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Invariant differential operators and holomorphic function spaces

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Abstract. In this paper, we are interested in studying the algebra $D(\Omega)$ of invariant differential operators on a symmetric cone Ω. We will give some sets of generators of $D(\Omega)$ and calculate the eigenvalues of spherical functions under those generators. The explicit construction of our invariant differential operators in $D(\Omega)$ leads to introducing some differential operators on an irreducible bounded symmetric domain D in a complex vector space Z. Some interesting results are obtained about these differential operators and their applications to the study of spaces of holomorphic functions on D are given.

Introduction

It is known that the algebra of the invariant differential operators on a symmetric space of rank r is generated by a set of r algebraically independent elements. An important problem is to give such a set of generators explicitly. In the case that a symmetric space is a symmetric cone, a set of generators has been given by Nomura [14] in terms of some invariant polynomials. In part I of this paper, we shall further study the algebra $D(\Omega)$ of the invariant differential operators on a symmetric cone Ω. For each complex number λ, an invariant differential operator D_λ is introduced. It is shown that for any r distinct numbers $\lambda_1, \ldots, \lambda_r$, $D_{\lambda_1}, \ldots, D_{\lambda_r}$ is a set of algebraically independent generators of $D(\Omega)$. We also introduce r "canonical" invariant differential operators $K_1, \ldots, K_r$ which are constructed from some canonical invariant polynomials, then express D_λ in terms of $K_1, \ldots, K_r$ and vice versa. For any spherical function, its eigenvalues under D_λ and K_j can be computed explicitly.

In part II, we consider an irreducible bounded symmetric domain D in a complex vector space Z in the standard Harish- Chandra realization. In D there is a subdomain D_T which is the unit ball of the complexification of a real simple Euclidean Jordan algebra. Let G be the the identity component of $Aut(D)$ and K the isotropy subgroup of G at 0. For each complex number λ, using results

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obtained for tube type domains, we introduce a differential operator $\mathcal{D}_\lambda$. For any distinct numbers $\lambda_1, \ldots, \lambda_r$, we also prove that $\mathcal{D}_{\lambda_1}, \ldots, \mathcal{D}_{\lambda_r}$ is a set of r algebraically independent generators of the algebra of the differential operators on Z that commute with the action of K. An important feature of $\mathcal{D}_\lambda$ is that as a consequence of the commutativity with K, polynomials in the irreducible subspaces of Schmid's decomposition are eigenfunctions of $\mathcal{D}_\lambda$, and, moreover, the eigenvalues can be calculated explicitly. The applications of this result will be given in part III.

In [4], for every $\lambda \in \mathbf{C}$, the space of holomorphic polynomials on the ambient space Z is equipped with the structure of a Harish-Chandra module, denoted by $\mathcal{P}^{(\lambda)}$, and a composition series of $\mathcal{P}^{(\lambda)}$,

$$M_0 \subset M_1 \subset \cdots \subset M_{q(\lambda)} = \mathcal{P}^{(\lambda)}$$

is determined. Each quotient $M_j/M_{j-1}, j = 0, 1, \ldots, q(\lambda), (M_{-1} = 0)$ has a natural invariant Hermitian form. Of particular interest is the case when the quotient is unitarizable, that is, the corresponding Hermitian form is an inner product. In this case, one has a corresponding Hilbert space of analytic functions on which G acts unitarily. It is known that M_j/M_{j-1} is unitarizable, if and only if $j = 0$ or $j = q(\lambda)$ with an appropriate λ. In part III, we shall express the invariant inner products in terms of integrals on D when the highest quotient is unitarizable. We shall also characterize M_0^λ, when it is unitarizable, by a corresponding canonical differential operator $\mathcal{K}_j$. The space of harmonic polynomials in the sense of Upmeier [17], which is equal to M_0^λ for a particular value of λ, is described in terms of a single differential operator in [17]. Our result generalizes that of Upmeier. We shall describe those Hilbert spaces of holomorphic functions corresponing to the cases that the quotients M_j/M_{j-1} are unitarizable and obtain a generalization of the classical Dirichlet space. Finally, we characterize the dual and predual of the Bergman space $L^1(D) \cap H(D)$ which generalize the results in [19] to the case of all bounded symmetric domains.

After the first version of this paper was finished, the author noticed a paper of R. Howe and T. Umeda [8], which is relevant to §1 and §2 in this paper. In particular, our Theorem 1.11 is motivated by a remark in [8].

Preprints of this paper were distributed in 1992. Some of its results were then incorporated in the book [5]. Following the referee's recommendation, the present paper is now shorter than the original preprint. We have omitted the proofs that appear also in [5].

1. Invariant Differential operators on Symmetric Cones

§1.1. Background and Notation of Symmetric Cones and Jordan Algebra

Let V be a real simple Euclidean Jordan algebra, Ω the symmetric cone in V, i.e., the interior of the set of all squares in V. It is known that every irreducible symmetric cone can be obtained in this way. We fix a complete system of orthogonal primitive idempotents $\{c_1, \ldots, c_r\}$ where r is the rank of V, then the identity element e is equal to $c_1 + \cdots + c_r$. We denote by $G(\Omega)$ the identity component of the subgroup of $GL(V)$ which preserves Ω, L the isotropy subgroup of $G(\Omega)$ at e. Then every element x in V can be written as

$$x = l.\sum_{i=1}^{r} t_i c_i, \qquad t_i \in \mathbf{R}, \quad l \in L, \tag{1}$$

and $x \in \Omega$ if and only if $t_i > 0$, $i = 1, \ldots, r$.

