathbb{R}([X, \theta J_1 J_2 X] - [J_1 X, \theta J_2 X]).$$

It is now clear that

$$\mathfrak{m}_{\alpha+\beta}^{\perp} \subseteq \mathcal{D}_0^+(\mathfrak{n}_2).$$

Since $\dim \mathcal{D}_0^+(\mathfrak{n}_2) = 1$ by Proposition 3.5, it follows that

$$\mathfrak{m}_{\alpha+\beta}^{\perp} = \mathcal{D}_0^+(\mathfrak{n}_2),$$

as required. ∎

Lemma 5.4. *Assume that $d \in \{4, 8\}$. If $\{Z_1, Z_2, Z_3, Z_4\}$ is an orthonormal system in $\mathfrak{g}_{\alpha+\beta}$ and X is a vector in $\mathfrak{g}_\alpha$ such that*

$$J_1 J_2 J_3 J_4 X = X, \tag{29}$$

then

$$\begin{gathered} Z_{12} + Z_{34} = Y_{12} + Y_{34}, \qquad Z_{13} + Z_{42} = Y_{13} + Y_{42}, \\ Z_{14} + Z_{23} = Y_{14} + Y_{23}, \qquad Z_{12} - Z_{34} = X_{34} - X_{12}, \\ Z_{13} - Z_{42} = X_{42} - X_{13}, \qquad Z_{14} - Z_{23} = X_{23} - X_{14}, \\ Y_{12} - Y_{34} = X_{12} + X_{34}, \qquad Y_{13} - Y_{42} = X_{13} + X_{42}, \\ Y_{14} - Y_{23} = X_{14} + X_{23}. \end{gathered}$$

Notice that when $d_\alpha = 4$, it is always possible to reorder any given orthonormal basis of $\mathfrak{g}_{\alpha+\beta}$ in such a way that (29) holds.

Proof. From (26) and (29) we get

$$[Z_3, \theta Z_4] = [X, \theta J_3 J_4 X] + [J_3 X, \theta J_4 X] = -[X, \theta J_1 J_2 X] + [J_3 X, \theta J_4 X].$$

Summing this formula and (26) we obtain

$$[Z_1, \theta Z_2] + [Z_3, \theta Z_4] = [J_1 X, \theta J_2 X] + [J_3 X, \theta J_4 X],$$

which gives the first equality enunciated. The others may be proved similarly. ∎

Proposition 5.5. *Suppose that $d_\alpha = 4$. Then*

$$\mathfrak{m} = (\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha) \oplus (\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\beta) \oplus (\mathfrak{m}_\alpha \cap \mathfrak{m}_\beta),$$

and

$$\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha \cong \mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\beta \cong \mathfrak{m}_\alpha \cap \mathfrak{m}_\beta \cong \mathfrak{so}(3).$$

If $\{Z_1, Z_2, Z_3, Z_4\}$ is an orthonormal basis of $\mathfrak{g}_{\alpha+\beta}$ such that $J_1 J_2 J_3 J_4 X = X$, then

$$\begin{aligned} \mathfrak{m}_\alpha \cap \mathfrak{m}_{\alpha+\beta} &= \operatorname{span}\{Z_{12} - Z_{34}, Z_{13} - Z_{42}, Z_{14} - Z_{23}\}, \\ \mathfrak{m}_\beta \cap \mathfrak{m}_{\alpha+\beta} &= \operatorname{span}\{Z_{12} + Z_{34}, Z_{13} + Z_{42}, Z_{14} + Z_{23}\}, \end{aligned}$$

and

$$\mathfrak{m}_\alpha \cap \mathfrak{m}_\beta = \operatorname{span}\{X_{12} + X_{34}, X_{13} + X_{42}, X_{14} + X_{23}\}.$$

Finally, $\mathcal{D}_0^+(\mathfrak{n}_4) = \mathfrak{m}_\alpha \cap \mathfrak{m}_\beta$.

Proof. From Proposition 5.3, it follows that $\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha$ is a nontrivial ideal in $\mathfrak{m}_\alpha$ and in $\mathfrak{m}_{\alpha+\beta}$. However, we see from (29) that

$$[Z_{12} + Z_{34}, X] = 0,$$

which implies that $\mathfrak{m}_{\alpha+\beta} \neq \mathfrak{m}_\alpha$, by Corollary 2.7. Since

$$\mathfrak{m}_{\alpha+\beta} \simeq \mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3),$$

it follows that $\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha \simeq \mathfrak{so}(3)$.

By Lemma 5.4 we have

$$Z_{12} + Z_{34} = Y_{12} + Y_{34},$$

so that

$$Z_{12} + Z_{34} \in \mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\beta.$$

Since $\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\beta$ is an ideal it follows from Proposition 2.4 that

$$Z_{13} + Z_{42} \in \mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\beta \qquad \text{and} \qquad Z_{14} + Z_{23} \in \mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\beta.$$

Hence,

$$\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\beta = \operatorname{span}\{Z_{12} + Z_{34}, Z_{13} + Z_{42}, Z_{14} + Z_{23}\}.$$

The rest of the proposition follows similarly. ∎

Proposition 5.6. *Suppose that $d_\alpha = 8$. Then*

$$\mathfrak{m} = \mathfrak{m}_\alpha = \mathfrak{m}_\beta = \mathfrak{m}_{\alpha+\beta} \cong \mathfrak{so}(8).$$

Proof. By Proposition 5.3, $\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha$ is a nontrivial ideal in $\mathfrak{m}_\alpha$ and $\mathfrak{m}_{\alpha+\beta}$. Since $\mathfrak{m}_\alpha \simeq \mathfrak{so}(8)$, which is simple, it follows that $\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha = \mathfrak{m}_\alpha$ and similarly that $\mathfrak{m}_{\alpha+\beta} \cap \mathfrak{m}_\alpha = \mathfrak{m}_{\alpha+\beta}$. ∎

Corollary 5.7. *Suppose that $d_\alpha = 8$. For all pairs (i,j) with $i \neq j$, there are uniquely determined coefficients $x_{ij,kl}$ and $y_{ij,kl}$ in $\mathbb{R}$ such that*

$$[X_i, \theta X_j] = \sum_{1 \le k < l \le 8} x_{ij,kl}\,[Z_k, \theta Z_l], \qquad \textit{and} \qquad [Y_i, \theta Y_j] = \sum_{1 \le k < l \le 8} y_{ij,kl}\,[Z_k, \theta Z_l].$$

Proof. By Proposition 5.6, $\mathfrak{m} = \mathfrak{m}_\alpha = \mathfrak{m}_\beta = \mathfrak{m}_{\alpha+\beta} \cong \mathfrak{so}(8)$. Therefore, each of the sets $\{[X_i, \theta X_j]\}$, $\{[Y_i, \theta Y_j]\}$ and $\{[Z_i, \theta Z_j]\}$ (where $1 \le i < j \le 8$) are bases of $\mathfrak{m}$. ∎

Let $\{Z_1, \dots, Z_8\}$ be an orthonormal basis of $\mathfrak{g}_{\alpha+\beta}$.

