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Nambu-Lie Groups

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Abstract. We extend the Nambu bracket to 1-forms. Following the Poisson-Lie case, we define Nambu-Lie groups as Lie groups endowed with a multiplicative Nambu structure. A Lie group G with a Nambu structure P is a Nambu-Lie group iff $P = 0$ at the unit, and the Nambu bracket of left (right) invariant forms is left (right) invariant. We define a corresponding notion of a Nambu-Lie algebra. We give several examples of Nambu-Lie groups and algebras.

In 1973, Nambu [15] studied a dynamical system which was defined as a Hamiltonian system with respect to a ternary, Poisson-like bracket defined by a Jacobian determinant. A few years ago, Takhtajan [16] reconsidered the subject, proposed a general, algebraic definition of a *Nambu-Poisson bracket of order n* which, for brevity, we call a *Nambu bracket*, and gave the basic properties of this operation. The Nambu bracket is an intriguing operation, in spite of its rather restrictive character, which follows from the fact conjectured in [16], and proven by several authors [6], [1], [14], [9], [12] namely, that, locally and with respect to well chosen coordinates, any nonzero Nambu bracket is just a Jacobian determinant.

In this paper, we show that a Nambu bracket induces a corresponding bracket of 1-forms, and use the latter for a characterization of *Nambu-Lie groups*, a natural generalization of the Poisson-Lie groups (e.g, [17]). The relation with a corresponding notion of a Nambu-Lie algebra is discussed, and several examples of Nambu-Lie groups and algebras are given.

Independently, the Nambu-Lie groups have been studied by J. Grabowski and G. Marmo, who determined the general structure of the multiplicative Nambu tensor fields on Lie groups [7]. A preliminary version of our paper circulated as a preprint before [7] was available. But, in the present, final version we will also use results from [7], with due quotation.

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1. Nambu brackets of functions and 1-forms

Since the subject is not classical, we first recall the notion of a Nambu bracket and the geometric structure behind it. Let M^m be an m-dimensional differentiable manifold (in this paper everything is of the C^∞ class), and $\mathcal{F}(M)$ its algebra of real valued functions. A *Nambu bracket or structure* of order n, $3 \leq n \leq m$, is an internal n-ary operation on $\mathcal{F}(M)$, denoted by $\{\ \}$, which satisfies the following axioms:

(i) $\{\ \}$ is $\mathbf{R}$-multilinear and totally skew-symmetric;

(ii) $$\{f_1,\ldots,f_{n-1},gh\} = \{f_1,\ldots,f_{n-1},g\}h + g\{f_1,\ldots,f_{n-1},h\}$$

(the *Leibniz rule*);

(iii) $$\{f_1,\ldots,f_{n-1},\{g_1,\ldots,g_n\}\} = \sum_{k=1}^{n}\{g_1,\ldots,g_{k-1},\{f_1,\ldots,f_{n-1},g_k\},g_{k+1},\ldots,g_n\}$$

(the *fundamental identity*). A manifold endowed with a Nambu bracket is a *Nambu manifold.* (Remember that if we use the same definition for $n=2$, we get a *Poisson bracket.*)

By (ii), $\{\ \}$ acts on each factor as a vector field, whence it must be of the form

$$\{f_1,\ldots,f_n\} = P(df_1,\ldots,df_n), \tag{1.1}$$

where P is a field of n-vectors on M. If such a field defines a Nambu bracket, it is called a *Nambu tensor (field).* P defines a bundle mapping

$$\sharp_P : \underbrace{T^*M\times\ldots\times T^*M}_{n-1\ times} \longrightarrow TM \tag{1.2}$$

given by

$$\langle\beta,\sharp_P(\alpha_1,\ldots,\alpha_{n-1})\rangle = P(\alpha_1,\ldots,\alpha_{n-1},\beta) \tag{1.3}$$

where all the arguments are covectors.

In what follows, we denote an n-sequence of functions or forms, say $f_1,\ldots,$ f_n, by $f_{(n)}$, and, if an index k is missing, by $f_{(n,\hat{k})}$.

The next basic notion is that of the P*-Hamiltonian vector field* of $(n-1)$ functions defined by

$$X_{f_{(n-1)}} = \sharp_P(df_{(n-1)}). \tag{1.4}$$

The fundamental identity (iii) means that *the Hamiltonian vector fields are derivations of the Nambu bracket.* Another interpretation of (iii) is

$$(L_{X_{f_{(n-1)}}}P)(dg_1,\ldots,dg_n) = 0, \tag{1.5}$$

where L is the Lie derivative, i.e., *the Hamiltonian vector fields are infinitesimal automorphisms of the Nambu tensor.*

A mapping $\varphi : (M_1, P_1) \to (M_2, P_2)$ between two Nambu manifolds of the same order n is a *Nambu morphism* if the tensor fields P_1 and P_2 are φ-related or, equivalently, for all $g_{(n)} \in \mathcal{F}(M_2)$, one has

$$\{g_1 \circ \varphi, \ldots, g_n \circ \varphi\}_1 = \{g_1, \ldots, g_n\}_2.$$

If, moreover, φ is a diffeomorphism , it is an *equivalence of Nambu manifolds.* The notion of a Nambu morphism also leads to the following definition: a submanifold N of a Nambu manifold (M, P) is a *Nambu submanifold* if N has a (necessarily unique) Nambu tensor Q such that the immersion of (N, Q) in (M, P) is a Nambu morphism. Like in the Poisson case, Q exists iff, along N, P vanishes whenever evaluated on n 1-forms one of which, at least, belongs to the annihilator space $Ann(TN)$, and then $im\sharp_P$ is a tangent distribution of N (e.g., [17]).