There is a determinant polynomial $\Delta(x)$ and a trace polynomial $tr(x)$ on V such that if x is written as in (1), then

$$\Delta(x) = \prod_{i=1}^{r} t_i,$$

and

$$tr(x) = \sum_{i=1}^{r} t_i.$$

For $x \in V$, one defines the multiplication operator $L(x) : V \to V$ by

$$L(x)y = xy, \quad \forall y \in V$$

and the quadratic representation $P(x) : V \to V$ by

$$P(x) = 2L(x)^2 - L(x^2).$$

For an idempotent c and $k \in \mathbf{R}$, let

$$V(c,k) = \{x \in V |\ L(c)x = k\,x\},$$

V has the Peirce decomposition

$$V = \sum_{1 \leq i \leq j \leq r} V_{ij},$$

where $V_{jj} = V(c_j, 1)$, and, for $i \neq j$, $V_{ij} = V(c_i, 1/2) \cap V(c_j, 1/2)$. For all $i \neq j$, all V_{ij} have the same dimension, which will be denoted by a.

Now the subspaces $V^{(k)} = V(c_1 + \cdots + c_k, 1)(1 \leq k \leq r)$ are subalgebras of V. Let P_k be the orthogonal projection onto $V^{(k)}$. The principal minor $\Delta_k(x)$ is the polynomial defined on V by

$$\Delta_k(x) = \Delta^{(k)}(P_k x),$$

where $\Delta^{(k)}$ is the determinant polynomial with respect to the algebra $V^{(k)}$.

For an $r-$ tuple of integers $\mathbf{m} = (m_1, \ldots, m_r)$ with $m_1 \geq \cdots m_r \geq 0$, abbreviated as $\mathbf{m} \geq 0$, one defines a polynomial $\Delta_{\mathbf{m}}(x)$ on V by

$$\Delta_{\mathbf{m}}(x) = \Delta_1^{m_1 - m_2}(x)\Delta_2^{m_2 - m_3}(x) \cdots \Delta_{r-1}^{m_{r-1} - m_r}(x)\Delta_r^{m_r}(x). \tag{2}$$

One observes that if $x \in \Omega$, then for any $r-$tuple of complex numbers $\mathbf{s} = (s_1, \ldots, s_r)$, replacing $\mathbf{m}$ by $\mathbf{s}$, (2) defines a function $\Delta_{\mathbf{s}}(x)$ on Ω.

For a linear transformation X on V, ${}^tX$ will be its transpose with respect to the inner product $(\ ,\ )_V$, where $(\ ,\ )_V$ is induced from the trace form on V, i.e., $(x, y)_V = tr(xy)$.

Let $\mathcal{G}_\Omega$ be the Lie algebra of $G(\Omega)$, and $\mathcal{L}_\Omega$ the Lie algebra of L. One has the following Cartan decomposition corresponding to the involution $\theta : \mathcal{G}_\Omega \to \mathcal{G}_\Omega$, $X \to -{}^tX$,

$$\mathcal{G}_\Omega = \mathcal{L}_\Omega + \mathcal{P}_\Omega. \tag{3}$$

Then

$$\mathcal{A}_\Omega = \{\sum_{i=1}^{r} t_i L(c_i) |\ t_i \in \mathbf{R}, i = 1, \ldots, r\}$$

is a maximal abelian subspace of $\mathcal{P}_\Omega$. One has

$$(\exp \mathcal{A}_\Omega).e = \{\sum_{i=1}^{r} u_i c_i |\ u_i > 0, i = 1, \ldots, r\}. \tag{4}$$

When there is no confusion caused, we will identify $(\exp \mathcal{A}_\Omega).e$ with the subgroup $A_\Omega = \exp \mathcal{A}_\Omega$ of $G(\Omega)$ or the subalgebra $\mathcal{A}_\Omega$ of $\mathcal{G}_\Omega$. The following two coordinate systems on $(\exp \mathcal{A}_\Omega).e$ will be used in our later calculations.

$$(I) \quad \varphi : (\exp \mathcal{A}_\Omega).e \to \mathbf{R}^r, \quad \varphi(\sum_{i=1}^{r} u_i c_i) = (u_1, \ldots, u_r),$$

$$(II) \quad \psi : (\exp \mathcal{A}_\Omega).e \to \mathbf{R}^r, \quad \psi(\sum_{i=1}^{r} u_i c_i) = (y_1, \ldots, y_r),$$

where $y_i = \log u_i, i = 1, \ldots, r$.

According to our convention, (I) and (II) will also be used as coordinate systems of A_Ω and $\mathcal{A}_\Omega$.

We define linear functionals $\alpha_{ij}, 1 \le i, j \le r, i \ne j$ on $\mathcal{A}_\Omega$ by

$$\alpha_{ij}(\sum_{i=1}^{r} t_i L(c_i)) = \frac{1}{2}(t_j - t_i),$$

then $\alpha_{ij}, 1 \le i, j \le r, i \ne j$ consist of all the restricted roots of the pair $(\mathcal{G}_\Omega, \mathcal{L}_\Omega)$.

Let $N = \exp \mathcal{N}$ where $\mathcal{N}$ is the direct sum of all the root spaces corresponding to α_{ij} with $1 \le i < j \le r$. Then $G(\Omega)$ has the following Iwasawa decomposition

$$G(\Omega) = L A_\Omega N. \tag{5}$$

Let $\rho = \frac{1}{2} a \sum_{1 \le i < j \le r} \alpha_{ij}$, and e^ρ be the function defined on A_Ω by

$$e^\rho(u) = e^{\rho(\log u)}$$

for $u \in A_\Omega$, where log is the inverse of $\exp : \mathcal{A}_\Omega \to A_\Omega$.

§*1.2. Differential Operators Associated with Polynomials*

In this section, most of our notation is from [5],[7] and [14].

