Journal of Lie Theory
Volume **10** (2000) 171–180

Some linear groups virtually having a free quotient

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Communicated by K. H. Hofmann

Abstract. It is proved that some discrete automorphism groups of convex cones have a finite index subgroup that maps onto a free group of rank 2. This generalizes recent results of A. Lubotzky.

We say that a group Γ virtually has some property, if some subgroup of finite index in Γ has this property. A group virtually having a non-abelian free quotient is called *large* [4]. In this paper we prove that some linear groups are large. This generalizes results of A. Lubotzky [5]. Except for some additions and simplifications, we essentially follow the lines of his proof.

Note that if a group Γ has a non-abelian free quotient, then any subgroup of finite index in Γ has such a quotient. It follows that if a group Γ is large, then any group commensurable with Γ shares this property. Clearly, if a group Γ has a virtually non-abelian free quotient, it is large. (The converse is not true.)

Let V be a finite-dimensional real vector space and V^* its dual space. If $K \subset V$ is an open convex cone, the *dual cone* $K^* \subset V^*$ is defined by
$$K^* = \{\alpha \in V^* : \alpha(v) > 0 \quad \forall v \in K\}.$$
Obviously, $K^* \cup \{0\}$ is a closed strictly convex cone in V^*. For a linear group $\Gamma \subset \mathrm{GL}(V)$ leaving invariant an open convex cone $K \subset V$, we consider the condition

$(*)$ $\quad\Gamma$ has no finite orbits in the projectivization PK^* of K^*.

Main Theorem. *Let $\Gamma \subset \mathrm{GL}(V)$ be a finitely generated linear group leaving invariant some open convex cone $K \subset V$ and satisfying the condition $(*)$. Suppose that there exists a subspace U of codimension 1 in V such that*

(H1) *$H = U \cap K \neq \varnothing$;*
(H2) *for any $\gamma \in \Gamma$, we have either $\gamma H = H$, or $\gamma H \cap H = \varnothing$;*
(H3) *any compact subset of K meets only finitely many sets of the form $\gamma H, \gamma \in \Gamma$;*
(H4) *both connected components of $K \backslash H$ contain some $\gamma H, \gamma \in \Gamma$.*

Then Γ has a subgroup of index $\leqslant 2$ that has a virtually non-abelian free quotient.

We shall deduce from this the theorems stated below, in which the term "reflection" always means a reflection in a hyperplane.

ISSN 0949–5932 / $2.50

Theorem 1. *Let $\Gamma \subset GL(V)$ be a finitely generated irreducible linear group leaving invariant some open convex cone $K \subset V$ and acting on it discretely. Suppose that Γ contains a linear reflection. Then one of the following alternatives takes place:*

(1) *$K = V$ and Γ is finite;*

(2) *K is a simplicial cone and Γ is virtually abelian;*

(3) *Γ is large.*

Corollary 1. *Let Γ be a finitely generated discrete group of motions of n-dimensional Lobachevsky space $L^n (n > 1)$, containing a reflection and leaving invariant no plane (of any dimension) or point at infinity. Then Γ is large.*

Corollary 2. *Any non-affine infinite indecomposable finitely generated Coxeter group is large.*

Let f be a quadratic form of signature $(n, 1)$ over $\mathbb{Q}$. Then the group $\mathrm{O}(f, \mathbb{Z})$ is a lattice in $\mathrm{O}(f, \mathbb{R})$.

Theorem 2. *Any finitely generated subgroup of $\mathrm{O}(f, \mathbb{Z})$ is either virtually abelian, or large.*

Before we started this work, we had been informed that C. Gonciulea had proved the assertion of Corollary 2 above. But unfortunately his proof was not available to us at the time we wrote this paper. (See [3] and [2] for a proof of a weaker theorem that any infinite finitely generated Coxeter group virtually maps onto $\mathbb{Z}$.)

We are grateful to H. Oh, A. Furman, and the referee for helpful remarks. This work was done during our stay at Bielefeld University. This stay was supported by the Humboldt Foundation. The work of the first author was supported in part by NSF Grant DMS-9800607, and the work of the second author was supported also by RFBR Grant 98-01-00598.

