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On minimal parabolic subgroups of exponential Lie groups

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Abstract. The conjecture in Problem 6.3 of [4] is refuted by a counterexample: minimal parabolic subgroups may not be used for testing the surjectivity of the exponential function. Specifically, the minimal parabolic subgroups of $\mathrm{GL}_n(\mathbb{H})$ are not exponential for $n \geq 8$. The same is true for $\mathrm{Sp}(p,q)$ if $p \geq q \geq 8$ and $\mathrm{SO}^*(2n)$ if $n \geq 15$. However, we show that the minimal parabolic subgroups of $\mathrm{SO}(p,1)^\circ$ and of $\mathrm{U}(p,q)$ are exponential.

1. Introduction

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Denote by E_G the image of the exponential map $\exp_G: \mathfrak{g} \to G$, i.e., the union of all one-parameter subgroups of G. If E_G is dense in G (which implies that G is connected) then we say that G is a *weakly exponential* group. We recall that if N is a closed normal subgroup of G such that both N and G/N are weakly exponential, then also G is weakly exponential (see [8]).

We say that a connected Lie group G is *exponential* if its exponential map is surjective, i.e., $E_G = G$. The problem of deciding which groups are exponential is still unresolved. Most of the known results are about groups that are either semisimple or solvable. We refer the reader to [3, 4] for the survey of this area of research and for some open problems. The case when G is neither semisimple nor solvable (the so called mixed case) is largely unexplored. In this note we consider an important class of such mixed groups: the minimal parabolic subgroups of real semisimple groups.

Let us recall the following result which is a part of a more comprehensive theorem of Jaworski [9].

Theorem J. *A connected real semisimple Lie group is weakly exponential if and only if its minimal parabolic subgroups are connected.*

Motivated in part by this theorem, the Problem 6.3 of [4] asks whether or not the following conjecture is true.

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Conjecture P. *A connected Lie group is exponential if and only if its minimal parabolic subgroups are exponential.*

We assume that the reader is familiar with the definition of minimal parabolic subgroups of a connected semisimple Lie group (see e.g. [9, 3, 4]). For reader's convenience, we now recall the definition of minimal parabolic subgroups of G, adopted in [3, 4], for an arbitrary connected Lie group G. Let R be the solvable radical of G. A subgroup P of G is called a minimal parabolic subgroup if $P \supset R$ and P/R is a minimal parabolic subgroup of the semisimple group G/R.

As a test case for Conjecture P, the Problem 6.3a of [4] asks whether or not the minimal parabolic subgroups of $\mathrm{SL}_n(\mathbb{H})$, $n \geq 2$, are exponential. (By $\mathbb{H}$ we denote the real quaternion division algebra.) We shall prove that they are not exponential if $n \geq 8$, and consequently the above conjecture is false.

It is now natural to raise the question: For which exponential semisimple Lie groups are the minimal parabolic subgroups exponential? The groups $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$ are known to be exponential, and we show that their minimal parabolic subgroups are not exponential if $p \geq q \geq 8$ and $n \geq 15$, respectively. It is also known that the identity component of $\mathrm{SO}(p,1)$ and the unitary groups $\mathrm{U}(p,q)$ are exponential groups. We shall prove that their minimal parabolic subgroups are exponential.

For any Lie group G we denote by G° its identity component. By ${}^t x$ we denote the transpose of a matrix x, and by x^* the transpose conjugate of x. By x_{ij} we denote the (i,j)-th entry of the matrix x. I_k denotes the identity matrix of order k. As usual we set $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. We denote by S_k the matrix of order k such that $(S_k)_{ij} = \delta_{i+j,k+1}$ for all $i, j = 1, \ldots, k$. We set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$.