For a real vector space E of dimension n with the inner product $(\,|\,)$, we will denote by $P(E)$ the space of all complex-valued polynomials on E. For every polynomial $p \in P(E)$ we define the unique linear differential operator $p(\frac{\partial}{\partial x})$ by

$$p(\frac{\partial}{\partial x}) e^{(x|y)} = p(y) e^{(x|y)}, \quad \forall y \in E. \tag{6}$$

The uniqueness of $p(\frac{\partial}{\partial x})$ follows immediately from (6); its existence can be seen by taking an orthonormal basis of E, expressing p in terms of the coordinates as $p(x_1, \ldots, x_n)$ and then formally replacing x_j by $\frac{\partial}{\partial x_j}, (1 \leq j \leq n)$.

For a polynomial $p(x, y)$ on $E \times E$, similarly, we can define the unique differential operator $p(x, \frac{\partial}{\partial x})$ by the following equation

$$p(x, \frac{\partial}{\partial x}) e^{(x|y)} = p(x, y) e^{(x|y)}. \tag{7}$$

We denote by W the Weyl group corresponding to the root system $\{\alpha_{ij}\}$, then W is isomorphic to the full permutation group S_r. Let $D(\Omega)$ be the algebra of the invariant differential operators on Ω, $D_W(A)$ the algebra of $W-$invariant differential operators on A_Ω with constant coefficients, and $I_W(\mathcal{A}_\Omega)$ the algebra of $W-$invariant polynomials on $\mathcal{A}_\Omega$. See [7, Ch.5]. In our case, $I_W(\mathcal{A}_\Omega)$ is the algebra of symmetric polynomials in r variables.

We write α for $(\alpha_1, \ldots, \alpha_r)$, u^α for $u_1^{\alpha_1} \cdots u_r^{\alpha_r}$ and $(\frac{\partial}{\partial u})^\alpha$ for $(\frac{\partial}{\partial u_1})^{\alpha_1} \cdots (\frac{\partial}{\partial u_r})^{\alpha_r}$.

In coordinate system I,

$$D_W(A_\Omega) = \{p(u\frac{\partial}{\partial u}) | p \in I_W(\mathcal{A}_\Omega)\}, \tag{8}$$

where $p(u\frac{\partial}{\partial u}) = \sum_\alpha b_\alpha u^\alpha (\frac{\partial}{\partial u})^\alpha$ if $p(y) = \sum_\alpha b_\alpha y^\alpha$.

In coordinate system II,

$$D_W(A_\Omega) = \{p(\frac{\partial}{\partial y}) | p \in I_W(\mathcal{A}_\Omega)\}. \tag{9}$$

In the following, we study the structure of $D(\Omega)$ as a vector space. The Fischer inner product on $P(V)$ is defined by

$$(p, q)_F = (p(\frac{\partial}{\partial x})\bar{q})(0),$$

for $p, q \in P(V)$. There is a natural representation π of the group $G(\Omega)$ defined on $P(V)$ by

$$(\pi(g)p)(x) = p(g^{-1}.x) \tag{10}$$

for $g \in G(\Omega), p \in P(V)$. The following is known, e.g. see [5].

Theorem I.1. *$P(V)$ is the orthogonal direct sum of the spaces $P_{\mathbf{m}}(V)$ ($\mathbf{m} \geq 0$) which are mutually inequivalent irreducible representation spaces of $G(\Omega)$. Moreover,*

$$P_{\mathbf{m}}(V) = span\{\pi(g)\Delta_{\mathbf{m}} : \quad g \in G(\Omega)\}.$$

For each $\mathbf{m} \geq 0$, there is a unique L-invariant polynomial $\varphi_{\mathbf{m}}$ in $P_{\mathbf{m}}(V)$ which is defined by

$$\varphi_{\mathbf{m}}(x) = \int_L \Delta_{\mathbf{m}}(l.x) dl.$$

Under the Fischer inner product $(\ ,\ )_F$ every $P_{\mathbf{m}}(V)$ is a Hilbert space and has a reproducing kernel $K^{\mathbf{m}}(x,y) = K_y^{\mathbf{m}}(x)$, that is,

$$p(y) = (p, K_y^{\mathbf{m}})_F \qquad \forall p \in \mathcal{P}_{\mathbf{m}}(V).$$

If $\{\psi_{\mathbf{m}}^{(i)}(x), i = 1, \ldots, d_{\mathbf{m}}\}$ is an orthonormal basis of $\mathcal{P}_{\mathbf{m}}(V)$, where $d_{\mathbf{m}}$ is the dimension of $\mathcal{P}_{\mathbf{m}}(V)$, then

$$K^{\mathbf{m}}(x,y) = \sum_{i=1}^{d_{\mathbf{m}}} \psi_{\mathbf{m}}^{(i)}(x)\overline{\psi_{\mathbf{m}}^{(i)}(y)}.$$

It is known, e.g.see [5], that

$$K^{\mathbf{m}}(g.x, {}^tg^{-1}.y) = K^{\mathbf{m}}(x,y) \tag{11}$$

for all $g \in G(\Omega)$, and

$$K^{\mathbf{m}}(x,y) = K^{\mathbf{m}}(y,x). \tag{12}$$

Let $P^{G(\Omega)}$ be the subspace of $P(V \times V)$ spanned by $\{K^{\mathbf{m}}(x,y), \mathbf{m} \geq 0\}$ and

$$P(V)^L = \{p \in P(V) | \pi(l)p = p, \forall l \in L\}.$$

Then the spherical polynomials $\{\varphi_{\mathbf{m}}\}, \mathbf{m} \geq 0$ form a basis of $P(V)^L$.

The following can be proved easily by using Proposition 14.1.1 in [5] or directly by using some ideas from [14].

Proposition 1.2. $\{K^{\mathbf{m}}(x, \frac{\partial}{\partial x}), \mathbf{m} \geq 0\}$ *is a basis of the vector space* $D(\Omega)$.