Lemma 5.8. *There is a unit vector X in $\mathfrak{g}_\alpha$, such that*

$$J_1J_2J_3J_4X = J_1J_2J_5J_6X = J_1J_2J_7J_8X = J_1J_3J_5J_7X = X.$$

Proof. The operators $J_1J_2J_3J_4$, $J_1J_2J_5J_6$, $J_1J_2J_7J_8$, and $J_1J_3J_5J_7$ commute with each other, are symmetric, and have square equal to the identity, so the lemma follows. ∎

We note that $J_1J_2J_3J_4X = X$ if and only if

$$Z_4 = [X, J_1J_2J_3X]. \tag{30}$$

We are interested in studying the subalgebra $\mathfrak{g}'$ generated by X, θX, Z_1, θZ_1, Z_2, θZ_2, Z_3, θZ_3, Z_4 and θZ_4. The set of vectors

$$\{Z_1, Z_2, Z_3, Z_4, X, J_1X, J_2X, J_3X, J_4X, J_1J_2X, J_1J_3X, J_1J_4X\}$$

spans a nilpotent Lie algebra $\mathfrak{n}'$. This subalgebra of $\mathfrak{n}$, endowed with the restriction of $\langle\cdot,\cdot\rangle$, is a generalized Heisenberg algebra. Since there is only one irreducible generalized Heisenberg algebra with four dimensional centre, up to equivalence, $\mathfrak{n}'$ is isomorphic to the irreducible generalized Heisenberg algebra $\mathfrak{n}_4$. Since $J_1J_2J_3J_4X = X$ the commutation relations in $\mathfrak{g}'$ take the same form as in the case $d_\alpha = 4$. Let

$$M_1 = X_{12} + X_{34}, \qquad M_2 = X_{13} + X_{42} \qquad \text{and} \qquad M_3 = X_{14} + X_{23}.$$

Lemma 5.9. *The vectors M_1, M_2 and M_3 act trivially on* $\operatorname{span}\{Z_1, Z_2, Z_3, Z_4\}$ *and satisfy*

$$\begin{aligned} M_1 &= X_{12} + X_{34} = Y_{12} - Y_{34} = Z_{58} - Z_{67}, \\ M_2 &= X_{13} + X_{42} = Y_{13} - Y_{42} = Z_{57} - Z_{86}, \\ M_3 &= X_{14} + X_{23} = Y_{14} - Y_{23} = Z_{56} - Z_{78}. \end{aligned} \tag{31}$$

Proof. Clearly M_1, M_2 and M_3 act trivially on $\operatorname{span}\{Z_1, Z_2, Z_3, Z_4\}$. Therefore, M_1, M_2, and M_3 lie in the linear span of $\{Z_{ij} : 5 \le i < j \le 8\}$, i.e.,

$$M_i = \sum_{5\le j<k\le 8} m_{i,jk}\, Z_{jk}. \tag{32}$$

Since the representation of $\mathfrak{so}(8)$ on $\mathbb{R}^8$ is faithful, by using Proposition 2.4 and Lemma 5.8 to compare the action of (31) and (32) on $\{X_1, \dots, X_8\}$, we obtain (31). ∎

6. Construction of the algebras

In this section, for any d in $\{1, 2, 4, 8\}$, we build a vector space $\mathfrak{g}_d$, and then we endow $\mathfrak{g}_d$ with a skew-symmetric product $[\cdot\,,\cdot]$. In the next two sections we prove that $(\mathfrak{g}_d, [\cdot,\cdot])$ is a Lie algebra. Our construction is motivated by the structural analysis of Sections 2 to 5.

Suppose that $d \in \{1,2,4,8\}$, and let $\mathfrak{z}_d$ denote $\mathbb{R}^d$ with its standard inner product $\langle \cdot , \cdot \rangle$. We fix an orthonormal basis $\{Z_1, \ldots, Z_d\}$ of $\mathfrak{z}_d$ once and for all. Let $\mathcal{C}(d)$ be the Clifford algebra associated to $(\mathbb{R}^d, \langle \cdot , \cdot \rangle)$. Take an irreducible module $\mathfrak{v}_d$ of $\mathcal{C}(d)$, and denote by $J : \mathcal{C}(d) \to \mathrm{End}(\mathfrak{v}_d)$ the corresponding representation. The module $\mathfrak{v}_d$ is naturally endowed with an inner product $\langle \cdot , \cdot \rangle$ with respect to which J_a is skew-symmetric for all a in $\mathcal{C}(d)$. As we observed in Section 5, for d in $\{1,2,4,8\}$ the dimension of $\mathfrak{v}_d$ is $2d$. By Proposition 3.2, up to equivalence, there is exactly one irreducible generalized Heisenberg algebra $(\mathfrak{n}_d, [\cdot,\cdot], \langle \cdot , \cdot \rangle)$ with centre $\mathfrak{z}_d$, where

$$\mathfrak{n}_d = \mathfrak{v}_d \oplus \mathfrak{z}_d.$$

Since $d \in \{1,2,4,8\}$, the module $\mathfrak{v}_d$ splits into the direct orthogonal sum of two irreducible $\mathcal{C}^+(d)$-modules $\mathfrak{v}_{d1}$ and $\mathfrak{v}_{d2}$. We know that $\mathcal{C}^+(d)$ is isomorphic to $\mathcal{C}(d-1)$, and that the $\mathcal{C}^+(d)$-modules $\mathfrak{v}_{d1}$ and $\mathfrak{v}_{d2}$ are equivalent when d is 1 or 2 and inequivalent when d is 4 or 8. If d is 4 or 8, then we may and shall assume that $\epsilon = J_1 \ldots J_d$ is the identity on $\mathfrak{v}_{d1}$ and minus the identity on $\mathfrak{v}_{d2}$ (as before we use the notation $J_i = J_{Z_i}$). Take X in $\mathfrak{v}_{d1}$ such that $\|X\| = 1$, and then