1. Residually Finite Actions

Definition 1. An action of a group Γ on a set X is called *residually finite*, if for any different points $x, x' \in X$ there exist an action of Γ on a finite set Y and a Γ-equivariant map $f : X \to Y$ such that $f(x) \neq f(x')$.

Obviously, an action is residually finite if an only if its restriction to each orbit is residually finite. If an action $\Gamma : X$ is residually finite, then for any finite number of points of X there exists a Γ-equivariant map of X to a finite set separating these points.

Let us also note that an action $\Gamma : X$ is residually finite if and only if for any different points $x, x' \in X$ there exists a normal subgroup Δ of finite index in Γ such that x and x' belong to different Δ-orbits. Indeed, let x and x' be different points of X. If the action $\Gamma : X$ is residually finite, then, in the above notation, one can take for Δ the kernel of the action $\Gamma : Y$. Conversely, if the action $\Gamma : X$ is transitive and $\Delta x \neq \Delta x'$, one can take for Y the quotient X/Δ.

The following fact is known, but we did not find a reference for it.

Lemma 1. *Let A be a finitely generated integral domain. Then for any nonzero element a of A there exists a homomorphism φ of A to a finite field such that $\varphi(a) \neq 0$.*

Proof. Let k denote the field of fractions of A, and let F be the minimal subfield of k. By Hilbert's Nullstellensatz there exists a homomorphism φ of A into a finite extension of F such that $\varphi(a) \neq 0$. If char $k > 0$ this finite extension is a finite field and φ is the desired homomorphism. Let char $k = 0$. Then k is an algebraic number field. There exists a (multiplicative) non-archimedian valuation v of k such that A belongs to the ring $\mathcal{O} = \{x \in k \mid v(x) \leq 1\}$ and $v(\varphi(a)) = 1$. Let $\mathfrak{m} = \{x \in k : v(x) < 1\}$ be the maximal ideal of $\mathcal{O}$ and $\pi : \mathcal{O} \to \mathcal{O}/\mathfrak{m}$ the natural projection. Then $\pi \circ \varphi$ is the desired homomorphism.■

Let V be a finite-dimensional vector space over a field k. Denote by PV the projective space associated to V, and, for any non-zero vector $v \in V$, denote by $[v]$ the corresponding point of PV.

Proposition 1. *Any linear action of a finitely generated group Γ in the space V and the induced action $\Gamma : PV$ are residually finite.*

Proof. Let $\gamma_1, \dots, \gamma_s$ be generators of Γ, and let v be a vector in V. Consider the subring A of k generated by the matrix coefficients of $\gamma_1, \dots, \gamma_s, \gamma_1^{-1}, \dots, \gamma_s^{-s}$ and the coordinates of v in some fixed basis of V. Any homomorphism φ of A to a field K defines a group homomorphism $\Gamma \to \mathrm{GL}_n(K)$, where $n = \dim V$, and thereby a linear action $\Gamma : K^n$. At the same time it defines a Γ-equivariant map $f : A^n \to K^n$. According to Lemma 1, for any $a \in A, a \neq 0$, there exists a homomorphism φ of A to a finite field K such that $\varphi(a) \neq 0$. In particular, if $v' \in \Gamma v$ is different from v, one can choose φ in such a way that $f(v) \neq f(v')$. Thus, the action $\Gamma : V$ is residually finite.

Let now $v \neq 0$. A vector $v' \in \Gamma v$ is proportional to v if and only if all the minors of order 2 of the matrix $M(v, v')$ of the co-ordinates of v and v' are equal to 0. In this case $f(v)$ and $f(v')$ are also proportional. We can choose φ in such a way that $f(v) \neq 0$. Then f gives rise to a Γ-equivariant map $\bar{f} : \Gamma[v] \to \Gamma[f(v)]$. For any $v' \in \Gamma v$, which is not proportional to v, we can choose φ in such a way that $\varphi(D) \neq 0$ for some minor D of order 2 of $M(v, v')$. Then $\bar{f}([v]) \neq \bar{f}([v'])$. Thus, the action $\Gamma : PV$ is also residually finite. ■