When we say that a function $F(x,y)$ defined on $\Omega \times V$ is polynomial in y, we mean that $F(x,y)$ can be expanded as $\sum_\alpha a_\alpha(x) y^\alpha$ with only finitely many nonzero terms.

Remark 1. Every linear differential operator D on Ω defines a function $F_D(x,y)$ on $\Omega \times V$, which is polynomial in y, by the following equation

$$D_x e^{(x|y)} = F_D(x,y) e^{(x|y)}. \tag{13}$$

Conversely, a function $F(x,y)$ on $\Omega \times V$ which is polynomial in y, determines a unique differential operator $F(x, \frac{\partial}{\partial x})$ by (13).

Remark 2. It follows from the proposition that every $D \in D(\Omega)$ can be extended to a differential operator on V with polynomial coefficients.

Among those $K^{\mathbf{m}}(x, \frac{\partial}{\partial x}), \mathbf{m} \geq 0$, of particular interest to our study are $K^{1_j}(x, \frac{\partial}{\partial x}), j = 1, \ldots, r$ where 1_j is the r-tuple of integers with 1 as its first jth components and 0 the remaining components.

§1.3. Generators of $D(\Omega)$

The purpose of this section is to study $D(\Omega)$ as an algebra and give some generators of $D(\Omega)$. This section contains the main results of part I.

Recall that $G(\Omega)$ has the Iwasawa decomposition

$$G(\Omega) = LA_\Omega N.$$

Let $R_N(D)$ denote the $N-$radial part of $D \in D(\Omega)$ defined as in [7,p.259]. One defines a linear mapping Γ from $D(\Omega)$ into the algebra of differential operators on A_Ω by

$$\Gamma(D) = e^{-\rho} R_N(D) \circ e^{\rho}, \quad \forall D \in D(\Omega).$$

One has the following special case of a result of Harish-Chandra.

Theorem 1.3. Γ is an isomorphism of $D(\Omega)$ onto $D_W(A)$.

Proof. See, e.g. [7, Cor. 5.19]. ∎

For $\lambda \in \mathbf{R}$, we define

$$D_\lambda = \Delta(x)^{1-\lambda} \Delta(\frac{\partial}{\partial x}) \circ \Delta(x)^{\lambda},$$

then it is easy to verify that $D_\lambda \in D(\Omega)$.

Now we have

Theorem 1.4. *The image of D_λ under the mapping Γ is given by*

$$\Gamma(D_\lambda) = p_\lambda(\frac{\partial}{\partial y})$$

or equivalently,

$$\Gamma(D_\lambda) = p_\lambda(u\frac{\partial}{\partial u})$$

where $p_\lambda(x) = \prod_{i=1}^r (x_i + \lambda + \frac{a}{4}(r-1)) \in I_W(\mathcal{A}_\Omega)$.

Proof. See p.296 in [5]. ∎

Let $S_j(x)$ be the jth elementary symmetric polynomial of $x_1, \ldots, x_r$, then

$$p_\lambda(x) = \sum_{j=0}^{r} (\lambda + \frac{a}{4}(r-1))^{r-j} S_j(x). \tag{14}$$

If $\lambda_1, \ldots, \lambda_r$ are all different, then it follows immediately from (14) that $p_{\lambda_1}(x), \ldots, p_{\lambda_r}(x)$ are algebraically independent generators of $I_W(\mathcal{A}_\Omega)$. As a corollary of Theorem 1.3 and 1.4, now we have

Theorem 1.5. *If $\lambda_1, \ldots, \lambda_r$ are distinct, then $D_{\lambda_1}, \ldots, D_{\lambda_r}$ are algebraically independent generators of $D(\Omega)$.*

Now, we wish to express D_λ as a linear combination of $K_1, \ldots, K_r$. Following Remark 1 after Proposition 1.2, we proceed to find the polynomial F_{D_λ} corresponding to D_λ as in next lemma.

Lemma 1.6. *For $\lambda \in \mathbf{R}$, $x, y \in \Omega$,*

$$D_{-\lambda} e^{(x|y)} = \sum_{j=1}^{r} (-1)^j \binom{r}{j} \prod_{l=1}^{j} (\lambda - \frac{l-1}{2} a) \frac{1}{c_{1_{r-j}}} K^{1_{r-j}}(x,y) e^{(x|y)} \tag{15}$$

Proof. See p.295 in [5]. ∎

Now we have the following expansion

Theorem 1.7. *For any real number λ,*

$$D_\lambda = \sum_{j=0}^{r} \binom{r}{j} \prod_{l=1}^{r-j} (\lambda + \frac{l-1}{2} a) K_j \tag{16}$$

where $K_j = \frac{1}{c_{1_j}} K^{1_j}(x, \frac{\partial}{\partial x})$.

Proof. The theorem follows from Lemma 1.6 and Remark 1 in §1.2. ∎

Remark 1. The above expansion has also been obtained independently by J.Arazy.

We let $\lambda = -\frac{i-1}{2}a, i = 1, \ldots, r$ in (16), we have r equations with a nonsingular coefficient matrix. Solving this system of equations, we obtain

Theorem 1.8. *$K_i, i = 1, \ldots, r$, are algebraically independent generators of $D(\Omega)$. Moreover,*

$$K_j = \frac{j!}{r!} (\frac{2}{a})^{r-j} \sum_{l=1}^{r-j+1} (-1)^l \binom{r-j}{l} D_{-\frac{l-1}{2}a}.$$

Corollary 1. *For $j = 1, \ldots, r$,*

$$\Gamma(K_j) = \frac{j!}{r!} (\frac{2}{a})^{r-j} \sum_{l=1}^{r-j+1} (-1)^l \binom{r-j}{l} p_{-\frac{l-1}{2}a}(\frac{\partial}{\partial y}).$$

Remark 2. Letting λ take distinct values $\lambda_1, \ldots, \lambda_r$ in (16), gives a system of r equations, as a consequence of Theorem 1.5 and 1.8, one obtains that the coefficient matrix of the system of r equations is nonsingular.