$$\{-X, J_1J_2X, J_1J_3X, \ldots, J_1J_dX\} \qquad \text{and} \qquad \{J_1X, J_2X, J_3X, \ldots, J_dX\}$$

are orthonormal bases of $\mathfrak{v}_{d1}$ and $\mathfrak{v}_{d2}$ respectively. We set

$$X_i = J_1J_iX \qquad \text{and} \qquad Y_i = J_iX.$$

We now build the algebra $\mathfrak{a} \oplus \mathfrak{m}$. Let $\mathfrak{a}$ be $\mathbb{R}^2$ endowed with the canonical inner product. Then let $\mathfrak{a}^*$ be the space of real linear forms on $\mathfrak{a}$ with its canonical inner product denoted by $(\cdot \mid \cdot)$. We fix two elements α and β in $\mathfrak{a}^*$ satisfying the conditions

$$(\alpha \mid \alpha) = (\beta \mid \beta) = 2 \qquad \text{and} \qquad (\alpha \mid \beta) = -1. \tag{33}$$

Once we have α, β and $\alpha+\beta$, we introduce the notation

$$\mathfrak{z} = \mathfrak{g}_{\alpha+\beta}, \qquad \mathfrak{v}_{d1} = \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{v}_{d2} = \mathfrak{g}_\beta,$$

and call $\mathfrak{g}_\alpha$, $\mathfrak{g}_\beta$, $\mathfrak{g}_{\alpha+\beta}$ root spaces. We now define the Lie bracket of an element of $\mathfrak{a}$ with an element of $\mathfrak{n}_d$.

Definition 6.1. For all H in $\mathfrak{a}$, we set

$$[H, X+Y+Z] = \alpha(H)X + \beta(H)Y + (\alpha+\beta)(H)Z,$$

for all X in $\mathfrak{g}_\alpha$, Y in $\mathfrak{g}_\beta$, and Z in $\mathfrak{g}_{\alpha+\beta}$.

It is clear that $\mathfrak{a}$ acts on $\mathfrak{n}_d$ by derivations. As in Section 2, we associate to α, β and $\alpha+\beta$ the vectors H_α, H_β, and $H_{\alpha+\beta}$ in $\mathfrak{a}$.

Definition 6.2. We next set

$$\mathfrak{m}_{\mathfrak{z}}(d) = \mathcal{D}_{\mathfrak{z}}(\mathfrak{n}_d), \qquad \mathfrak{m}_0^+(d) = \mathcal{D}_0^+(\mathfrak{n}_d) \qquad \text{and} \qquad \mathfrak{m}_d = \mathfrak{m}_{\mathfrak{z}}(d) \oplus \mathfrak{m}_0^+(d),$$

and let the Lie bracket in $\mathfrak{a} \oplus \mathfrak{m}_d$, which is an algebra of derivations of $\mathfrak{n}_d$, be the ordinary commutator.

Note that the sum in $\mathfrak{m}_d = \mathfrak{m}_{\mathfrak{z}}(d) \oplus \mathfrak{m}_0^+(d)$ is direct, by Proposition 3.3. In view of Corollary 2.6 and Proposition 3.5, set

$$\mathfrak{m}_{\mathfrak{z}}(1) = \mathfrak{m}_0^+(1) = \{0\} \qquad \text{and} \qquad \mathfrak{m}_0^+(8) = \{0\}.$$

We choose now a specific basis of $\mathcal{D}_0^+(\mathfrak{n}_d)$. In doing this, given a linear map $D : \mathfrak{g}_\alpha \to \mathfrak{g}_\alpha$, we will extend it to a map on $\mathfrak{g}_\beta$ by

$$D = -J_Z D J_Z,$$

where Z is any unit vector in $\mathfrak{z}$, and then we extend it to be zero on $\mathfrak{z}$.

When $d = 2$, we set

$$DX_1 = 3X_2 \qquad \text{and} \qquad DX_2 = -3X_1.$$

It is easy to verify that the extended map D on $\mathfrak{n}_2$, obtained as described above, is a derivation.

When $d = 4$, we set

$$\begin{aligned} D_1 X_k &= 2(\delta_{1k}X_2 - \delta_{2k}X_1 - \delta_{3k}X_4 + \delta_{4k}X_3), \\ D_2 X_k &= 2(\delta_{3k}X_1 - \delta_{1k}X_3 + \delta_{4k}X_2 - \delta_{2k}X_4), \\ D_3 X_k &= 2(\delta_{1k}X_4 - \delta_{4k}X_1 + \delta_{2k}X_3 - \delta_{3k}X_2). \end{aligned}$$

Again it is easy to verify that the extensions to $\mathfrak{n}_4$ of these maps are derivations, still called D_1, D_2, and D_3 respectively.

Lemma 6.3. *The subalgebras $\mathfrak{m}_{\mathfrak{z}}(d)$, $\mathfrak{m}_0^+(d)$ and $\mathfrak{a}$ are ideals in $\mathfrak{a} \oplus \mathfrak{m}_d$.*

We are now in a position to begin the construction of $\mathfrak{g}_d$. We need two copies of $\mathfrak{n}_d$, so we take the Cartesian product

$$\mathfrak{n}_d \times \{+, -\}.$$

To simplify notation, we set

$$\mathfrak{n}_{d+} = \mathfrak{n}_d \times \{+\} \qquad \text{and} \qquad \mathfrak{n}_{d-} = \mathfrak{n}_d \times \{-\},$$

and write

$$\mathfrak{n}_d \times \{+, -\} = \mathfrak{n}_{d+} \oplus \mathfrak{n}_{d-}.$$

We also set

$$\begin{aligned} \mathfrak{g}_{\alpha+\beta} &= \mathfrak{z}_d \times \{+\}, &\qquad \mathfrak{g}_{-\alpha-\beta} &= \mathfrak{z}_d \times \{-\}, \\ \mathfrak{g}_{\alpha} &= \mathfrak{v}_{d1} \times \{+\}, &\qquad \mathfrak{g}_{-\alpha} &= \mathfrak{v}_{d2} \times \{-\}, \\ \mathfrak{g}_{\beta} &= \mathfrak{v}_{d1} \times \{+\}, &\qquad \mathfrak{g}_{-\beta} &= \mathfrak{v}_{d2} \times \{-\}. \end{aligned}$$

Correspondingly, for V in $\mathfrak{n}_d$, we denote by V^+ the pair $(V, +)$ and by V^- the pair $(V, -)$.