Remark 1. The proposition implies that, for any finitely generated subgroup Γ of an algebraic group G and any action of G on an algebraic variety X, the induced action $\Gamma : X$ is residually finite. Indeed, it suffices to prove this for transitive G-actions. But due to a theorem of Chevalley any transitive G-action is embedded into the natural G-action on PV, where V is a G-module. ■

2. Some Group Actions on Trees

For any graph S (possibly with loops and multiple edges) denote by $[S]$ its topological space. If S is connected, set $\pi_1(S) = \pi_1([S])$. If a group Γ acts on a graph T without reversing edges, one can naturally define a topological action of Γ on $[T]$ in such a way that the topological quotient $[T]/\Gamma$ is identified with $[T/\Gamma]$. In particular, if T is a tree, this gives rise to a surjective homomorphism $\varphi : \Gamma \to \pi_1(T/\Gamma)$, taking each $\gamma \in \Gamma$ to the homotopy class of the image in $[T/\Gamma]$ of a path l connecting the base vertex o with γo in $[T]$.

Lemma 2. *The kernel Γ_0 of φ does not depend on the choice of the base vertex and is normalized by any automorphism of T normalizing Γ.*

Proof. Let o' be any other vertex of T and m a path connecting o' with o. In the above notation, $l' = ml(\gamma m)^{-1}$ is a path connecting o' with $\gamma o'$. If the image of l in $[T/\Gamma]$ is homotopic to a point, then the image of l' is also homotopic to a point. This proves the first assertion of the lemma. The second assertion immediately follows. ∎

Remark 2. In fact it is easy to show that Γ_0 is the subgroup of Γ generated by all the stabilizers of vertices of T. ∎

It is useful to have in mind the following

Lemma 3. *Any group Γ acting on a tree T contains a subgroup Γ_+ of index 1 or 2 that does not reverse edges.*

Proof. Note that if the group Γ preserves some orientation of the edges of T, it cannot reverse edges. There are two distinguished opposite orientations of the edges of T, under which any vertex is either a source or a sink. Any automorphism of T can only permute them. It follows that the subgroup Γ_+ of Γ preserving one of these orientations has index 1 or 2, and it does not reverse edges. ∎

A tree will be called a *star* if all its edges have a common vertex, and a *line* if its topological space is homeomorphic to $\mathbb{R}$. A graph will be called a *cycle* if its topological space is homeomorphic to a circle.

Denote by $V(S)$ (resp. $E(S)$) the set of vertices (resp. edges) of a graph S. For a connected finite graph S, we consider the Euler characteristic

$$e(S) = \#V(S) - \#E(S).$$

Obviously, $e(S) \leq 1$. If $e(S) = 1$, the graph S is acyclic (i.e. a tree); if $e(S) = 0$, it contains exactly one cycle; if $e(S) < 0$, it contains at least two different cycles. In the latter case $[S]$ is mapped onto the figure eight, whose fundamental group is a free group of rank 2 which implies that $\pi_1(S)$ has a non-abelian free quotient.

Lemma 4. *Let S be a connected finite graph whose automorphism group acts transitively on $E(S)$. Then*

1) *if $e(S) = 1$, S is a star;*

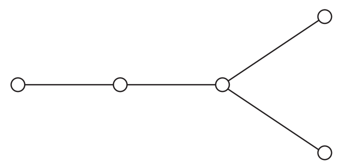
2) *if* $e(S)=0$, S *is a cycle.*

Proof. If S has an extreme vertex, then each edge contains an extreme vertex. The other ends of the edges must coincide, so S is a star.