1.1 Theorem. *[6], [1], [14], [9], [12]. P is a Nambu tensor field of order n iff for all $p \in M$ where $P_p \neq 0$ there are local coordinates (x^k, y^α) $(k = 1, \ldots, n,$ $\alpha = 1, \ldots, m - n)$ around p such that*

$$P = \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \qquad (\{f_1, \ldots, f_n\} = \frac{\partial(f_1, \ldots, f_n)}{\partial(x^1, \ldots, x^n)}). \tag{1.6}$$

On the canonical coordinate neighborhood where (1.6) holds we have

$$\mathcal{D} := span(im\,\sharp_P) = span\{\partial/\partial x^k\}.$$

Hence, globally $\mathcal{D}$ is a foliation with singularities whose leaves are either points, called the *singular points* of P, or n-dimensional submanifolds with a Nambu bracket induced by P. $\mathcal{D}$ is the *canonical foliation* of the Nambu structure P. The canonical foliation is regular i.e., all the leaves are n-dimensional, iff P never vanishes, and then we say that P is a *regular Nambu structure.*

1.2 Theorem. *[6], [9]. A regular Nambu structure of order n on a differentiable manifold M^m is equivalent with a regular n-dimensional foliation S of M, and a bracket operation defined by the formula*

$$d_S f_1 \wedge \ldots \wedge d_S f_n = \{f_1, \ldots, f_n\}\omega, \tag{1.7}$$

where ω is an S-leafwise volume form, and d_S is differentiation along the leaves of S.

Theorem 1.2 allows us to associate Nambu structures P to the n-dimensional orientable foliations. Furthermore, for each $f \in C^\infty(M)$, fP also is Nambu structure [4] with singular points at the zeroes of f.

It is also essential to stress the following fundamental consequence of Theorem 1.1 (e.g., [8]): a Nambu tensor field P necessarily is locally decomposable around the regular points and, conversely, a tensor field $P = V_1 \wedge \ldots \wedge V_n$ is Nambu iff $span\{V_1, \ldots, V_n\}$ is an involutive distribution on the subset where $P \neq 0$.

Given a Nambu bracket of order n, if p of its arguments are fixed, one gets a Nambu bracket of order $n-p$ (a Poisson bracket if $n-p=2$). The converse is also true [8]. On R^m, any constant, decomposable n-vector field $k^{i_1\dots i_n}$ is a Nambu tensor [2], and if we keep the fixed function $(1/2)\sum_{j=1}^m (x^j)^2$, we get a Nambu tensor, of order $n-1$, with the natural components

$$P^{i_1\dots i_{n-1}} = \sum_{j=1}^{m} k^{i_1\dots i_{n-1}j} x^j. \tag{1.8}$$

A Nambu structure defined on a vector space V^m by a tensor such that its components with respect to a linear basis of V are linear functions is called a *linear Nambu structure*. The classification of linear Nambu structures was done by Dufour and Zung [4] (see also [7]).

Several authors [5, 16, 6, 13, 7] etc. have studied vector spaces endowed with an internal, n-ary, skew symmetric bracket which satisfies the fundamental identity of a Nambu bracket. Following [7] we call these *Filippov algebras* since they were first studied by Filippov [5]. By looking at brackets of linear functions, it easily follows that a linear Nambu structure of order n on a vector space V induces a Filippov algebra structure on the dual space V^*. (The converse may not be true since the structure constants of a Filippov algebra may form a non decomposable n-vector.)

For instance, if k of (1.8) is the canonical volume tensor of R^{n+1} we get the linear Nambu structure of order n discussed in [2]. The corresponding Filippov algebra is the vector space R^{n+1} endowed with the operation of a *vector product* of n vectors given by

$$v_1 \times \dots \times v_n = *(v_1 \wedge \dots \wedge v_n), \tag{1.9}$$

where $*$ is the Hodge star operator of the canonical Euclidean metric of R^{n+1}. The canonical foliation of the linear Nambu structure of R^{n+1} defined above has the origin as a 0-dimensional leaf, and the spheres with center at the origin as n-dimensional leaves. (For $n=2$, this is the dual of the Lie algebra $o(3)$ with its well known Lie-Poisson structure.)

As in the case of Poisson structures, if (M,P) is a Nambu manifold, and if $p \in M$ is a *singular point* of P, the linear part of the Taylor development of P at p defines a linear Nambu structure on T_pM, and a corresponding Filippov algebra structure on T_p^*M, which are independent of the choice of the local coordinates at p. This structure is the *linear approximation* of P at p, and P is *linearizable* at p if P is equivalent with its linear approximation on some neighbourhood of p. Linearization theorems were proven in [4].

This ends the announced recall on Nambu brackets.