Definition 6.4. Let

$$\mathfrak{g}_d = \mathfrak{n}_{d-} \oplus (\mathfrak{a} \oplus \mathfrak{m}_d) \oplus \mathfrak{n}_{d+},$$

and let $\theta : \mathfrak{g}_d \to \mathfrak{g}_d$ be the linear involution defined to be the identity on $\mathfrak{m}_d$, minus the identity on $\mathfrak{a}$, and by

$$\theta V^+ = V^- \qquad \text{and} \qquad \theta V^- = V^+ \qquad \forall V \in \mathfrak{n}_d.$$

Clearly θ^2 is the identity on $\mathfrak{g}_d$. Notice also that the image under θ of an orthonormal basis of $\mathfrak{n}_d$ is an orthonormal basis of $\mathfrak{n}_{d-}$.

Definition 6.5. Let $\mathfrak{a} \oplus \mathfrak{m}_d$ act on $\mathfrak{n}_{d+}$ by the action defined in Definitions 6.1 and 6.2. For all H in $\mathfrak{a}$, M in $\mathfrak{m}_d$ and W^- in $\mathfrak{n}_{d-}$, we set

$$[H + M, W^-] = \theta[\theta H + \theta M, \theta W^-] = \theta[-H + M, W^+].$$

Then $\mathfrak{a} \oplus \mathfrak{m}_d$ acts by derivations not only on $\mathfrak{n}_{d+}$, but also on $\mathfrak{n}_{d-}$. We will denote by $\mathfrak{g}_{d+}$ and $\mathfrak{g}_{d-}$ the Lie algebras consisting respectively of the semi-direct product of $\mathfrak{n}_{d+}$ and $\mathfrak{n}_{d-}$ with $\mathfrak{a} \oplus \mathfrak{m}_d$.

We observe that when $d \in \{1, 2\}$, since there is only one irreducible $\mathcal{C}^+(d)$-module, the distinction between $\mathfrak{v}_{d1}$ and $\mathfrak{v}_{d2}$ is irrelevant in the subsequent construction. When $d \in \{4, 8\}$, the $\mathcal{C}^+(d)$-modules $\mathfrak{v}_{d1}$ and $\mathfrak{v}_{d2}$ are inequivalent and we are free to identify one of the root spaces $\{\mathfrak{g}_\alpha, \mathfrak{g}_\beta\}$ with either of them. But, after that, in accordance with Proposition 4.5, all the other root spaces are determined. For instance, we may choose $\mathfrak{v}_{d1}$ as the root space $\mathfrak{g}_\alpha$, then we have to take $\mathfrak{g}_\beta$ isomorphic to $\mathfrak{v}_{d2}$, $\mathfrak{g}_{-\alpha}$ isomorphic to $\mathfrak{v}_{d1}$, and $\mathfrak{g}_{-\beta}$ isomorphic to $\mathfrak{v}_{d2}$.

It remains to define the Lie bracket of pairs of vectors lying in root spaces $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{-\delta}$ with γ and δ in Δ_+. We do this by analogy with the results of Sections 4 and 5.

Definition 6.6. Take Z^+ in $\mathfrak{g}_{\alpha+\beta}$, X^+ in $\mathfrak{g}_\alpha$, and $Y^+ \in \mathfrak{g}_\beta$. We set

$$\begin{gathered}
[Z^+, X^-] = J_Z X^+, \qquad [Z^+, Y^-] = J_Z Y^+, \\
[Y^+, X^-] = 0, \qquad [X^-, X^+] = \|X^+\|^2 H_\alpha, \\
[Y^-, Y^+] = \|Y^+\|^2 H_\beta \qquad \text{and} \qquad [Z^-, Z^+] = H_{\alpha+\beta} \|Z\|^2.
\end{gathered}$$

Suppose also that $d \in \{2, 4, 8\}$. For all (i, k) with $1 \leq i < k \leq d$, we set

$$[Z_i^+, Z_k^-] = D_{ik}^0. \tag{34}$$

Further, if $d = 2$, we impose the conditions

$$-[X_1^+, X_2^-] + [Y_1^+, Y_2^-] = [X, \theta J_1 J_2 X] + [J_1 X, \theta J_2 X] = D_{12}^0$$

and

$$[X_1^+, X_2^-] + [Y_1^+, Y_2^-] = -[X, \theta J_1 J_2 X] + [J_1 X, \theta J_2 X] = D,$$

by analogy with Proposition 5.3. These relations define the single Lie brackets $[X_i^+, X_k^-]$ and $[Y_i^+, Y_k^-]$. If $d = 4$, we impose the conditions

$$\begin{aligned} D_{12}^0 - D_{34}^0 &= [X_1^+, X_2^-] - [X_3^+, X_4^-], \\ D_{13}^0 - D_{42}^0 &= [X_1^+, X_3^-] - [X_4^+, X_2^-], \\ D_{14}^0 - D_{23}^0 &= [X_1^+, X_4^-] - [X_2^+, X_3^-], \end{aligned} \tag{35}$$

and

$$\begin{aligned} D_{12}^0 + D_{34}^0 &= [Y_1^+, Y_2^-] + [Y_3^+, Y_4^-], \\ D_{13}^0 + D_{42}^0 &= [Y_1^+, Y_3^-] + [Y_4^+, Y_2^-], \\ D_{14}^0 + D_{23}^0 &= [Y_1^+, Y_4^-] + [Y_2^+, Y_3^-], \end{aligned} \tag{36}$$

by analogy with Lemma 5.4, and then we set

$$\begin{aligned} D_1 &= [X_1^+, X_2^-] + [X_3^+, X_4^-] = [Y_1^+, Y_2^-] - [Y_3^+, Y_4^-], \\ D_2 &= [X_1^+, X_3^-] + [X_4^+, X_2^-] = [Y_1^+, Y_3^-] - [Y_4^+, Y_2^-], \\ D_3 &= [X_1^+, X_4^-] + [X_2^+, X_3^-] = [Y_1^+, Y_4^-] - [Y_2^+, Y_3^-]. \end{aligned} \tag{37}$$