If $e(S)=1$, S has an extreme vertex and hence is a star. If $e(S)=0, S$ contains exactly one cycle and no extreme vertices, so it is a cycle. ∎

Proposition 2. *Let a group* Γ *act on a tree* T *which is not a star or a line. Suppose that* Γ *does not reverse edges and the action* $\Gamma : E(T)$ *is transitive and residually finite. Then* Γ *has a virtually non-abelian free quotient.*

Proof. Since the tree T is not a star or a line, it contains a subgraph T_0 of the form



There exists a normal subgroup Δ of finite index in Γ such that the edges of T_0 belong to different Δ-orbits and thereby go to different edges of T/Δ (but some vertices of T_0 may glue together). This configuration of edges does not allow T/Δ to be a star or a cycle. Since Γ acts transitively on $E(T/\Delta)$, Lemma 4 shows that under such a choice of Δ we have $e(T/\Delta)<0$. Let $\varphi : \Delta \to \pi_1(T/\Delta)$ be the homomorphism defined by the action $\Delta : T$. It follows from Lemma 2 that $\operatorname{Ker}\varphi = \Delta_0$ is a normal subgroup of Γ. Since $\Delta/\Delta_0 \simeq \pi_1(T/\Delta)$ is a non-abelian free group, Γ/Δ_0 is a virtually non-abelian free group. ∎

3. Some Virtually Abelian Linear Groups

Let V be a finite-dimensional vector space.

Lemma 5. *Any abelian linear group* $\Gamma \subset \mathrm{GL}(V)$ *leaving invariant a non-trivial strictly convex cone* $K \subset V$, *has a weight vector in* $\overline{K}$.

Proof. If all the operators of Γ are scalar, there is nothing to prove. Otherwise, let $\gamma \in \Gamma$ be a non-scalar operator. By the Brouwer theorem on a fixed point γ has an eigenvector in $\overline{K}$, with some eigenvalue $\lambda > 0$. The eigenspace V_1 of γ corresponding to λ, is invariant under Γ and does not coincide with V. Moreover, $K_1 = \overline{K} \cap V_1$ is a non-trivial strictly convex cone in V_1 invariant under Γ. Proceeding by induction in $\dim V$, we may conclude that Γ has a weight vector in K_1 and, hence, in $\overline{K}$. ∎

Let now $\Gamma \subset \mathrm{GL}(V)$ be an irreducible linear group.

Lemma 6. *If* Γ *has a finite orbit in* PV *(or* PV^**), then it is virtually abelian.*

Proof. Let $e \in V$ be a non-zero vector such that the orbit of the corresponding point $[e] \in PV$ is finite. The vectors γe, $\gamma \in \Gamma$, span a Γ-invariant subspace, which must coincide with V. The kernel Δ of the action of Γ on the orbit of $[e]$ is a (normal) subgroup of finite index in Γ. All the vectors γe, $\gamma \in \Gamma$, are weight vectors of Δ. This implies that Δ is abelian. ∎

Note that if K is a Γ-invariant convex cone in V,then $\overline{K} \cap (-\overline{K})$ is a Γ-invariant subspace. If follows that any Γ-invariant convex cone either is strictly convex, or coincides with V.

Let $\mathbb{R}_+^*$ denote the subgroup of $\mathrm{GL}(V)$ consisting of the multiplications by positive scalars.

Lemma 7. *If Γ leaves invariant a convex cone $K \neq V$ and $\Gamma/(\Gamma \cap \mathbb{R}_+^*)$ is finite, then* $\dim V = 1$.

Proof. Multiplying the elements of Γ by suitable positive numbers, we may assume that $\det \gamma = \pm 1$ for any $\gamma \in \Gamma$. Then Γ is finite and, hence, has a fixed non-zero vector in K. The linear span of this vector must coincide with V. ∎

Lemma 8. *If Γ is virtually abelian, then every Γ-invariant convex cone $K \neq V$ is simplicial and the one-dimensional subspaces spanned by its edges are the weight subspaces of any normal abelian subgroup Δ of finite index in Γ.*

Proof. According to Lemma 5, Δ has a weight subspace W such that $\overline{K} \cap W \neq \{0\}$. It follows that the corresponding weight χ of Δ is positive valued, and the space V decomposes into a direct sum of weight subspaces $\gamma W, \gamma \in \Gamma$, of Δ. Moreover, the stabilizer of the subspace W acts on it irreducibly. Applying Lemma 7, we get that $\dim\, W = 1$.