Finally, motivated by Theorem 1.7, for any complex number λ, we define

$$D_\lambda = \sum_{j=0}^{r} \binom{r}{j} \prod_{l=1}^{r-j} (\lambda + \frac{l-1}{2} a) K_j.$$

By Proposition 7.1.6 in [5], analytic continuation, Theorem 1.1 and Schur's lemma, we have

Theorem 1.9. *For $\mathbf{m} \geq 0$ and any complex number λ,*

$$D_\lambda p = \prod_{i=1}^{r} (m_i + \lambda + \frac{r-i}{2} a) p, \quad \forall p \in P_{\mathbf{m}}(V). \tag{17}$$

Next, we have

Theorem 1.10. *For* $\mathbf{m} \geq 0$ *and* $j = 1, \ldots, r$,

$$K_j p = \binom{r}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq r} \prod_{l=1}^{j} (m_{i_l} + \frac{j-l}{2} a) p, \quad \forall p \in P_{\mathbf{m}}(V). \tag{18}$$

Proof. By Theorems 1.8 and 1.9, we have

$$K_j \Delta_{\mathbf{m}} = \frac{j!}{r!} (\frac{2}{a})^{r-j} \sum_{l=1}^{r-j+1} (-1)^l \binom{r-j}{l} \prod_{i=1}^{r} (m_i + \frac{r-l-i}{2} a) \Delta_{\mathbf{m}}. \tag{19}$$

Now, it is sufficient to show that

$$(r-j)! \sum_{1 \leq i_1 < \cdots < i_j \leq r} \prod_{l=1}^{j} (m_{i_l} + \frac{j-l}{2} a) = (\frac{2}{a})^{r-j} \sum_{l=1}^{r-j+1} (-1)^l \binom{r-j}{l} \prod_{i=1}^{r} (m_i + \frac{r-l-i}{2} a). \tag{20}$$

When $a = 2$, Ω is the cone of positive definite Hermitian matrices. In this case, $G(\Omega) = GL(n, \mathbf{C})$ and $L = U(n)$, then by (11.1.15) in [8]

$$K_j \Delta_{\mathbf{m}} = \binom{r}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq r} \prod_{l=1}^{j} (m_{i_l} + j - l) \Delta_{\mathbf{m}}.$$

This and (19) implies that we have obtained a special case of (20)

$$(r-j)! \sum_{1 \leq i_1 < \cdots < i_j \leq r} \prod_{l=1}^{j} (m_{i_l} + j - l) = \sum_{l=1}^{r-j+1} (-1)^l \binom{r-j}{l} \prod_{i=1}^{r} (m_i + r - l - i). \tag{21}$$

However, both sides of (21) are polynomials of r variables $m_1, \ldots, m_r$, and they are equal for all $\mathbf{m} \geq 0$. It follows easily that (21) holds for all $\mathbf{m} \in \mathbf{C}^r$. In particular, we have for all $\mathbf{m} \geq 0, a \neq 0$

$$\sum_{1 \leq i_1 < i_2 \cdots < i_j \leq r} \prod_{l=1}^{j} (\frac{2}{a} m_{i_l} + j - l) = \sum_{l=1}^{r-j+1} (-1)^l \binom{r-j}{l} \prod_{i=1}^{r} (\frac{2}{a} m_i + r - l - i). \tag{22}$$

This yields

$$(\frac{2}{a})^j \sum_{1 \leq i_1 < \cdots < i_j \leq r} \prod_{l=1}^{j} (m_{i_l} + \frac{j-l}{2} a) = (\frac{2}{a})^r \sum_{l=1}^{r-j+1} (-1)^l \binom{r-j}{l} \prod_{i=1}^{r} (m_i + \frac{r-l-i}{2} a) \tag{23}$$

proving (20). ■

It is known, for instance see [5], that every spherical function on Ω can be written as

$$\varphi_{\mathbf{s}}(x) = \int_L \Delta_{\mathbf{s}}(l.x) dl$$

for some $\mathbf{s} \in \mathbf{C}^r$.

One can readily see that replacing p by $\varphi_{\mathbf{s}}$ and $\mathbf{m}$ by $\mathbf{s}$, (17) and (18) still hold. Therefore, we have

Theorem 1.11. *For every* $\mathbf{s} \in \mathbf{C}^r$, *then*

(i) *for any complex number* λ

$$D_\lambda \varphi_{\mathbf{s}} = \prod_{i=1}^{r} (s_i + \lambda + \frac{r-i}{2} a) \varphi_{\mathbf{s}}; \tag{24}$$

(ii) *if* $j = 1, \ldots, r$,

$$K_j \varphi_{\mathbf{s}} = \binom{r}{j}^{-1} \sum_{1 \le i_1 < \cdots < i_j \le r} \prod_{l=1}^{j} (s_{i_l} + \frac{j-l}{2} a) \varphi_{\mathbf{s}}. \tag{25}$$

2. Differential operators commuting with the action of K

In this part, we shall introduce and study some differential operators on Z which commute with the action of the isotropy group K. Their applications will be given in the last part of this paper.