These relations define the single brackets $[X_i^+, X_k^-]$ and $[Y_i^+, Y_k^-]$. Finally, if $d = 8$, we define, for $i \neq j$,

$$[X_i^+, X_j^-] = \sum_{1 \le k < l \le 8} x_{ij,kl}\, D_{kl}^0,$$

and

$$[Y_i^+, Y_j^-] = \sum_{1 \le k < l \le 8} y_{ij,kl}\, D_{kl}^0,$$

where $x_{ij,kl}$ and $y_{ij,kl}$ are given by Corollary 5.7. All the remaining brackets are defined by imposing the condition that

$$[\theta U, \theta V] = \theta[U, V] \qquad \forall U, V \in \mathfrak{g}_d.$$

In the last two sections we prove that the skew-symmetric product just defined really is a Lie bracket. Then, since there is exactly one irreducible generalized Heisenberg algebra with d-dimensional centre, up to equivalence, by Proposition 3.2, the considerations in Section 5 determine the commutation relations uniquely, proving that for any d in $\{1, 2, 4, 8\}$, there is only one real semisimple Lie algebra with $d_\alpha = d$.

7. An explicit model for $\mathfrak{g}_d$ when $d \in \{1, 2, 4\}$

In this section we will prove that the vector space $\mathfrak{g}_d$ endowed with the product $[\cdot, \cdot]$ is a Lie algebra when $d \in \{1, 2, 4\}$, by verifying the Jacobi identity.

We denote by $\mathbb{F}_d$ the real numbers, the complex numbers and the quaternions, when $d = 1$, $d = 2$, and $d = 4$ respectively. We write $z = a_0 + a_1 i + a_2 j + a_3 k$ with a_1, a_2, a_3, a_4 in $\mathbb{R}$ for z in $\mathbb{F}_d$, with the understanding that $a_1 = a_2 = a_3 = 0$ when $d = 1$ and $a_2 = a_3 = 0$ when $d = 2$. Correspondingly, we write $\operatorname{Im} \mathbb{F}_d$ for

$\{z \in \mathbb{F}_d \ : a_0 = 0\}$. If $x \in \mathbb{F}_d$ we denote by $\overline{x}$ the conjugate of x and by $|x|$ the modulus of x, i.e.,

$$\overline{x} = a_0 - a_1 i - a_2 j - a_3 k \qquad \text{and} \qquad |x|^2 = x\overline{x} = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

Let $\mathfrak{sl}(3, \mathbb{F}_d)$ be the Lie algebra of 3×3 matrices X with entries in $\mathbb{F}_d$ such that $\operatorname{tr} X = 0$ when $d \in \{1, 2\}$ and $\operatorname{Re}\operatorname{tr} X = 0$ when $d = 4$. We identify $(\mathfrak{g}_d, [\cdot,\cdot], \theta)$ with $\mathfrak{sl}(3, \mathbb{F}_d)$, endowed with the matrix commutator, written $[\cdot,\cdot]$, and with the Cartan involution Θ, given by

$$\Theta X = -X^*,$$

for X in $\mathfrak{sl}(3, \mathbb{F}_d)$, where $X^* = {}^t\overline{X}$.

For x, y and z in $\mathbb{F}_d$, we set

$$N(x, y, z) = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

Define

$$\begin{aligned} \mathfrak{z}_d &= \{N(0, 0, z) : z \in \mathbb{F}_d\}, \\ \mathfrak{v}_{d1} &= \{N(x, 0, 0) : x \in \mathbb{F}_d\}, \\ \mathfrak{v}_{d2} &= \{N(0, y, 0) : y \in \mathbb{F}_d\}, \\ \mathfrak{n}_{d+} &= \mathfrak{v}_{d1} \oplus \mathfrak{v}_{d2} \oplus \mathfrak{z}_d, \\ \mathfrak{n}_{d-} &= \Theta\mathfrak{n}_{d+}. \end{aligned}$$

We define the inner product $\langle \cdot,\cdot \rangle$ on $\mathfrak{n}_{d+}$ by

$$\begin{aligned} \langle N(x, y, z), N(x', y', z') \rangle &= \operatorname{Re}(x\overline{x'} + y\overline{y'} + z\overline{z'}) \\ &= \frac{1}{2}(x\overline{x'} + y\overline{y'} + z\overline{z'} + x'\overline{x} + y'\overline{y} + z'\overline{z}). \end{aligned}$$

It may be easily verified that $\mathfrak{n}_d$, with this inner product, is a generalized Heisenberg algebra and that the operator J is given by

$$J_{N(0,0,z)} N(x, y, 0) = N(-z\overline{y}, \overline{x}z, 0).$$

For q_1, q_2, q_3 in $\mathbb{F}_d$, we denote by $\operatorname{diag}(q_1, q_2, q_3)$ the 3×3 diagonal matrix with diagonal entries q_1, q_2, q_3 in $\mathbb{F}_d$. Then we define $\mathfrak{a}$ to be the space of real diagonal 3×3 matrices of trace zero, and define H_α, H_β, and $H_{\alpha+\beta}$ in $\mathfrak{a}$ by

$$H_\alpha = \operatorname{diag}(1, -1, 0), \qquad H_\beta = (0, 1, -1) \qquad \text{and} \qquad H_{\alpha+\beta} = (1, 0, -1).$$

Next, if $d = 2$, then we define

$$\mathfrak{m}_\mathfrak{z} = \{\operatorname{diag}(ix, 0, -ix) : x \in \mathbb{R}\} \qquad \text{and} \qquad \mathfrak{m}_0^+ = \{\operatorname{diag}(ix, -2ix, ix) : x \in \mathbb{R}\}.$$

If $d = 4$, then we define

$$\mathfrak{m}_\mathfrak{z} = \{\operatorname{diag}(q_1, 0, q_2) : q_1, q_2 \in \operatorname{Im}\mathbb{F}_4\} \qquad \text{and} \qquad \mathfrak{m}_0^+ = \{\operatorname{diag}(0, q, 0) : q \in \operatorname{Im}\mathbb{F}_4\}.$$

In both cases we put

$$\mathfrak{m} = \mathfrak{m}_\mathfrak{z} \oplus \mathfrak{m}_0^+.$$

It is easy to show that the spaces of matrices introduced in this section may be identified with those introduced in Section 6 and that the commutation relations are the same. This proves that $\mathfrak{g}_d$ is a Lie algebra (i.e., the Jacobi identity holds).