Now, following the Poisson model (e.g., [17]), we will extend the bracket of functions to a bracket of 1-forms. Namely, for a Nambu structure P of order n on M^m we define

$$\begin{aligned}\{\alpha_1,\dots,\alpha_n\} &= d(P(\alpha_{(n)})) + \sum_{k=1}^{n}(-1)^{n+k} i(\sharp_P(\alpha_{n,\hat{k}}))d\alpha_k \\ &= \sum_{k=1}^{n}(-1)^{n+k} L_{\sharp_P(\alpha_{n,\hat{k}})}\alpha_k - (n-1)d(P(\alpha_{(n)})),\end{aligned} \tag{1.10}$$

where α_k $(k = 1, \ldots, n)$ are 1-forms on M. The equality of the two expressions of the new bracket follows from the classical relation $L_X = di(X) + i(X)d$.

The bracket (1.10) will be called the *Nambu form-bracket*, and we have

1.3 Proposition. *The Nambu form-bracket satisfies the following properties:*
i) the form-bracket is totally skew-symmetric;
ii) for all $f_{(n)} \in \mathcal{F}(M)$, *one has*

$$\{df_1, \ldots, df_n\} = d\{f_1, \ldots, f_n\}; \tag{1.11}$$

iii) for any 1-forms $\alpha_{(n)}$, *and for all* $f \in \mathcal{F}(M)$ *one has*

$$\{f\alpha_1, \alpha_2, \ldots, \alpha_n\} = f\{\alpha_1, \alpha_2, \ldots, \alpha_n\} + P(df, \alpha_2, \ldots, \alpha_n)\alpha_1. \tag{1.12}$$

iv) for all $f_{(n-1)} \in \mathcal{F}(M)$ *and for any 1-form* α *one has*

$$\{df_1, \ldots, df_{n-1}, \alpha\} = L_{X_{f_{(n-1)}}}\alpha. \tag{1.13}$$

Proof. i) is obvious. ii) and iii) follow from the first expression of (1.10). iv) is a consequence of the first expression (1.10) and of the commutativity of d and L. ■

It would be nice if the form-bracket would also satisfy the fundamental identity of Nambu brackets. This happens for $n = 2$ but, generally, we only have the following weaker result

1.4 Proposition. *The Lie derivative with respect to a Hamiltonian vector field is a derivation of the Nambu form-bracket.*

Proof. Suppose that the required property holds for the 1-forms $\alpha_{(n)}$ i.e.,

$$L_{X_{f_{(n-1)}}}\{\alpha_1, \ldots, \alpha_n\} = \sum_{k=1}^{n}\{\alpha_1, \ldots, L_{X_{f_{(n-1)}}}\alpha_k, \ldots, \alpha_n\}. \tag{1.14}$$

Then, a straightforward computation which uses (1.12) and (1.5) shows that $L_{X_{f_{(n-1)}}}$ also acts as a derivation of the bracket $\{f\alpha_1, \alpha_2, \ldots, \alpha_n\}$, for all $f \in \mathcal{F}(M)$.

This remark shows that the proposition is true if the result holds for a bracket of the form $\{dg_1, \ldots, dg_n\}$, for all $g_k \in \mathcal{F}(M)$. But, this is a consequence of the fundamental identity for functions since by (1.11) we have

$$L_{X_{f_{(n-1)}}}\{dg_1, \ldots, dg_n\} = L_{X_{f_{(n-1)}}}d\{g_1, \ldots, g_n\} = dL_{X_{f_{(n-1)}}}\{g_1, \ldots, g_n\}.$$

■

The relation between (1.14) and the fundamental identity for 1-forms is given by (1.13). Since any closed form locally is an exact form, we see that the *fundamental identity*

$$\begin{aligned}\{\beta_1, \ldots, \beta_{n-1}, \{\alpha_1, \ldots, \alpha_n\}\} \\ = \sum_{k=1}^{n}\{\alpha_1, \ldots, \alpha_{k-1}, \{\beta_1, \ldots, \beta_{n-1}, \alpha_k\}, \alpha_{k+1}, \ldots, \alpha_n\}\end{aligned} \tag{1.15}$$

holds whenever the 1-forms β *are closed.*

In [7], the bracket (1.10) was interpreted as a dual bracket of the complete lift of P to TM .

2. Nambu-Lie Groups

Since Poisson-Lie groups play an important role in Poisson geometry (e.g., [17]), we are motivated to discuss similarly defined Nambu-Lie groups. These cannot be defined by the demand that the multiplication is a Nambu morphism since, on a product manifold, the sum of Nambu tensors may not be a Nambu tensor. But, it makes sense to say that a Nambu tensor P of order n endows the Lie group G with the structure of a *Nambu-Lie group* if P is a *multiplicative tensor field* i.e. (e.g., [17]), for all $g_1, g_2 \in G$, one has

$$P_{g_1 g_2} = L_{g_1^*} P_{g_2} + R_{g_2^*} P_{g_1}, \tag{2.1}$$

where L and R denote left and right translations in G, respectively.

The multiplicativity of P implies $P(e) = 0$, where e is the unit of G. Moreover, if G is connected, P is multiplicative iff $P(e) = 0$, and the Lie derivative $L_X P$ is a left (right) invariant tensor field whenever X is left (right) invariant. As an immediate consequence it follows that the Nambu-Lie group structures on the additive Lie group R^m are exactly the linear Nambu structures of R^m.

From (2.1), it follows that the set

$$G_0 := \{g \in G \mid P_g = 0\}$$

is a closed subgroup of G. Indeed, (2.1) shows that if $g_1, g_2 \in G_0$, the product $g_1 g_2 \in G_0$. Furthermore, if $g \in G_0$, then

$$0 = P_e = P_{gg^{-1}} = L_{g^*} P_{g^{-1}},$$

hence $g^{-1} \in G_0$.

