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Supplements to the papers entitled "On a Theorem of S. Banach" and "The separable representations of U(H)"

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Abstract. We explain how the theorem of Banach discussed in [Ne97] can be generalized to target spaces which are not necessarily path connected. Moreover, we correct errors in the literature concerning the classification of separable unitary representations for the Banach Lie group $U(H)$ [Pi88, Pi90].

1. Foreword and Acknowledgements

After the publication of the first author's note [Ne97] on a theorem of Banach, C. C. Moore communicated several interesting remarks to the first author. He suggested that the result in [Ne97] could be derived in the greater generality described in Corollary 3 below. He pointed out that the theorem on the continuity of homomorphisms in the form that he used it in [Mo76] was in fact firmly grounded in the literature and had been stated in [Ku52, pp. 399/400], a reference that was specifically cited in [Mo76]. He also pointed out that the second author's use of Banach's theorem in [Pi90], and reproduced in [Ne97], was fallacious, as explained in Part 3 below.

In his review [Pe98] of the paper [Ne97], Pestov mentions a possible way to get around the assumption of path connectedness of the target group by embedding it into a path connected group. This method is more complicated than the one used here.

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2. Continuity of Homomorphisms

Kuratowski, following Baire and Lebesgue, says that a subset S of a topological space X has the *Baire property* if there exists an open set $O \subseteq X$ such that the symmetric difference of S and O is of first category in X. It is easily verified ([Ku52, §11, p.54]) that the set $\mathfrak{B}$ of subsets with the Baire propery is a σ-algebra and hence contains the Borel sets.

Kuratowski says that a function $f: X \to Y$ between topological spaces has the *Baire property* if the inverse image of every open set has the Baire property ([Ku52, §28, p. 306]). Note that Borel measurable functions do have this property because Borel sets have the Baire property.

The key observation is the following ([Ka52, §28, p.306]):

Proposition 1. (Kuratowski) *A function $f: X \to Y$ from a topological space to a second countable space has the Baire property if and only there exists a subset $P \subseteq X$ of first category such that $f|_{X \setminus P}$ is continuous.* ∎

The following theorem can be found in [Ku52, pp. 399/400]:

Theorem 2. *Every group homomorphism $f: G \to H$ from a metrizable group G which is a Baire space into a second countable group H with the Baire property is continuous.*

Proof. In view of the fact that G is a Baire space and Proposition 1 above, this follows by the same reasoning as [Ba32], respectively, in the proof of Theorem I.5 in [Ne97]. ∎

Corollary 3. *Every Borel measurable group homomorphism $f: G \to H$ from a completely metrizable group G into a separable metric group H is continuous.* ∎

3. Separable Representations of $U(H)$

Let H denote a separable Hilbert space (over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$), let K denote the group of automorphisms, and let τ_n and τ_s denote the operator norm and strong operator topologies, respectively. The identity map $(K, \tau_n) \to (K, \tau_s)$ is continuous (uniform convergence implies pointwise convergence), hence given a complex Hilbert space $\mathcal{H}$, a representation $\pi : (K, \tau_s) \to (U(\mathcal{H}), \tau_s^{\mathcal{H}})$ naturally induces a representation $\pi : (K, \tau_n) \to (U(\mathcal{H}), \tau_s^{\mathcal{H}})$. The second part of this note concerns the converse:

Theorem 4. *Suppose $\mathcal{H}$ is separable. Then a representation $\pi : (K, \tau_n) \to (U(\mathcal{H}), \tau_s^{\mathcal{H}})$ is also a representation for (K, τ_s).* ∎

The second author has attempted to give two proofs of this. The first proof in [Pi88] has a gap in the argument which was detected and filled by the second author. In the proof of Lemma 1 of [Pi88], the third sentence refers to a generalization of a Lemma 1.3 of Kirillov and Olshansky. It had been established that the closed convex hull of $\rho(K).\xi$ did not contain the origin, and it remained to be explained that this implied the existence of a $\rho(K)$ fixed point in the closed convex hull. This is a consequence of Birkhoff's Ergodic Theorem ([Bi39]).

The second proof in [Pi90] (the third paragraph following Proposition (5.1)) is fatally flawed, as observed by C. C. Moore. It was claimed that the identity map $(K, \tau_n) \to (K, \tau_s)$ induces an isomorphism of Borel structures, hence that Theorem 4 follows from Banach's Theorem. It was shown that balls in the norm topology are Borel with respect to τ_s; but this is not enough, because τ_n open sets are generally uncountable unions of balls, hence balls do not necessarily generate the Borel structure for τ_n. In fact Moore has pointed out that the Borel structure for τ_n cannot possibly be standard; he observes that there exists a family of cardinality $\mathfrak{c}$ of disjoint open sets, and from this he deduces that the Borel structure for τ_n contains $2^{\mathfrak{c}}$ Borel sets, and this is not the case for a standard Borel structure.

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