Let χ_γ denote the weight of Δ corresponding to γW. There exists an element $\delta_0 \in \Delta$ such that all the weights χ_γ take different values on it. By symmetry, we may assume that the maximum of these values is taken by $\chi = \chi_e$ itself. Then for any $v \in V$ we have

$$\lim \frac{\delta_0^n v}{\chi(\delta_0)^n} = w,$$

where w is the projection of v to W. It follows that the cone $\overline{K}$ contains its projection to W and, by symmetry , its projection to each subspace γW. This means that $\overline{K}$ is spanned by these projections and, hence, is simplicial. ∎

4. Proof of Main Theorem

The sets $\gamma H, \gamma \in \Gamma$, which we shall call *walls*, decompose K into some (closed in K) convex bodies, which we shall call *chambers*. Consider the graph T, whose vertices (resp. edges) are the chambers (resp. the walls) and the incidence is defined by inclusion. Clearly, T is a tree. The group Γ naturally acts on it. The action $\Gamma : E(T)$ is transitive and residually finite, as it follows from the definition of walls and Proposition 1. Possibly passing to a subgroup of index 2, we may assume that Γ does not reverse edges (see Lemma 3). Condition 4) provides that T is not a star. So, if T is not a line, the group Γ has a virtually non-abelian free quotient according to Proposition 2.

Suppose T is a line. Possibly passing to a subgroup of index 2 and deleting half of the walls, we may assume that Γ acts on T just by shifts. Let

Γ_1 be the kernel of this action. Then $\Gamma/\Gamma_1 \simeq \mathbb{Z}$. The linear form α defining the subspace U is a weight vector of Γ_1 in the dual space V^* of V. Since there are only finitely many weight subspaces of Γ_1 in V^* and Γ can only permute them, there is a subgroup Γ_0 of finite index in Γ containing Γ_1 such that $\Gamma_0\alpha$ lies in one weight subspace. Denote by A the linear span of $\Gamma_0\alpha$. Obviously, it is a Γ_0-invariant subspace. Take any $\gamma_0 \in \Gamma_0\backslash\Gamma_1$. Since

$$(\gamma_0 U \cap U) \cap K = \varnothing,$$

some non-trivial linear combination of α and $\gamma_0\alpha$ belongs to K^* (here we use one of the versions of the Hahn–Banach theorem). Hence,

$$K^* \cap A \neq \varnothing.$$

By the Brouwer theorem γ_0 has a fixed point in $P(K^* \cap A)$. The Γ-orbit of this point is finite, which contradicts our condition $(*)$. ■

5. Proof of Theorem 1

If $K = V$, then Γ is finite as the stabilizer of $0 \in K$. In the further consideration, let us assume that $K \neq V$ (and, hence, K is strictly convex). By the Selberg lemma [6] Γ has a torsion free normal subgroup of finite index, say, Γ_1. Apply the Main Theorem to Γ_1, taking for U the mirror of a linear reflection r contained in Γ. We are to prove conditions 1)-4) and $(*)$.

To prove 1), take any point $x \in K$. Then $x + r(x) \in H$.

To prove 2), note that for $\gamma \in \Gamma$ the subspace γU is the mirror of the reflection $\gamma r \gamma^{-1}$. If $\gamma H \cap H \neq \varnothing$, then $\delta = (\gamma r \gamma^{-1})r$ has a fixed point in K and, hence, is of finite order. Let $\gamma \in \Gamma_1$. Then $\delta = \gamma(r\gamma r)^{-1} \in \Gamma_1$ and hence, $\delta = id$; this means that $\gamma U = U$ or, equivalently, $\gamma H = H$.

Let us prove 3). By the definition of a discrete action, for any compact set $C \subset K$ we have $\#\{\gamma \in \Gamma : \gamma C \cap C \neq \varnothing\} < \infty$. But if $\gamma H \cap C \neq \varnothing$, the reflection $r_\gamma = \gamma r \gamma^{-1}$ has a fixed point in C so $r_\gamma C \cap C \neq \varnothing$. Therefore, C meets only finitely many sets of the form $\gamma H, \gamma \in \Gamma$.