The following theorem extends the characterization of Poisson-Lie groups given by Dazord and Sondaz [3].

2.1 Theorem. *If G is a connected Lie group endowed with a Nambu tensor field P which vanishes at the unit e of G, then (G, P) is a Nambu-Lie group iff the P-bracket of any n left (right) invariant 1-forms of G is a left (right) invariant 1-form.*

Proof. As in the Poisson case (e.g., [17]), the evaluation of the Lie derivative via (1.10) yields

$$(L_Y\{\alpha_1, \ldots, \alpha_n\})(X) = Y((L_X P)(\alpha_{(n)}))$$

for any left invariant vector field X, right invariant vector field Y, and left invariant 1-forms $\alpha_{(n)}$. (Same if left and right are interchanged.) Hence, the condition of the theorem is equivalent with the fact that $L_X P$ is left invariant whenever X is left invariant. ∎

Many other properties of Poisson-Lie groups also have a straightforward generalization.

Since $P(e) = 0$, the linear approximation of P at e defines a linear Poisson structure on the Lie algebra $\mathcal{G}$ of G and a dual Filippov algebra structure on the dual space $\mathcal{G}^*$. Furthermore, as for $n=2$, a compatibility relation between the Lie bracket and the linear Nambu structure of $\mathcal{G}$ exists.

Following [11], let us consider the *intrinsic derivative* $\pi_e := d_eP : \mathcal{G} \to \wedge^n\mathcal{G}$ defined by

$$\pi_e(X)(\alpha_{(n)}) = (L_{\check{X}}P)_e(\alpha_{(n)}), \tag{2.2}$$

where $\alpha_{(n)} \in \mathcal{G}^*$, $X \in \mathcal{G}$, and $\check{X}$ is any vector field on G with the value X at e. It is easy to understand that, if Π is the Nambu tensor on $\mathcal{G}$ associated with the linear approximation of P at e, then, for all $X \in \mathcal{G}$, $\Pi_X = \pi_e(X)$. Furthermore, we have

2.2 Theorem. *i) The bracket of the dual Filippov algebra structure of $\mathcal{G}^*$ is the dual π_e^* of the mapping π_e, and it has each of the following expressions*

$$\begin{aligned}[\alpha_1, \ldots, \alpha_n] &= d_e(P(\check{\alpha}_{(n)})) = \pi_e^*(\wedge_{k=1}^n \alpha_k) \\ &= \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}_e = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}_e,\end{aligned} \tag{2.3}$$

where $\alpha_{(n)} \in \mathcal{G}^*$, $\check{\alpha}_{(n)}$ are 1-forms on G which are equal to $\alpha_{(n)}$ at e, and $\bar{\alpha}_{(n)}$, $\tilde{\alpha}_{(n)}$ are the left and right invariant 1-forms, defined by $\alpha_{(n)}$, respectively.
ii) The mapping π_e is a $\wedge^n\mathcal{G}$-valued 1-cocycle of $\mathcal{G}$ with respect to the adjoint representation

$$ad_X(Y_1 \wedge \ldots \wedge Y_n) = \sum_{k=1}^n Y_1 \wedge \ldots Y_{k-1} \wedge [X, Y_k]_{\mathcal{G}} \wedge Y_{k+1} \wedge \ldots \wedge Y_n,$$

$(X, Y_{(n)} \in \mathcal{G})$.

Proof. The proofs are as for $n=2$; see [11] or Chapter 10 of [17].

i) The first equality sign is just the definition of the linear approximation, and the second follows since from $P_e = 0$ we have

$$\langle \pi_e^*(\wedge_{k=1}^n \alpha_{(k)}), X\rangle = \pi_e(X)(\alpha_{(n)}) = (L_{\check{X}}P)_e(\alpha_{(n)}) = X(P(\check{\alpha}_{(n)})) = \langle d_e(P(\check{\alpha}_{(n)}), X\rangle.$$

The remaining part of (2.3) is proven by the following computation with left (similarly, right) invariant forms:

$$\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}_e(X) \overset{(1.10)}{=} X(P(\bar{\alpha}_{(n)})) + \sum_{k=1}^n (-1)^{n+k}(d\bar{\alpha}_k)_e(\sharp_P(\alpha_{(n,\hat{k})}), X)$$

$$= X(P(\bar{\alpha}_{(n)})) - \sum_{k=1}^n (-1)^{n+k}(L_{\tilde{X}}\bar{\alpha}_k)_e(\sharp_P(\alpha_{(n,\hat{k})})) + \sum_{k=1}^n \sharp_P(\alpha_{(n,\hat{k})})_e(\bar{\alpha}_k(\tilde{X}))$$
$$= X(P(\bar{\alpha}_{(n)})),$$

where $\tilde{X}$ is the right invariant vector field defined by X, and we used the equalities $P_e = 0$, $L_{\tilde{X}}\bar{\alpha}_k = 0$.