8. The Jacobi identity for $(\mathfrak{g}_8, [\cdot,\cdot]_8, \theta_8)$

Recall from Definition 6.5 that $\mathfrak{g}_{8+}$ and $\mathfrak{g}_{8-}$ are the semi-direct products of $\mathfrak{a}\oplus\mathfrak{m}_8$ with $\mathfrak{n}_{8+}$ and $\mathfrak{n}_{8-}$ respectively, hence they are Lie algebras. By linearity, to prove that $\mathfrak{g}_8$ is a Lie algebra, it therefore suffices to check that

$$[U,[V,W]]+[V,[W,U]]+[W,[U,V]]=0$$

(the Jacobi identity) for (U,V,W) in $\mathfrak{n}_{8+}\times\mathfrak{n}_{8-}\times\mathfrak{g}_8$.

Suppose that $(U,V,W)\in\mathfrak{n}_{8+}\times\mathfrak{n}_{8-}\times\mathfrak{g}_8$. There are essentially three different cases which have to be taken into account. The first two, which are equivalent, are those in which $U,V\in\mathfrak{n}_{8+}$ and $W\in\mathfrak{n}_{8-}$, or $U\in\mathfrak{n}_{8+}$ and $V,W\in\mathfrak{n}_{8-}$. The last case is that in which $U\in\mathfrak{n}_{8+}$, $V\in\mathfrak{n}_{8-}$, and $W\in\mathfrak{m}_8\oplus\mathfrak{a}$. By linearity, in proving the identity for these cases, we may assume that each of U, V, and W lies in a root space or in $\mathfrak{m}\oplus\mathfrak{a}$. In what follows, we use the term "root vector" to indicate an element of $\cup_{\gamma\in\Delta}\mathfrak{g}_\gamma$.

The next results will enable us to avoid direct verification of the Jacobi identity.

Lemma 8.1. *Suppose that $d=8$ and that γ is a root in Δ_+. The vector space*

$$\mathfrak{g}_{(\gamma)}=\mathfrak{g}_{-\gamma}\oplus(\mathbb{R}H_\gamma\oplus\mathfrak{m}_8)\oplus\mathfrak{g}_\gamma,$$

equipped with $[\cdot,\cdot]_8$, is a real simple Lie algebra of real rank one isomorphic to $\mathfrak{so}(1,9)$.

Proof. Let $\{V_1,\ldots,V_8\}$ be an orthonormal basis of $\mathfrak{g}_\gamma$. Then by (3)

$$H_\gamma=[\theta V_k,V_k],$$

and further

$$\{[V_i,\theta V_j]\ :\ 1\le i<j\le 8\}$$

is a basis of $\mathfrak{m}_8$ acting on $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{-\gamma}$ by formulae (4) and (5).

Let e_i be the row vector whose entry in the i^{th} place is equal to 1 and all other entries are equal to 0. Define

$$E_i=\begin{pmatrix}0 & e_i & 0\\ {}^te_i & 0 & {}^te_i\\ 0 & -e_i & 0\end{pmatrix}\qquad\forall i\in\{1,\ldots,8\}.$$

Then set

$$A=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}.$$

It is clear that

$$[E_i,\Theta E_j]=2\begin{pmatrix}0&0&0\\0&e_{ij}&0\\0&0&0\end{pmatrix}\qquad\forall i,j\in\{1,\ldots,8\}\quad i\neq j,$$

where Θ was introduced in Section 7, and e_{ij} is the matrix with $(i,j)^{\text{th}}$-entry equal to 1 and $(j,i)^{\text{th}}$ entry equal to -1, and all other entries zero. Then it may be easily seen that the linear map defined by the conditions that $V_i\mapsto E_i$, $\theta V_i\mapsto\Theta E_i$, $H_\gamma\mapsto A$, and $[V_i,\theta V_j]\mapsto[E_i,\Theta E_j]$, is a Lie algebra isomorphism. ∎

Lemma 8.2. *Suppose that $\{Z_1, Z_2, Z_3, Z_4\}$ is orthonormal in $\mathfrak{g}_{\alpha+\beta}$ and X is a unit vector in $\mathfrak{g}_\alpha$. Suppose further that*

$$J_1 J_2 J_3 J_4 X = X. \tag{38}$$

The smallest $[\cdot,\cdot]_8$-closed vector subspace of $\mathfrak{g}_8$ containing Z_1, Z_2, Z_3, Z_4, X and θX is denoted by $\mathfrak{g}^\dagger$. This subspace, endowed with the restriction $[\cdot,\cdot]'$ of $[\cdot,\cdot]_8$, is a Lie algebra isomorphic to $\mathfrak{g}_4$.

Proof. Set

$$\begin{aligned} \mathfrak{g}^\dagger_{\alpha+\beta} &= \operatorname{span}\{Z_1, Z_2, Z_3, Z_4\}, \\ \mathfrak{g}^\dagger_{\alpha} &= \operatorname{span}\{X, J_1J_2X, J_1J_3X, J_1J_4X\}, \\ \mathfrak{g}^\dagger_{\beta} &= \operatorname{span}\{J_1X, J_2X, J_3X, J_4X\} \end{aligned}$$

and

$$\mathfrak{n}^\dagger = \mathfrak{g}^\dagger_\alpha \oplus \mathfrak{g}^\dagger_\beta \oplus \mathfrak{g}^\dagger_{\alpha+\beta}.$$