§2.1. Some Background on Bounded Symmetric Domains

Our notation follows that of [4]. Let $\mathcal{G}$ be the Lie algebra of G, and $\mathcal{K}$ the Lie algebra of K, then $\mathcal{G}$ is a simple real Lie algebra with Cartan decomposition

$$\mathcal{G} = \mathcal{K} + \mathcal{P}.$$

$\mathcal{G}^{\mathbf{C}}$ will be its complexification and $G^{\mathbf{C}}$ will be the adjoint group of $\mathcal{G}^{\mathbf{C}}$. A basis of root vectors $\{e_\alpha\}$ will be so chosen that $\tau e_\alpha = -e_{-\alpha}, [e_\alpha, e_{-\alpha}] = h_\alpha$, $[h_\alpha, e_{\pm\alpha}] = 2e_{\pm\alpha}$, where τ is the conjugation with respect to the real form $\mathcal{K} + i\mathcal{P}$. Φ^+ will denote the set of positive non-compact roots, and setting

$$\mathcal{P}^{\pm} = \sum_{\alpha \in \Phi^+} \mathbf{C} e_{\pm\alpha},$$

one has

$$\mathcal{G}^{\mathbf{C}} = \mathcal{P}^- + \mathcal{K}^{\mathbf{C}} + \mathcal{P}^+.$$

Define a Hermitian inner product $(\,|\,)$ on $\mathcal{P}^+$ by $(z|w) = -B(z, \tau w)$ where B is the Killing form.

It is known that in the Harish-Chandra realization, D is a bounded symmetric domain in $\mathcal{P}^+$, and K acts on $\mathcal{P}^+$ by unitary transformations which coincide with the adjoint action. Let $\gamma_1, \ldots, \gamma_r$ be the strongly orthogonal roots of Harish-Chandra with the ordering $\gamma_1 > \cdots > \gamma_r$. We simply write

$$e_j = e_{\gamma_j}, \ \ (j = 1, \ldots, r), \quad e = \sum_{j=1}^{r} e_j.$$

The Cayley transform is defined by $c = \exp i(\frac{\pi}{4})(e - \tau e)$. We write ${}^c\mathcal{G}$ for the Lie algebra of cGc^{-1} and $\mathcal{G}_T$ for the fixed point set of $\mathcal{G}$ under $Ad(c^4)$. Let $\mathcal{K}_T, \mathcal{P}_1, \mathcal{P}_1^+$ and $\mathcal{P}_1^-$ denote the intersections of $\mathcal{K}, \mathcal{P}, \mathcal{P}^+, \mathcal{P}^-$ with $\mathcal{G}_T^{\mathbf{C}}$ respectively, then one has the corresponding decompositions $\mathcal{G}_T = \mathcal{K}_T + \mathcal{P}_1$, $\mathcal{G}_T^{\mathbf{C}} = \mathcal{P}_1^- + \mathcal{K}_T^{\mathbf{C}} +$

$\mathcal{P}_1^+$. $Ad(c^2)$ is the Cartan involution of $\mathcal{K}_T$, the corresponding decomposition is $\mathcal{K}_T = \mathcal{L}_T + \mathcal{Q}_1$. Let $\mathcal{K}_T^* = \mathcal{K}_T + i\mathcal{Q}_1$ be its noncompact dual.

Now $\mathcal{N}_1^+ = {}^c\mathcal{G} \cap \mathcal{P}_1^+$ is a real form of $\mathcal{P}_1^+$. In particular, $\mathcal{N}_1^+$ has the structure of a real simple Euclidean Jordan algebra as described in [11], e coincides with the identity element of the Jordan algebra $\mathcal{N}_1^+$ and $e_1, \ldots, e_r$ form of a complete system of orthogonal promitive idempotents. $\mathcal{P}_1^+$ becomes a complex Jordan algebra. The operator $D(w, \overline{w})$, for $w \in \mathcal{P}_1^+$, is defined by

$$D(w, \overline{w}) = L(w\,\overline{w}) + [L(w), L(\overline{w})],$$

where $\overline{w}$ is the conjugate of w with respect to the real form $\mathcal{N}_1^+$. Let $\| D(w, \overline{w}) \|$ denote the operator norm, then the unit ball

$$D_T = \{z \in \mathcal{P}_1^+ | \parallel D(z, \overline{z}) \parallel < 1\}$$

is equal to $D \cap \mathcal{P}_1^+$. D_T is a bounded symmetric domain in $\mathcal{P}_1^+$ (the "tube type subdomain" of D).

K_T, K_T^* and G_T will denote the analytic subgroups in $G^{\mathbf{C}}$ corresponding to the Lie algebras $\mathcal{K}_T, \mathcal{K}_T^*$ and $\mathcal{G}_T$ respectively. Then D_T is the standard realization of G_T/K_T as a bounded symmetric domain. $K_T^*.e$ is the symmetric cone in $\mathcal{N}_1^+$, that is, the interior of the set of all squares in $\mathcal{N}_1^+$.

Let $\mathcal{H}^-$ be the real span of $h_{\gamma_1}, \ldots, h_{\gamma_r}$, then $i\mathcal{H}^-$ is a Cartan subalgebra of the pair $({}^c\mathcal{G}, {}^c\mathcal{K})$, and the $i\mathcal{H}^-$-roots of ${}^c\mathcal{G}$ are $\pm\frac{1}{2}(\gamma_j \pm \gamma_k)$, $\pm\gamma_j, \pm\frac{1}{2}\gamma_j$ $(1 \leq j, k \leq r)$ with respective multiplicities $a, 1$ and $2b$. See [13].

Let $\mathcal{P}^{+j/2}$ be the root space in $\mathcal{P}^+$ for $\frac{1}{2}\gamma_j$, and $\mathcal{P}_2^+ = \sum_j \mathcal{P}^{+j/2}$. Then $\mathcal{P}^+ = \mathcal{P}_1^+ + \mathcal{P}_2^+$.

§2.2. Polynomials and their corresponding differential operators

Let U be a complex vector space of dimension n with a Hermitian inner product $(\ |\ )$ and coordinates $(z_1, \ldots, z_n)$, $P(U)$ the space of holomorphic polynomials on U, and $P(U \times \overline{U})$ the space of polynomials on $U \times U$ which are holomorphic in the first variable and antiholomorphic in the second variable.