ii) The fact that π_e is a 1-cocycle means that we have

$$\partial\pi_e(X, Y) := ad_X(\pi_e(Y)) - ad_Y(\pi_e(X)) - \pi_e([X, Y]_{\mathcal{G}}) = 0, \tag{2.4}$$

where $X, Y \in \mathcal{G}$, or equivalently,

$$(2.4') \qquad ad_X(\Pi_Y) - ad_Y(\Pi_X) = \Pi_{[X,Y]},$$

with the earlier defined Π. We always use the notation with bars and tildes for left and right invariant objects on Lie groups as we did above. Then,

$$ad_X(\pi_e(Y)) = \frac{d}{ds}\,|_{s=0}\, Ad\, exp(sX)((L_{\bar{Y}}P)_e) = (L_{\bar{X}}L_{\bar{Y}}P)_e,$$

and (2.4) is a consequence of this result. ∎

The second part of Theorem 2.2 yields

2.3 Corollary. *For all $\alpha_{(n)} \in \mathcal{G}^*$ and for all $X, Y \in \mathcal{G}$ the following relation holds*

$$(2.5) \qquad \langle[\alpha_1, \ldots, \alpha_n], [X, Y]_{\mathcal{G}}\rangle = \sum_{k=1}^{n}(\langle[\alpha_1, \ldots, \alpha_{k-1}, coad_Y\alpha_k, \alpha_{k+1}, \ldots, \alpha_n], X\rangle$$
$$-\langle[\alpha_1, \ldots, \alpha_{k-1}, coad_X\alpha_k, \alpha_{k+1}, \ldots, \alpha_n], Y\rangle).$$

Proof. The result is nothing but a reformulation of the cocycle condition (2.4). ∎

On the other hand, the first part of Theorem 2.2 allows us to get a result on subgroups just as in the Poisson case. A Lie subgroup H of a Nambu-Lie group (G, P) will be called a *Nambu-Lie subgroup* if H has a (necessarily unique) multiplicative Nambu tensor Q such that (H, Q) is a Nambu submanifold of (G, P). For instance, the vanishing subgroup G_0 of P with the Nambu structure $Q = 0$ is a Nambu-Lie subgroup of (G, P). If H is connected, it is a Nambu-Lie subgroup of (G, P) iff $Ann(\mathcal{H})$, where $\mathcal{H}$ is the Lie algebra of H, is an ideal in $(\mathcal{G}^*, [., \ldots, .])$. By this we mean that the bracket (2.3) is in $Ann(\mathcal{H})$ whenever one of the arguments (at least) is in $Ann(\mathcal{H})$. The proof is the same as for $n = 2$ e.g., [17].

Furthermore, if (H, Q) is a Nambu-Lie subgroup of (G, P), the homogeneous space $M = G/H$ inherits a Nambu structure S of the same order as P, Q such that the natural projection $p : (G, P) \to (M, S)$ is a Nambu morphism. This holds since the brackets $\{f_1 \circ p, \ldots, f_n \circ p\}_P$ $(f_i \in C^\infty(M))$ are constant along the fibers of p, which is easy to check using (2.1). (E.g., see Proposition 10.30 in [17] for the case $n = 2$.) Moreover, as a consequence of (2.1), the natural left action of G on M satisfies the multiplicativity condition

$$(2.1') \qquad S_{g(x)} = \varphi_{g*}(S_x) + \varphi^x_*(P_g),$$

where $g \in G, x \in M$, and $\varphi_g : M \to M$, $\varphi^x : G \to M$ are defined by $\varphi_g(x) = \varphi^x(g) = g(x)$. An interesting example is $H = G_0$, $Q = 0$, $M = G/G_0$, where G_0 is the vanishing subgroup of P. In agreement with the above situation, any action of a Nambu-Lie group (G, P) on a Nambu manifold (M, S) which satisfies (2.1′) will be called a *Nambu action.* If G is connected, one has the same infinitesimal characteristic properties of Nambu actions as in the Poisson case e.g., Proposition

10.27 in [17]. In particular, that for all $X \in \mathcal{G}$, $L_{X_M} S = -[(d_e P)(X)]_M$, where e is the unit of G, and the index M denotes the infinitesimal action on M.

At this point, one may ask whether a Lie algebra $\mathcal{G}$ with a linear Nambu structure Π that satisfies the cocycle condition (2.4), (2.5) can be integrated to a Nambu-Lie group (G, P)?

Some of the results known for $n = 2$ still hold. If G is connected and simply connected, for any 1-cocycle π_e as in Theorem 2.2 ii), there exists a unique multiplicative n-vector field P on G, called the *integral field* of π_e such that $d_e P$ is the given cocycle. Indeed, for the given cocycle π_e,

$$\pi_g(X_g) := Ad\, g(\pi_e(L_{g^{-1}*} X_g)) \quad (g \in G,\ X_g \in T_g G)$$

defines a $\wedge^n \mathcal{G}$-valued 1-form π on G which satisfies the equivariance condition $L_g^* \pi = (Ad\, g) \circ \pi$. This implies that $d\pi = 0$, and, since G is connected and simply connected, $\pi = dP$ for a unique n-vector field P on G, which can be seen to be multiplicative [11, 17]. If this field is Nambu, we are done. But, this final part is not ensured if $n \geq 3$.