If one of the connected components of $K\backslash H$ contains some γH, $\gamma \in \Gamma_1$, then the other one contains $r\gamma H = (r\gamma r)H$, where $r\gamma r \in \Gamma_1$, so condition 4) is satisfied. If no one of the connected components of $K\backslash H$ contains any γH, $\gamma \in \Gamma_1$, then $\gamma H = H$ for any $\gamma \in \Gamma_1$. This implies that the Γ-orbit of the point of PV^* corresponding to U, is finite, so according to Lemmas 6 and 8 the second alternative of the theorem takes place. The same is true, if Γ has a finite orbit in PK^*. Thus, either the second alternative of the theorem takes place, or all the conditions of the Main Theorem are satisfied for Γ_1. In the latter case Γ is large. ■

6. Proof of Corollary 1

The space L^n can be realized as the projectivization of the cone K of the future in the Minkowski space $E^{n,1}$ in such a way that the motions of L^n be induced by pseudo-orthogonal operators. Under this realization the planes of L^n correspond to subspaces of $E^{n,1}$ intersecting K, while the points at infinity correspond to isotropic one-dimensional subspaces. The conditions of Corollary 1 mean that Γ is an irreducible subgroup of the pseudo-orthogonal group $O_{n,1}$ acting on K discretely and containing a reflection. By Theorem 1 it is large.

7. Proof of Corollary 2

It is known [7] that any non-affine indecomposable finitely generated Coxeter group Γ can be realized as a linear group in a finite-dimensional real vector space V in such a way that

1) Γ is irreducible;
2) Γ leaves invariant a non-degenerate symmetric bilinear form f in V;
3) the generators of Γ are represented by orthogonal reflections (with respect to f);
4) Γ leaves invariant an open convex cone $K \subset V$ and acts on it discretely.

Applying Theorem 1, we see that if Γ is infinite, one of alternatives 2) and 3) takes place, so we only need to exclude 2).

Suppose that the cone K is simplicial. The group Γ can only permute the edges of K. Let Δ be the kernel of the action of Γ on the set of edges of K. Then Δ is abelian and the one-dimensional subspaces $V_1, \dots, V_m$ spanned by the edges of K, are the weight subspaces of Δ. They are isotropic with respect to f and decompose in pairs in such a way that the subspaces of one pair correspond to opposite weights and are not orthogonal, while any subspaces of different pairs are orthogonal. Any reflection of Γ permutes the subspaces of one pair, leaving all the others invariant. It follows that the group Γ leaves any pair of the subspaces $V_1, \dots, V_m$ invariant, which contradicts its irreducibility, unless $m = 2$. But the case $m = 2$ occurs only for dihedral groups, which are finite or affine. ∎

8. Proof of Theorem 2

Consider the space $V = \mathbb{R}^{n+1}$ with the scalar product defined by the quadratic form f. Denote by K one of two connected components of the cone $f < 0$. The subgroup

$$\mathrm{O}'(f, \mathbb{R}) = \{g \in \mathrm{O}(f, \mathbb{R}) : gK = K\}$$

has index 2 in $\mathrm{O}(f, \mathbb{R})$. Set

$$\mathrm{O}'(f, \mathbb{Z}) = \mathrm{O}(f, \mathbb{Z}) \cap \mathrm{O}'(f, \mathbb{R}).$$

Take any vector $e \in \mathbb{Z}^{n+1}$ with $f(e) = m > 0$ and set

$$\begin{aligned} Q &= \{v \in V : \quad f(v) = m\}, \\ U &= \{v \in V : \quad (v, e) = 0\}. \end{aligned}$$

Obviously,

$$H = U \cap K \neq \varnothing.$$

Lemma 9. *The sets $\gamma H, \gamma \in \mathrm{O}'(f,\mathbb{Z})$, constitute a discrete family in K, i.e. any compact subset $C \subset K$ meets only finitely many of them.*

Proof. We have

$$\gamma H = \{v \in K : (v, \gamma e) = 0\}.$$

Clearly, $\gamma e \in Q \cap \mathbb{Z}^{n+1}$. The orthogonal subspace of any vector $v \in K$ is Euclidean, so its intersection with Q is compact. It follows that the set

$$Q_C = \{v \in Q : (v, u) = 0 \text{ for some } v \in C\},$$

is also compact and, hence, contains only finitely many integral vectors. Therefore, C meets only finitely many sets γH, $\gamma \in \mathrm{O}'(f, \mathbb{Z})$. ∎