It is easy to check that (38) ensures that the subspace $\mathfrak{n}^\dagger$, equipped with the restrictions of the bracket and of the inner product of $\mathfrak{g}_8$, is a subalgebra of the generalized Heisenberg algebra $\mathfrak{n}_{8+}$. Since, up to equivalence, there is only one irreducible generalized Heisenberg algebra with four-dimensional centre, $\mathfrak{n}^\dagger$ is isomorphic to $\mathfrak{n}_4$. Hence the construction of Section 6 yields a Lie algebra $\mathfrak{g}^\dagger$ isomorphic to $\mathfrak{g}_4$, by the results of Section 7. We prove that $\mathfrak{g}^\dagger$ is a subalgebra of $\mathfrak{g}_8$. We observe that for U, V in $\mathfrak{g}^\dagger$ the definitions of the brackets $[U,V]_4$ and $[U,V]_8$ are the same with the exception of those brackets which are defined in $\mathfrak{g}_4$ by (37). However, from Lemma 5.9 we see that $[X_1, \theta X_2]_8 + [X_3, \theta X_4]_8$ acts on $\mathfrak{g}^\dagger_\alpha$ in exactly the same way as $[X_1, \theta X_2]_4 + [X_3, \theta X_4]_4$. Since the action of $\mathfrak{m}_\alpha$ on $\mathfrak{g}_\alpha$ is faithful by Lemma 8.1 and Corollary 2.7, this ensures that

$$[X_1, \theta X_2]_4 + [X_3, \theta X_4]_4 = [X_1, \theta X_2]_8 + [X_3, \theta X_4]_8,$$

where $[\cdot,\cdot]_4$ stands for the Lie bracket in $\mathfrak{g}_4$ and $[\cdot,\cdot]_8$ stands for the bracket of $\mathfrak{g}_8$. Similarly we deduce that

$$\begin{aligned} [X_1, \theta X_3]_4 + [X_2, \theta X_4]_4 &= [X_1, \theta X_3]_8 + [X_2, \theta X_4]_8 \\ [X_1, \theta X_4]_4 + [X_2, \theta X_3]_4 &= [X_1, \theta X_4]_8 + [X_2, \theta X_3]_8, \end{aligned}$$

as required. ∎

Proposition 8.3. *Take root vectors U, V in $\mathfrak{n}_{8+}$ and a root vector W in $\mathfrak{n}_{8-}$. There exists a subspace of $\mathfrak{g}_8$ containing U, V, W which, equipped with $[\cdot,\cdot]_8$, is a Lie algebra isomorphic to $\mathfrak{g}_4$.*

Proof. Given any triple of root vectors in $\mathfrak{n}_{8+} \times \mathfrak{n}_{8+} \times \mathfrak{n}_{8-}$ we will use Lemma 5.8 to see that there exist an orthonormal system $\{Z_1, Z_2, Z_3, Z_4\}$ in $\mathfrak{g}_{\alpha+\beta}$ and a unit vector X in $\mathfrak{g}_\alpha$ such that U, V, and W lie in the span of $\{Z_1, Z_2, Z_3, Z_4\}$,

$\{-X, J_1J_2X, J_1J_3X, J_1J_4X\}$, $\{J_1X, J_2X, J_3X, J_4X\}$, and of the vectors obtained acting on these by means of θ.

Take U and V in $\mathfrak{n}_{8+}$ and W in $\mathfrak{n}_{8-}$. We assume without loss of generality that they are unit vectors. There are essentially six different cases to consider.

Case 1. Suppose that $U, V \in \mathfrak{g}_{\alpha+\beta}$ and that $W \in \mathfrak{g}_{-\alpha-\beta}$. Consider an orthonormal system $\{Z_1, Z_2, Z_3\}$ in $\mathfrak{g}_{\alpha+\beta}$ such that

$$U = Z_1, \qquad V = v_1Z_1 + v_2Z_2 \qquad \text{and} \qquad \theta W = w_1Z_1 + w_2Z_2 + w_3Z_3,$$

where $v_1, v_2, w_1, w_2, w_3 \in \mathbb{R}$. We fix a unit vector X in $\mathfrak{g}_\alpha$, and define Z_4 by (30).

Case 2. Suppose that $U, V \in \mathfrak{g}_\alpha$ and that $W \in \mathfrak{g}_{-\alpha}$. There is an orthonormal system $\{Z_1, Z_2, Z_3\}$ in $\mathfrak{g}_{\alpha+\beta}$ such that, writing X for U,

$$V = v_1X + v_2J_1J_2X \qquad \text{and} \qquad \theta W = w_1X + w_2J_1J_2X + w_3J_1J_3X,$$

where $v_1, v_2, w_1, w_2, w_3 \in \mathbb{R}$. We define Z_4 by (30).

Case 3. Suppose that $U \in \mathfrak{g}_{\alpha+\beta}$, $V \in \mathfrak{g}_\alpha$, and $W \in \mathfrak{g}_{-\alpha-\beta}$. There exist orthonormal vectors Z_1 and Z_2 in $\mathfrak{g}_{\alpha+\beta}$ such that

$$U = Z_1 \qquad \text{and} \qquad \theta W = w_1Z_1 + w_2Z_2,$$

where $w_1, w_2 \in \mathbb{R}$. Setting $V = X$, we take any unit vector Z_3 in $\mathfrak{g}_{\alpha+\beta}$ orthogonal to Z_1 and Z_2, and we define Z_4 by (30).

Case 4. Suppose that $U \in \mathfrak{g}_{\alpha+\beta}$, $V \in \mathfrak{g}_\alpha$ and $W \in \mathfrak{g}_{-\alpha}$. There exist orthonormal vectors Z_1 and Z_2 in $\mathfrak{g}_{\alpha+\beta}$ such that, setting $V = X$, then

$$U = Z_1 \qquad \text{and} \qquad \theta W = w_1X + w_2J_1J_2X,$$

where $w_1, w_2 \in \mathbb{R}$. We take any unit vector Z_3 in $\mathfrak{g}_{\alpha+\beta}$ orthogonal to Z_1 and Z_2, and we define Z_4 by (30).

Case 5. Suppose that $U \in \mathfrak{g}_\alpha$, $V \in \mathfrak{g}_\beta$, and $W \in \mathfrak{g}_{-\alpha}$. There exist two orthonormal vectors Z_1 and Z_2 in $\mathfrak{g}_{\alpha+\beta}$ such that, if we set $U = X$, then

$$V = J_1X \qquad \text{and} \qquad \theta W = w_1X + w_2J_1J_2X,$$

where $w_1, w_2 \in \mathbb{R}$. We take any unit vector Z_3 in $\mathfrak{g}_{\alpha+\beta}$ orthogonal to Z_1 and Z_2, and we define Z_4 by (30).