Moreover, the structure theory of multiplicative Nambu tensors of [7] leads to one more, important, necessary condition. Recalling from [7], let (G, P) be a connected Nambu-Lie group, and for all $g \in G$ where $P_g \neq 0$ put $\tilde{P}(g) = R_{g^{-1}*} P_g$. $\tilde{P}(g)$ is a decomposable element of $\wedge^n \mathcal{G}$, and yields a subspace $V(g) :=$ span of the factors of $\tilde{P}(g) \subseteq \mathcal{G}$. Furthermore, $V_\cap := \cap_g V(g)$, and $V_\cup := \sum_g V(g)$ are ideals in $\mathcal{G}$, and a sum-intersection lemma ([4], Lemma 3.2 and Theorem 3.1, [7], Lemma 1) tells that either $dim\, V_\cap \geq n - 1$ or $dim\, V_\cup = n + 1$. In the first case, put $\mathcal{H} = V_\cap$, and if $dim\, V_\cup = n + 1$ put $\mathcal{H} = V_\cup$. In both cases, the ideal $\mathcal{H}$ will be called the *core ideal* of P, and if $0 \neq \Lambda_0 \in \wedge^{dim\, \mathcal{H}} \mathcal{H}$, we call Λ_0 a *core* of P. According to the cases $dim\, \mathcal{H} = n, n-1, n+1$ we have either a) $\tilde{P}(g) = \theta(g)\Lambda_0$, with $\theta \in C^\infty(G)$, or b) $\tilde{P}(g) = X(g) \wedge \Lambda_0$ with $X : G \to \mathcal{G}$, or c) $\tilde{P}(g) = i(\alpha(g))\Lambda_0$ with $\alpha : G \to \mathcal{G}^*$, and the conditions which θ, X, α must satisfy in order to provide a multiplicative Nambu tensor P on G are determined in [7]. In case a) the canonical foliation $\mathcal{D}$ of P is the same as the left (and right) invariant foliation $\mathcal{F}_\mathcal{H}$ defined by translating $\mathcal{H}$ along G, in case b) $\mathcal{F}_\mathcal{H}$ is a subfoliation of codimension 1 of $\mathcal{D}$, and in case c) $\mathcal{D}$ is a subfoliation of codimension 1 of $\mathcal{F}_\mathcal{H}$.

In particular, since any linear Nambu structure Π on a vector space V is multiplicative with respect to the additive structure of V, Π has a core linear subspace $\mathcal{H}$ (ideal of a commutative Lie algebra) and a core Λ_0. Moreover, we get

2.4 Theorem. *Let (G, P) be a connected, m-dimensional, Nambu-Lie group with the core ideal $\mathcal{H}$ and a core Λ_0, and let Π be the linear approximation of P at the unit $e \in G$. Then, $\mathcal{H}, \Lambda_0$ also are the core subspace and a core of Π, respectively. Furthermore, the core of Π is an ideal of $\mathcal{G}$, and $\mathcal{G}$ must have an ideal $\mathcal{H}$ of one of the dimensions $n, n-1, n+1$, where n is the order of P.*

Proof. By its very definition, $\pi_X = \pi_e(X) = X\tilde{P}$, where $\tilde{P}$ is seen as a $\wedge^n \mathcal{G}$-valued function on G (e.g., [17], p. 166). Then, by derivating the three possible expressions of $\tilde{P}(g)$ as recalled from [7] above, we get the conclusion. ∎

Theorem 2.4 reduces the finding of the core ideal and the core of a Nambu-Lie group to the same problem for a linear Nambu structure.

Now, in agreement with Theorems 2.2 and 2.4, we define a *Nambu-Lie algebra* as being a Lie algebra $\mathcal{G}$ endowed with a linear Nambu structure Π such that: i) Π is a 1-cocycle , and ii) the core subspace of Π is an ideal of $\mathcal{G}$.

The method of [7] can also be used for the determination of the Nambu-Lie algebras $(\mathcal{G}, \Pi)$. Namely, if $\mathcal{G}$ is given, we have to consider the ideals $\mathcal{H}$ of $\mathcal{G}$. Then, for an ideal $\mathcal{H}$ of dimension n, and with $0 \neq \Lambda_0 \in \wedge^n \mathcal{H}$, we have to look for Π_X under one of the following forms:

$$\Pi_X = \varphi(X)\Lambda_0, \qquad \varphi \in \mathcal{G}^*, \tag{2.6}$$

$$\Pi_X = \mathcal{X}(X) \wedge \Lambda_0, \qquad \mathcal{X} \in End(\mathcal{G}), \tag{2.7}$$

$$\Pi_X = i(\alpha(X))\Lambda_0, \qquad \alpha \in Hom(\mathcal{G}, \mathcal{G}^*). \tag{2.8}$$

Finally, we must ask the cocycle condition $(2.4')$ which, respectively, will give

$$\varphi([X,Y]) = \varphi(Y)\gamma(X) - \varphi(X)\gamma(Y), \tag{2.6'}$$

$$\mathcal{X}([X,Y]) - [X, \mathcal{X}(Y)] + [Y, \mathcal{X}(X)] - \gamma(X)\mathcal{X}(Y) + \gamma(Y)\mathcal{X}(X) \in \mathcal{H}, \tag{2.7'}$$

$$\begin{aligned} i[\alpha[X,Y]) + (coad_X\alpha)(Y) - (coad_Y\alpha)(X) \\ +\gamma(X)\alpha(Y) - \gamma(Y)\alpha(X)]\Lambda_0 = 0, \end{aligned} \tag{2.8'}$$