It is a pleasure to thank W. H. Hesselink for sending us a list of 2-dimensional sections for the action of the Borel subgroup B of $\mathrm{GL}_8(\mathbb{C})$ on the space N of nilpotent upper triangular complex matrices of size 8. One of these sections (Nr. 7169) was used, in modified form, in the proof of Theorem 2.1. Our original proof of that theorem was valid only for $n \geq 10$.

2. The case of $\mathbf{GL}_n(\mathbb{H})$ and $\mathbf{SL}_n(\mathbb{H})$

Before concentrating on the quaternionic general linear group, let us review the situation for complex and real general linear groups.

The group $\mathrm{GL}_n(\mathbb{C})$ is exponential. Its minimal parabolic subgroups are the Borel subgroups, i.e., the conjugates of the subgroup B of invertible upper triangular matrices. It is an elementary fact that B is also exponential, i.e., if $b \in B$ then $b = e^x$ for some upper triangular matrix x (this follows from [1]).

The group $\mathrm{GL}_n(\mathbb{R})$ is not connected. Its identity component $\mathrm{GL}_n(\mathbb{R})^\circ$ is not exponential if $n \geq 2$. For instance, if $n = 2$, the matrix

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not of the form e^x for any real 2 by 2 matrix x. The group P of all upper triangular matrices in $\mathrm{GL}_n(\mathbb{R})^\circ$, $n \geq 2$, is a minimal parabolic subgroup and is not connected (in accordance with Theorem J). The group P° is exponential.

Until further notice, we denote by G the group $\mathrm{GL}_n(\mathbb{H})$, $n \geq 1$, of all invertible quaternionic matrices of order n. The group T of all upper triangular matrices in G is a minimal parabolic subgroup. It admits a Levi decomposition $T = DU$ where U (resp. D) is the subgroup consisting of all unitriangular (resp. diagonal) matrices in T. Let N denote the space of nilpotent upper triangular quaternionic matrices of order n.

Theorem 2.1. *If $n \geq 8$ then the minimal parabolic subgroups of $\mathrm{GL}_n(\mathbb{H})$ are not exponential groups.*

Proof. Witout any loss of generality we may assume that $n = 8$. Let x be the nilpotent upper triangular matrix given by:

$$x = \begin{pmatrix} 0 & 1 & & 1 & & & & \\ & 0 & & & & 1 & \eta & \\ & & 0 & & & \xi & 1 & \\ & & & 0 & 1 & & & \\ & & & & 0 & & & 1 \\ & & & & & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{pmatrix},$$

where $\xi, \eta \in \mathbb{H}$ are chosen so that $\xi\eta \neq \eta\xi$. (The supressed entries in the above matrix are zeroes.) Let $a \in Z_T(x) := \{y \in T : xy = yx\}$. We claim that all the diagonal entries of a are equal and are real.

We equate the entries in the matrix equation $xa = ax$. The (1,2)-entries give that $a_{11} = a_{22}$. The (1,3)-entries give $a_{23} = 0$. Now the (2,6)-entries give $a_{22} = a_{66}$. The (2,5)-entries give $a_{24} = 0$. Then the (1,4)-entries give $a_{11} = a_{44}$. By symmetry we have $a_{88} = a_{77}$, $a_{67} = 0$, $a_{77} = a_{33}$, $a_{57} = 0$, and $a_{88} = a_{55}$. The (4,5)-entries give $a_{44} = a_{55}$. From (3,6)-entries we obtain $\xi a_{66} = a_{33}\xi$, and from (2,7)-entries $\eta a_{77} = a_{22}\eta$. Thus all the diagonal entries of a are equal and must be in $\mathbb{R}^*$ because they commute with both ξ and η. Hence our claim is proved.

Assume that the matrix $b := x - I_8 \in T$ belongs to E_T, i.e., that there exists a 1-parameter subgroup $\alpha\colon \mathbb{R} \to T$ such that $\alpha(1) = b$. Then for arbitrary $t \in \mathbb{R}$ the matrix $\alpha(t)$ commutes with b, and also with x. Our claim above implies that all diagonal entries of $\alpha(t)$ are real. As $\alpha(0) = I_8$, they are also positive. Since the diagonal entries of b are all -1, we have a contradiction.