Lemma 10. *There is a subgroup Θ of finite index in $\mathrm{O}'(f,\mathbb{Z})$ such that for any $\gamma \in \Theta$ we have either $\gamma H = H$, or $\gamma H \cap H = \varnothing$.*

Proof. For $\gamma \in \mathrm{O}'(f,\mathbb{R})$ the sets H and γH intersect but do not coincide if and only if the vectors e and γe span a two-dimensional Euclidean subspace in V, i.e. if

$$|(e, \gamma e)| < m. \tag{1}$$

Let Θ be the congruence subgroup of $\mathrm{O}'(f,\mathbb{Z})$ modulo $2m$. For $\gamma \in \Theta$ we have $\gamma e \equiv e \pmod{2m}$, whence

$$(e, \gamma e) \equiv m \pmod{2m}.$$

This makes (1) impossible. ∎

Now let Γ be a finitely generated subgroup of $\mathrm{O}(f,\mathbb{Z})$. Passing to a subgroup of finite index, we may assume that $\Gamma \subset \Theta$. Then conditions 1)–3) of the Main Theorem hold. To ensure condition 4), we choose the vector e depending on Γ as follows.

The group $\mathrm{O}'(f,\mathbb{R})$ can be considered as the group of motions of Lobachevsky space L^n modelled on the projectivization of the cone K. In this interpretation, an element $\gamma \in \mathrm{O}'(f,\mathbb{R})$ is called *elliptic*, if it has a fixed point in L^n, *hyperbolic*, if it has no fixed point in L^n but has an invariant line, and *parabolic*, if it has no fixed points or invariant lines in L^n, but has a fixed point at infinity. Any element of $\mathrm{O}'(f,\mathbb{R})$ is one of these three types (see, e.g., [1]).

Passing to a subgroup of finite index, we may assume that Γ is torsion-free and, hence, contains no elliptic elements but the identity.

Let $\gamma_0 \in \Gamma$ be a hyperbolic element and l the (unique) invariant line of it in L^n. Since the projectivization of $\mathbb{Z}^{n+1}$ is dense in PV, we can choose e in such a way that H intersects the line l but does not contain it. Then $\gamma_0^d H$ and $\gamma_0^{-d} H$ are in different connected components of $K \backslash H$ for any $d \in \mathbb{N}$. Some γ_0^d survives after possible passing to a subgroup of finite index in Γ, so condition 4) will hold for the obtained group.

Now let $\gamma_0 \in \Gamma$ be a parabolic element and p the (unique) fixed point at infinity of it. The point p is represented by an isotropic vector with rational co-ordinates. Any horosphere of L^n centered at p is invariant under γ_0, and there exists an invariant Euclidean line l on it. We can choose e in such a way that (the closure of) H passes through p and intersects the line l but does not contain it. Then again $\gamma_0^d H$ and $\gamma_0^{-d} H$ are in different connected components of $K \backslash H$, and we can apply the above argument.

Thus, we can conclude that if the group Γ satisfies the condition $(*)$, it is large. If it does not satisfy $(*)$, then a subgroup of finite index in Γ has a fixed point in L^n or in ∂L^n. In both cases Γ is virtually abelian. ∎

Remark 3. Theorem 2 implies the following generalization of its own. Let $k \subset \mathbb{R}$ be a totally real algebraic number field and f a quadratic form of signature $(n, 1)$ over k that becomes positive definite under any non-identity embedding $\sigma : k \to \mathbb{R}$. Let $\mathcal{O}$ be the ring of integers of k. Then any finitely generated subgroup of $\mathrm{O}(f, \mathcal{O})$ is either virtually abelian, or large. Indeed, if $[k : \mathbb{Q}] = d$, then there are exactly d embeddings $\sigma : k \to \mathbb{R}$, and the restriction of scalars permits us to embed $\mathrm{O}(f, \mathcal{O})$ into $\mathrm{O}(F, \mathbb{Z})$, where F is a rational quadratic form of signature $(d(n + 1) - 1, 1)$. ∎

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Received December 8, 1998
and in final form July 17, 1998