Case 6. Suppose that $U \in \mathfrak{g}_{\alpha+\beta}$, $V \in \mathfrak{g}_\beta$, and $W \in \mathfrak{g}_{-\alpha}$. Setting $U = Z_1$, there exists a unit vector Z_2 orthogonal to Z_1 and a unit vector X in $\mathfrak{g}_\alpha$ such that

$$V = J_1X \qquad \text{and} \qquad W = w_1\theta X + w_2\theta J_1J_2X,$$

where $w_1, w_2 \in \mathbb{R}$. We take any unit vector Z_3 in $\mathfrak{g}_{\alpha+\beta}$ orthogonal to Z_1 and Z_2, and we define Z_4 by (30).

Since in any case $J_1J_2J_3J_4X = X$ by Lemma 5.8, the proposition follows from Lemma 8.2. ∎

Proposition 8.4. *Suppose that U is a root vector in $\mathfrak{n}_{8+}$, V is a root vector in $\mathfrak{n}_{8-}$, and $L \in \mathfrak{a}$ or $L = D^0_{ij}$. Then there exists a subspace of $\mathfrak{g}_8$ containing U, V, L which is isomorphic to $\mathfrak{g}_4$.*

Proof. The case where $L \in \mathfrak{a}$ is easy. Indeed, take a unit vector Z in $\mathfrak{g}_{\alpha+\beta}$. As in the proof of Proposition 8.3, we find a linear subspace of $\mathfrak{g}_8$ which, endowed with the bracket inherited from $\mathfrak{g}_8$, is a Lie algebra isomorphic to $\mathfrak{g}_4$. Now, this algebra, being of real rank two, contains $\mathfrak{a}$, and hence also L.

When $L \in \mathfrak{m}_8$, we may assume that L is equal to D^0_{12}. Therefore, there are two orthonormal vectors Z_1 and Z_2 in $\mathfrak{g}_{\alpha+\beta}$ such that

$$L = [Z_1, \theta Z_2].$$

There are essentially three different cases.

Case 1. Suppose that $U \in \mathfrak{g}_{\alpha+\beta}$ and that $V \in \mathfrak{g}_{-\alpha-\beta}$. The orthonormal system $\{Z_1, Z_2\}$ may be extended to an orthonormal system $\{Z_1, Z_2, Z_3, Z_4\}$ such that

$$U = u_1 Z_1 + u_2 Z_2 + u_3 Z_3 \qquad \text{and} \qquad V = v_1 \theta Z_1 + v_2 \theta Z_2 + v_3 \theta Z_3 + v_4 \theta Z_4,$$

where $u_1, u_2, u_3, v_1, v_2, v_3, v_4 \in \mathbb{R}$. Now, since $(J_1 J_2 J_3 J_4)^2 = I$, we may assume that $J_1 J_2 J_3 J_4$ has a unit eigenvector X in $\mathfrak{g}_\alpha$ with eigenvalue 1.

Case 2. Suppose that $U \in \mathfrak{g}_\alpha$ and that $V \in \mathfrak{g}_{-\alpha}$. There is an orthonormal system $\{Z_1, Z_2, Z_3\}$ in $\mathfrak{g}_{\alpha+\beta}$ extending $\{Z_1, Z_2\}$, such that, setting $U = X$,

$$V = v_1 \theta X + v_2 \theta J_1 J_2 X + v_3 \theta J_1 J_3 X,$$

where $v_1, v_2, v_3 \in \mathbb{R}$. We define Z_4 by (30).

Case 3. Suppose that $U \in \mathfrak{g}_{\alpha+\beta}$ and that $V \in \mathfrak{g}_{-\alpha}$. There is an orthonormal system $\{Z_1, Z_2, Z_3\}$ in $\mathfrak{g}_{\alpha+\beta}$ extending $\{Z_1, Z_2\}$, such that, taking $X = \theta V$,

$$U = u_1 Z_1 + u_2 Z_2 + u_3 Z_3,$$

where $u_1, u_2, u_3 \in \mathbb{R}$. We define Z_4 by (30).

Since $J_1 J_2 J_3 J_4 X = X$ in all cases, the result follows from Lemma 8.2. ∎

Theorem 8.5. *The vector space $\mathfrak{g}_8$ endowed with the bracket $[\cdot,\cdot]_8$ defined in Section 6, is a simple real Lie algebra.*

Proof. The Jacobi identity holds in $\mathfrak{g}_8$ by the results above.

We prove that $\mathfrak{g}_8$ is simple. Suppose that $\mathfrak{h}$ is a nontrivial ideal in $\mathfrak{g}_8$, and let U be a nonzero vector in $\mathfrak{h}$. Take Z in $\mathfrak{g}_{\alpha+\beta}$. Since $\operatorname{ad} Z$ is nilpotent, there exists a nonnegative integer k such that

$$(\operatorname{ad} Z)^{k+1} U = 0 \qquad \text{and} \qquad V = (\operatorname{ad} Z)^k U \neq 0.$$

Then $V \in \mathfrak{h} \cap \mathfrak{n}_8$, since $[Z, V] = 0$. Write V as $X + Y + Z'$, where $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, and $Z' \in \mathfrak{g}_{\alpha+\beta}$. Then

$$[H_\alpha, V] = 2X - Y + Z' \in \mathfrak{h}, \qquad [H_\beta, V] = -X + 2Y + Z' \in \mathfrak{h}$$

and

$$[H_{\alpha+\beta}, V] = X + Y + 2Z' \in \mathfrak{h}.$$

From these relations it follows easily that $X, Y, Z' \in \mathfrak{h}$. Assume that $X \neq 0$. Since the representation of $\mathfrak{m}$ on $\mathfrak{g}_\alpha$ is irreducible, it follows that $\mathfrak{g}_\alpha \subseteq \mathfrak{h}$. Therefore we deduce from Corollary 4.2 that all root spaces are contained in $\mathfrak{h}$, from which it follows that $\mathfrak{h} = \mathfrak{g}_8$.

Finally, denoting by B the Killing form of $\mathfrak{g}_8$, it is easy to verify that

$$\langle \cdot, \cdot \rangle = -B(\cdot, \theta \cdot),$$

as required. ∎

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