Thus $b \in T \setminus E_T$, and so T is not exponential. ∎

The group $P = T \cap \mathrm{SL}_n(\mathbb{H})$ is a minimal parabolic subgroup of $\mathrm{SL}_n(\mathbb{H})$. The map $f\colon \mathbb{R} \times P \to T$ defined by $f(t, x) = e^t x$ is an isomorphism, and so we deduce the following result.

Corollary 1. *The minimal parabolic subgroups of $\mathrm{SL}_n(\mathbb{H})$, $n \geq 8$, are not exponential groups.*

It is a routine matter to derive two more corollaries.

Corollary 2. *The minimal parabolic subgroups of $\mathrm{Sp}(p, q)$, $p \geq q \geq 8$, are not exponential groups.*

Proof. We set $n = p + q$ and let $S = S_q$ (as defined in the introduction). Let

$$G := \mathrm{Sp}(p,q) = \{x \in \mathrm{GL}_n(\mathbb{H}) : x^* J x = J\},$$

where

$$J = \begin{pmatrix} 0 & 0 & S \\ 0 & I_{p-q} & 0 \\ S & 0 & 0 \end{pmatrix}.$$

The group P consisting of all quaternionic matrices

$$A = \begin{pmatrix} a & c & d \\ 0 & b & -bc^*(a^*)^{-1}S \\ 0 & 0 & S(a^*)^{-1}S \end{pmatrix} \tag{1}$$

with $a \in \mathrm{GL}_q(\mathbb{H})$ upper triangular, $b \in \mathrm{Sp}(p-q)$, and such that $aSd^* + dSa^* + cc^* = 0$, is a minimal parabolic subgroup of G. The Lie algebra of P consists of all quaternionic matrices

$$X = \begin{pmatrix} x & z & w \\ 0 & y & -z^*S \\ 0 & 0 & -Sx^*S \end{pmatrix} \tag{2}$$

where x is an upper triangular matrix of order q, y is a skew-hermitian matrix of order $p-q$, and $Sw^* + wS = 0$. Since the blocks a in (1) and x in (2) are upper triangular matrices, with no further restrictions, and $q \geq 8$, the assertion of the corollary follows immediately from the theorem. ∎

Corollary 3. *The minimal parabolic subgroups of* $\mathrm{SO}^*(2n)$, $n \geq 15$, *are not exponential groups.*

Proof. Let $J = iS_n$ where $i \in \mathbb{C}$ is the imaginary unit, and let

$$G := \mathrm{SO}^*(2n) = \{x \in \mathrm{GL}_n(\mathbb{H}) : x^* J x = J\}.$$

Then the group P of all upper triangular matrices in G is a minimal parabolic subgroup. The rest of the argument is similar to the proof just given above. ∎

Remark 1. T acts on N by $(t,x) \to txt^{-1}$. For $n \leq 5$ the number of T-orbits in N is finite: It is equal to 1,2,5,16,61 for $n = 1,2,3,4,5$, respectively. Each of these orbits contains a $\{0,1\}$-matrix. For $n = 6$ the number of T-orbits in N is infinite. Indeed the matrices

$$x_\xi = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & & 1 & 1 & \\ & & 0 & \xi & 1 & \\ & & & 0 & & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}$$

and x_η belong to the same T-orbit if and only if $\eta = \zeta\xi\zeta^{-1}$ for some $\zeta \in \mathbb{H}^*$. There are only finitely many orbits that do not contain any matrix x_ξ and each of them contains a $\{0,1\}$-matrix. By using these facts, one can show that T and P

are exponential groups for $n \leq 6$. (We believe that they are also exponential for $n = 7$.)