We call D a holomorphic differential operator if in coordinates D can be expressed as

$$D = \sum_\alpha A_\alpha(z)(\frac{\partial}{\partial z})^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $(\frac{\partial}{\partial z})^\alpha = (\frac{\partial}{\partial z_1})^{\alpha_1} \cdots (\frac{\partial}{\partial z_n})^{\alpha_n}$.

Each $p \in P(U)$ defines a unique holomorphic differential operator $p(\frac{\partial}{\partial z})$ by

$$p(\frac{\partial}{\partial z})e^{(z|w)} = p(\bar{w})e^{(z|w)} \quad \forall z, w \in U.$$

Similarly, each $p(z, w) \in P(U \times \overline{U})$ defines a unique holomorphic differential operator $p(z, \frac{\partial}{\partial z})$ by

$$p(z, \frac{\partial}{\partial z})e^{(z|w)} = p(z, w)e^{(z|w)} \quad \forall z, w \in U.$$

Two such differential operators $p(z, \frac{\partial}{\partial z}), q(z, \frac{\partial}{\partial z})$ are equal if and only if

$$p(z, w) = q(z, w).$$

Let u be a unitary operator on U and let it act on functions defined on U by $(u.f)(z) = f(u^{-1}.z)$. Then for $p \in P(U \times \overline{U})$, $p(z, \frac{\partial}{\partial z})$ commutes with u if and only if

$$p(u.z, u.w) = p(z, w), \quad \forall z, w \in U. \tag{1}$$

The Fischer inner product on $P(U)$ is defined as follows

$$(p, q)_{F,U} = (p(\frac{\partial}{\partial z})\overline{q})(0) \tag{2}$$

where $\overline{q}(z) = \overline{q(\overline{z})}$.

In the following, we will apply the above discussion to the complex vector spaces $\mathcal{P}_1^+$ and $\mathcal{P}^+$ without further mentioning. Since $\mathcal{N}_1^+$ is a real form of $\mathcal{P}_1^+$, a holomorphic polynomial is determined by its restriction to $\mathcal{N}_1^+$, thus there is one-to-one correspondence between $P(\mathcal{P}_1^+)$ and $P(\mathcal{N}_1^+)$.

Similarly, a complex-valued polynomial $p(x, y)$ on $\mathcal{N}_1^+ \times \mathcal{N}_1^+$ determines a unique polynomial $p(z, w)$ in $P(\mathcal{P}_1^+ \times \overline{\mathcal{P}_1^+})$.

The Fischer inner products on $P(\mathcal{N}_1^+)$ and $P(\mathcal{P}_1^+)$ are denoted respectively by $(\,,\,)_{F,\mathcal{N}_1^+}$ and $(\,,\,)_{F,\mathcal{P}_1^+}$.

For simplicity, we will use the same p to denote a polynomial in $P(\mathcal{N}_1^+)$ or in $P(\mathcal{N}_1^+ \times \mathcal{N}_1^+)$ and its corresponding polynomial in $P(\mathcal{P}_1^+)$ or $P(\mathcal{P}_1^+ \times \overline{\mathcal{P}_1^+})$. Under this convention, it is easy to see that

$$(p, q)_{F,\mathcal{N}_1^+} = (p, q)_{F,\mathcal{P}_1^+} \tag{3}$$

The following result is known, e.g., see [4], [16], [17].

Theorem 2.1. *The space $P(\mathcal{P}^+)$ of holomorphic polynomials on $\mathcal{P}^+$ (resp. $\mathcal{P}_1^+$) decomposes into irreducible subspaces under $Ad(K)$ (resp. $Ad(K_T)$) as*

$$P(\mathcal{P}^+) = \bigoplus_{\mathbf{m} \geq 0} P_{\mathbf{m}}(\mathcal{P}^+)$$

and

$$P(\mathcal{P}_1^+) = \bigoplus_{\mathbf{m} \geq 0} P_{\mathbf{m}}(\mathcal{P}_1^+).$$

For each $\mathbf{m} \geq 0$, $\Delta_{\mathbf{m}} \in P_{\mathbf{m}}(\mathcal{P}_1^+)$, and its extension $\Delta_{\mathbf{m}}^E$ to $\mathcal{P}^+$ is in $P_{\mathbf{m}}(\mathcal{P}^+)$. For each $\mathbf{m} \geq 0$, restriction of polynomials maps $P_{\mathbf{m}}(\mathcal{P}^+)$ onto $P_{\mathbf{m}}(\mathcal{P}_1^+)$.

For $\mathbf{m} \geq 0$, we denote by $P_{\mathbf{m}}^R(\mathcal{N}_1^+)$ the restriction of holomorphic polynomials in $P_{\mathbf{m}}(\mathcal{P}_1^+)$ to $\mathcal{N}_1^+$. Then it follows from Theorem XI.2.4 in [5] that $P_{\mathbf{m}}^R(\mathcal{N}_1^+)$ is equal to $P_{\mathbf{m}}(\mathcal{N}_1^+)$ where $P_{\mathbf{m}}(\mathcal{N}_1^+)$ is the corresponding irreducible subspace occuring in the decomposition in Theorem 1.1.

Let $K_1^{\mathbf{m}}(x, y)$, for $\mathbf{m} \geq 0$, be the reproducing kernel of $P_{\mathbf{m}}(\mathcal{N}_1^+)$ with respect to the Fischer inner product $(\,,\,)_{F,\mathcal{N}_1^+}$, then it follows from (3) that $K_1^{\mathbf{m}}(z, w)$ is the reproducing kernel of $P_{\mathbf{m}}(\mathcal{P}_1^+)$ with respect to the Fischer inner product $(\,,\,)_{F,\mathcal{P}_1^+}$. It is this fact that relates our study in part I to the following work.