where $X, Y \in \mathcal{G}$, *coad* is naturally extended to the twice covariant tensor α, and $\gamma \in \mathcal{G}^*$ is determined by the condition $ad_X\Lambda_0 = \gamma(X)\Lambda_0$, therefore, it must satisfy

$$\gamma([X,Y]) = 0, \qquad \forall X, Y \in \mathcal{G}. \tag{2.9}$$

For an example, let us consider the unitary Lie algebra $u(2)$. In $u(2)$ we have the basis

$$X_1 = \frac{\sqrt{-1}}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; X_2 = -\frac{\sqrt{-1}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$X_3 = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; X_4 = -\frac{\sqrt{-1}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

such that X_1 spans the center, and

$$[X_2, X_3] = X_4, \; [X_3, X_4] = X_2, \; [X_4, X_2] = X_3, \tag{2.10}$$

and we will denote by (x^a) $(a = 1,2,3,4)$, the corresponding linear coordinates.The relevant ideals are $\mathcal{H}_1 = u(2)$, $\mathcal{H}_2 = su(2) = span\{X_2, X_3, X_4\}$, with the cores

$$\Lambda_{01} = X_1 \wedge X_2 \wedge X_3 \wedge X_4, \; \Lambda_{02} = X_2 \wedge X_3 \wedge X_4,$$

respectively.

From (2.9), (2.10), we see that $\gamma = 0$, and any φ which satisfies (2.6′) vanishes at X_2, X_3, X_4. It follows that on $u(2)$ there is only one Nambu-Lie algebra structure of type (2.6) with the core Λ_{01}, up to a constant factor, and this is

$$\Pi = x^1 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4}. \tag{2.11}$$

But, it is easy to see that there is no non zero Nambu-Lie group structure P on the unitary group $U(2)$ with the core Λ_{01}. Such a structure would have $\tilde{P}(g) = \theta(g)\Lambda_{01}$ where multiplicativity implies that for all $g_1, g_2 \in U(2)$ one has $\theta(g_1 g_2) = \theta(g_1) + \theta(g_2)$ [7]. And, there is no non zero θ with this property since by Theorem 5 of [7] one should have $\theta(unit) = 0$, $d\theta =$ a bi-invariant 1-form, i.e., $d\theta = kdx^1$ $(k = const.)$. Hence, θ would be an additive character of the circle subgroup S^1 of $U(2)$ and, thus, $\theta = 0$.

Therefore, above we have an example of a Nambu-Lie algebra which does not integrate to a Nambu-Lie group.

Furthermore, for Λ_{01}, Π of (2.7) is zero. But, it is possible to find Nambu-Lie structures of the type (2.8). An example is

$$\Pi = (x^2 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^4}) \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^1} = \partial(X_4 \wedge X_2 \wedge X_1), \tag{2.12}$$

which is a coboundary hence, a cocycle. Here, $\partial X_1 = 0$ since X_1 is ad-invariant, and $\partial(X_4 \wedge X_2)$ is the cocycle of a well known example of a Poisson-Lie structure of $SU(2)$ namely [10],

$$W(g) = L_{g^*}(X_4 \wedge X_2) - R_{g^*}(X_4 \wedge X_2), \;\; (g \in SU(2)). \tag{2.13}$$

Accordingly, (2.12) is the cocycle of a Nambu-Lie structure on $U(2)$ which is defined by

$$\begin{aligned} P(g) &= W(g) \wedge X_1 \\ &= L_{g^*}(X_4 \wedge X_2 \wedge X_1) - R_{g^*}(X_4 \wedge X_2 \wedge X_1), \;\; (g \in U(2)). \end{aligned} \tag{2.14}$$

Indeed, it is easy to check that P is multiplicative [17]. It is decomposable since, if $W(g) \neq 0$, $rank\, W(g) = 2$, and the factors span an involutive distribution.

It is interesting that we have thereby obtained an example of a compact Nambu-Lie group. The construction does not extend to $U(n)$ with $n > 2$ since the similar structure P is not decomposable.

Now, comming back to the Lie algebra $u(2)$, we should look at cocycles with the core ideal $\mathcal{H}_2 = su(2)$. Like for (2.11), it follows again that, up to a constant factor, the only structure of type (2.6) is

$$P = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}, \tag{2.15}$$

and it does not come from a Nambu-Lie group structure on $U(2)$. The structures (2.7) which can be obtained reduce to (2.11), and (2.8) may lead to Poisson-Lie structures on $U(2)$. In particular, we can refind (2.13) with $g \in U(2)$.

In principle, a systematic search for the structures II should be possible by using the canonical forms of the linear Nambu structures given in [4, 7]. One has to look for structure constants of Lie algebras which, together with the canonical structures of [4], satisfy the cocycle condition (2.5), and such that the core of the linear Nambu structure is an ideal of the considered Lie algebra.

We finish by giving some more examples of non commutative Nambu-Lie groups.

A first example is the 3-dimensional solvable Lie group

$$G_3 := \Big\{ \begin{pmatrix} x & 0 & y \\ 0 & x & z \\ 0 & 0 & 1 \end{pmatrix} \ \mid x, y, z \in R,\ x \neq 0 \Big\}. \tag{2.16}$$

The left invariant forms of this group are $dx/x, dy/x, dz/x$, and if we look for a Nambu tensor of the form

$$P = f(x)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \tag{2.17}$$

such that $\{dx/x, dy/x, dz/x\}$ is left-invariant, and $f(1) = 0$, we see that $f = x(x^2 - 1)/2$ does the job. The corresponding Nambu-Lie algebra is R^3 with the linear Nambu structure $x^1(\partial/\partial x^1) \wedge (\partial/\partial x^2) \wedge (\partial/\partial x^3)$.