Remark 2. Let us now replace $\mathbb{H}$ by an infinite (commutative) field F. The problem of describing the T-orbits in N was raised in 1978 by M. Roitman [10]. He observed that for $n \leq 5$ there are only finitely many orbits and that each orbit contains a $\{0,1\}$-matrix. He also showed that for $n \geq 12$ there are infinitely many orbits. In fact the number of orbits is infinite for $n \geq 6$ (see [5]). The same problem, in a more general context, was studied somewhat later by H. Bürgstein and W. H. Hesselink [7, 2]. In particular they classified the T-orbits in N for $n \leq 7$.

3. The case of $\mathbf{SO}(p,1)^\circ$

In this section we consider another class of exponential groups, namely the identity components of the special orthogonal groups $\mathrm{SO}(p,1)$ of real rank 1.

Theorem 3.1. *The minimal parabolic subgroups of* $\mathrm{SO}(p,1)^\circ$ *are exponential groups.*

Proof. Let J be the symmetric matrix of order $p+1$ defined by

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{p-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$G := \mathrm{SO}(p,1) = \{x \in \mathrm{SL}_{p+1}(\mathbb{R}) : {}^t x J x = J\}.$$

Then

$$P := \left\{ \begin{pmatrix} \lambda & -\lambda\, {}^t v a & -\lambda {}^t v v/2 \\ 0 & a & v \\ 0 & 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R}^*,\ a \in \mathrm{SO}(p-1),\ v \in \mathbb{R}^{p-1} \right\}$$

is a minimal parabolic subgroup of G, and P° is a minimal parabolic subgroup of G°. We have the Levi decomposition $P = LU$ with

$$L = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R}^*,\ a \in \mathrm{SO}(p-1) \right\} = \mathbb{R}^* \times \mathrm{SO}(p-1)$$

reductive, and

$$U = \left\{ \begin{pmatrix} 1 & -{}^t v & -{}^t v v/2 \\ 0 & I_{p-1} & v \\ 0 & 0 & 1 \end{pmatrix} : v \in \mathbb{R}^{p-1} \right\} \cong \mathbb{R}^{p-1}$$

the unipotent radical. Since U is abelian, for $u \in U$ we have $Z_{P^\circ}(u) = Z_{L^\circ}(u)U$, and if $u \neq 1$ then $Z_{L^\circ}(u) = Z_{\mathrm{SO}(p-1)}(u) \cong \mathrm{SO}(p-2)$. Hence $Z_{P^\circ}(u)$ is weakly exponential for all $u \in U$, and by [6, Theorem 2.2] P° is exponential. ∎

4. The case of $\mathbf{U}(p,q)$

We set $n=p+q$ and let S and J be as in the proof of Corollary 2. For $G:=\mathrm{U}(p,q)$ we choose the following matrix realization:

$$G=\{x\in \mathrm{GL}_n(\mathbb{C}) : x^*Jx=J\}.$$

The group P consisting of all complex matrices of the form (1) with $a\in \mathrm{GL}_q(\mathbb{C})$ upper triangular, $b\in \mathrm{U}(p-q)$, and such that $aSd^*+dSa^*+cc^*=0$, is a minimal parabolic subgroup of G.

Theorem 4.1. *The minimal parabolic subgroups of* $\mathrm{U}(p,q)$ *are exponential groups.*

Proof. The Lie algebra $\mathfrak{p}$ of P consists of all complex matrices of the form (2) where x is an upper triangular matrix of order q, y is a skew-hermitian matrix of order $p-q$, and $Sw^*+wS=0$.

Let $A\in P$ be arbitrary, as in (1). We have to prove that $A\in E_P$. Without any loss of generality, we may assume that b is diagonal. We use induction on $q\geq 0$ to prove the following stronger assertion: If A is as above (with b diagonal) then there exists $X\in\mathfrak{p}$, as in (2), with y diagonal and such that $A=e^X$ and

$$-\pi<\mathrm{Im}(X_{ii})\leq\pi,\quad i=1,2,\dots,n.$$

Clearly the assertion is true for $q=0$. Assume now that $q>0$. We shall use the new partitioning of J :

$$J=\begin{pmatrix}0&0&1\\0&J_1&0\\1&0&0\end{pmatrix},$$

where J_1 is now of order $n-2$. We partition $A\in P$ and $X\in\mathfrak{p}$ accordingly:

$$A=\begin{pmatrix}\lambda&v&\zeta\\0&B&-(\bar\lambda)^{-1}J_1(B^*)^{-1}v^*\\0&0&(\bar\lambda)^{-1}\end{pmatrix},\quad X=\begin{pmatrix}\mu&u&i\alpha\\0&Y&-J_1u^*\\0&0&-\bar\mu\end{pmatrix},$$

where $\zeta\in\mathbb{C}$, $\alpha\in\mathbb{R}$,

$$2\mathrm{Re}(\bar\lambda\zeta)+vJ_1v^*=0, \tag{3}$$

and B belongs to the obvious minimal parabolic subgroup Q of $\mathrm{U}(p-1,q-1)$.

By induction hypothesis we can choose Y in the Lie algebra of Q such that $e^Y=B$, Y is upper triangular, and

$$-\pi<\mathrm{Im}(Y_{ii})\leq\pi,\quad i=1,2,\dots,n-2.$$

We also choose $\mu\in\mathbb{C}$ such that $e^\mu=\lambda$ and $-\pi<\mathrm{Im}(\mu)\leq\pi$. Then e^X has the form

$$e^X=\begin{pmatrix}\lambda&w&\zeta'\\0&B&-(\bar\lambda)^{-1}J_1(B^*)^{-1}w^*\\0&0&(\bar\lambda)^{-1}\end{pmatrix}. \tag{4}$$

The row-vector w is given by $w = u\eta$, where η is the triangular matrix

$$\eta = \sum_{k\geq 0} \frac{1}{(k+1)!}(Y^k + \mu Y^{k-1} + \cdots + \mu^k I_{n-2}).$$

If $\xi = Y_{ii}$ is one of the diagonal entries of Y, then the corresponding diagonal entry of η is given by:

$$\eta_{ii} = \begin{cases} (e^\xi - e^\mu)/(\xi - \mu), & \xi \neq \mu \\ e^\xi, & \xi = \mu. \end{cases}$$

The above mentioned conditions on the imaginary parts of the diagonal entries Y_{ii} and μ imply that $\eta_{ii} \neq 0$ for all i, and so the matrix η is nonsingular.

Consequently, there is a unique row-vector u such that $w = u\eta = v$. We assume from now on that u has been chosen so that $w = v$. The complex number ζ' in the formula (4) has the form:

$$\zeta' = \zeta'_0 + i\alpha \sum_{k\geq 0} \frac{1}{(k+1)!}(\mu^k + \mu^{k-1}(-\bar{\mu}) + \cdots + \mu(-\bar{\mu})^{k-1} + (-\bar{\mu})^k),$$

where ζ'_0 is independent of α. The coefficient of $i\alpha$ is equal to $(e^\mu - e^{-\bar{\mu}})/(\mu + \bar{\mu})$ if $\mu + \bar{\mu} \neq 0$, and is equal to e^μ otherwise. Hence it is never 0.

As the equation (3) remains valid when ζ is replaced by ζ', we have $\mathrm{Re}(\bar{\lambda}\zeta') = \mathrm{Re}(\bar{\lambda}\zeta)$. Since the coefficient of $i\alpha$ in $\bar{\lambda}\zeta'$ is real and nonzero, we can choose $\alpha \in \mathbb{R}$ such that $\bar{\lambda}\zeta' = \bar{\lambda}\zeta$, i.e., $\zeta' = \zeta$. Then we obtain that $e^X = A$. ■

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