The next example is the *generalized Heisenberg group*

$$H(1,p) := \Big\{ \begin{pmatrix} Id_p & X & Z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \Big\}, \tag{2.18}$$

where $X = {}^t(x_1, \ldots, x_p)$, $Z = {}^t(z_1, \ldots, z_p)$ (t means transposition of matrices). The left invariant 1-forms of this group are

$$dx_1, \ldots, dx_p, dy, dz_1 - x_1 dy, \ldots, dz_p - x_p dy, \tag{2.19}$$

and

$$P = y\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial y} \tag{2.20}$$

makes $H(1,p)$ into a Nambu-Lie group. Indeed, it vanishes at the unit, and it follows easily that the brackets of the left invariant 1-forms are left invariant. The corresponding Nambu-Lie algebra is R^{2p+1} with the same Nambu tensor (2.20).

A third example is the direct product $G = H(1,1) \times R_+$, where R_+ is the multiplicative group of the positive real numbers t. The left invariant 1-forms of the group are those given by (2.19), and dt/t. The tensor

$$P = t(\ln t)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t} \tag{2.21}$$

makes G into a Nambu-Lie group for the same reasons as in the previous examples. The corresponding Nambu-Lie algebra is R^4 with the linear Nambu structure $P = x_4(\partial/\partial x_2) \wedge (\partial/\partial x_3) \wedge (\partial/\partial x_4)$.

We also notice that if (G_1, P) is a Nambu-Lie group, and G_2 is any other Lie group, fP, where $f \in C^\infty(G_2)$, is a Nambu-Lie structure on $G_1 \times G_2$ (check (2.1)).

Finally, we quote the following important result proven in [7]: there are no Nambu-Lie structures of order $n \geq 3$ on simple Lie groups, and if $G = G_1 \times \cdots \times G_s$ is a semisimple Lie group with the simple factors G_i $(i = 1, \ldots, s)$, the only multiplicative Nambu tensors on G are wedge products of contravariant volume tensor fields on a part of the factors with either multiplicative Poisson bivectors or multiplicative vector fields on other factors.

References

[1] Alekseevsky, D., and P. Guha, *On Decomposability of Nambu-Poisson Tensor*, Acta Math. Univ. Commenianae **65** (1996), 1–10.

[2] Chatterjee, R., and L. Takhtajan, *Aspects of Classical and Quantum Mechanics*, Lett. Math. Phys. **37** (1996), 475–482.

[3] Dazord, P., and D. Sondaz, *Groupes de Poisson Affines*, In: "Symplectic Geometry, Groupoids and Integrable Systems," Séminaire Sud-Rhodanien de Géométrie à Berkeley (1989) (P. Dazord and A. Weinstein, eds.) MSRI Publ. **20**, Springer, Berlin-New York, 1991, 99–128.

[4] Dufour, J.-P., and N. T. Zung, *Linearization of Nambu structures*, Compositio Mathematica, to appear.

[5] Filippov, V. T., *n-ary Lie algebras*, Russian Sibirskii Math. J. **24** (1985), 126–140.

[6] Gautheron, Ph., *Some remarks concerning Nambu mechanics*, Lett. in Math. Phys. **37** (1996), 103–116.

[7] Grabowski, J. and G. Marmo, *On Filippov algebroids and multiplicative Nambu-Poisson structures*, Diff. Geom. Appl., to appear, and preprint math@xxx.lanl.gov, DG/9902127.

[8] Grabowski, J., and G. Marmo, *Remarks on Nambu-Poisson and Nambu-Jacobi brackets*, Preprint math@xxx.lanl.gov DG/9902128.

[9] Ibáñez, R., M. de León, J. C. Marrero, and D. M. de Diego, *Dynamics of generalized Poisson and Nambu-Poisson brackets*, J. of Math. Physics **38** (1997), 2332–2344.

[10] Lu, J.-H., "Multiplicative and affine Poisson structures on Lie groups, " Thesis, Univ. of California, Berkeley, 1990.

[11] Lu, J.-H., and A. Weinstein, *Poisson-Lie groups, dressing transformations and Bruhat decompositions*, J. Diff. Geometry **31** (1990), 501–526.

[12] Marmo, G., G. Vilasi, and A. M. Vinogradov, *The local structure of n-Poisson and n-Jacobi manifolds*, J. Geom. Physics **25** (1998), 141–182.

[13] Michor, P. W., and A. M. Vinogradov, *n-ary Lie and asociative algebras*, Rend. Sem. Mat. Torino **53** (1996), 373–392.

[14] Nakanishi, N., *On Nambu-Poisson Manifolds*, Reviews in Mathematical Physics, **10** (1998), 499–510.

[15] Nambu, Y., *Generalized Hamiltonian Mechanics*, Phys. Rev. D **7** (1973), 2405–2412.

[16] Takhtajan, L., *On Foundations of Generalized Nambu Mechanics*, Comm. Math. Phys. **160** (1994), 295–315.

[17] Vaisman, I., "Lectures on the Geometry of Poisson Manifolds, " Progress in Math. **118**, Birkhäuser, Basel, 1994.

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