For $j = 1, \ldots, r$, we define

$$\mathcal{K}_j = \frac{1}{c_{1_j}} K^{1_j}(z, \frac{\partial}{\partial z})$$

and

$$\mathcal{K}_j^T = \frac{1}{c_{1_j}} K_1^{1_j}(z_1, \frac{\partial}{\partial z_1}).$$

For each $\mathbf{m} \geq 0$, let $K^{\mathbf{m}}(z, w)$ be the reproducing kernel of $P_{\mathbf{m}}(\mathcal{P}^+)$. Following Theorem 1.7, we define, for any complex number λ, holomorphic differential operators $\mathcal{D}_\lambda$ and $\mathcal{D}_\lambda^T$ respectively by

$$\mathcal{D}_\lambda = \sum_{j=0}^{r} \binom{r}{j} \prod_{l=1}^{r-j} (\lambda - \frac{r-l}{2} a) \mathcal{K}_j,$$

and

$$\mathcal{D}_\lambda^T = \sum_{j=0}^{r} \binom{r}{j} \prod_{l=1}^{r-j} (\lambda - \frac{r-l}{2} a) \mathcal{K}_j^T.$$

(We note that the parameter λ has been shifted by $-\frac{r-1}{2}a$.)

More generally, we define, for any positive integer k,

$$\mathcal{D}_\lambda^k = \mathcal{D}_\lambda \circ \mathcal{D}_{\lambda+1} \circ \cdots \mathcal{D}_{\lambda+k-1}.$$

It turns out, as one may expect, that $\mathcal{D}_\lambda$ (resp. $\mathcal{D}_\lambda^T$) is diagonal on the polynomial space $P(\mathcal{P}^+)$ (resp. $P(\mathcal{P}_1^+)$) corresponding to the Schimid decomposition. The main purpose of this section is to calculate the eigenvalues of $\mathcal{D}_\lambda$ on the corresponding irreducible spaces.

The idea is as follows: roughly speaking, the action of $\mathcal{D}_\lambda^T$ on $P_{\mathbf{m}}(\mathcal{P}_1^+)$ is almost the same as that of D_λ on $P_{\mathbf{m}}(\mathcal{N}_1^+)$, then the eigenvalues of $\mathcal{D}_\lambda^T$ can be immediatedly obtained from the results in §1.2. Thus what is left is to find the relation between the eigenvalues of $K^{\mathbf{m}}(z, \frac{\partial}{\partial z})$ and $K_1^{\mathbf{m}}(z_1, \frac{\partial}{\partial z_1})$. Fortunately, they will be seen to have the same eigenvalues on the corresponding irreducible polynomial spaces.

We write (z_1, z_2) for $z \in \mathcal{P}^+$ with $z_1 \in \mathcal{P}_1^+, z_2 \in \mathcal{P}_2^+$. For a function f on $\mathcal{P}_1^+$, we define its extension f^E on $\mathcal{P}^+$ by

$$f^E(z_1, z_2) = f(z_1).$$

For a function F on $\mathcal{P}^+$, we define its retriction F^R to $\mathcal{P}_1^+$ by

$$F^R(z_1) = F(z_1, 0).$$

We note that if $p \in P(\mathcal{P}_1^+)$, then $p^E \in P(\mathcal{P}^+)$; if $p \in P(\mathcal{P}^+)$, then $p^R \in P(\mathcal{P}_1^+)$.

Similarly we define p^R for $p \in P(\mathcal{P}^+ \times \overline{\mathcal{P}^+})$.

It is easy to verify that for $p \in P(\mathcal{P}_1^+)$, $q \in P(\mathcal{P}^+)$,

$$(p, q^R)_{F, \mathcal{P}_1^+} = (p^E, q)_{F, \mathcal{P}^+} \tag{4}$$

The relation between $K^{\mathbf{m}}(z, w)$ and $K_1^{\mathbf{m}}(z_1, w_1)$ is given as follows

Lemma 2.2. *For each* $\mathbf{m} \geq 0$,

$$K^{\mathbf{m}}(z_1, w_1) = K_1^{\mathbf{m}}(z_1, w_1), \quad \forall z_1, w_1 \in \mathcal{P}_1^+.$$

Proof. It follows from (4) and the reproducing property that for all z_1, w_1,

$$\begin{aligned} K_1^{\mathbf{m}}(z_1, w_1) &= ((K_{1,w_1}^{\mathbf{m}})^E, K_{z_1}^{\mathbf{m}})_{F,\mathcal{P}^+} = (K_{1,w_1}^{\mathbf{m}}, (K_{z_1}^{\mathbf{m}})^R)_{F,\mathcal{P}_1^+} \\ &= \overline{((K_{z_1}^{\mathbf{m}})^R, K_{1,w_1}^{\mathbf{m}})_{F,\mathcal{P}_1^+}} = K^{\mathbf{m}}(z_1, w_1). \end{aligned}$$

■

For a differential operator D on $\mathcal{P}^+$, following [7], we define *its projection* D_P by

$$(D_P f)(z_1) = (Df^E)(z_1)$$

for any function f defined on $\mathcal{P}_1^+$. Writing a polynomial $p(z, w) \in P(\mathcal{P}^+ \times \overline{\mathcal{P}^+})$ in terms of coordinates $(z_1, \ldots, z_n)$, it is easy to see that

$$(p(z, \frac{\partial}{\partial z}))_P = p^R(z_1, \frac{\partial}{\partial z_1}) \tag{5}$$

As a consequence of (5) and Lemma 2.2, we have

Lemma 2.3. *For all* $\mathbf{m} \geq 0$,

$$K^{\mathbf{m}}(z, \frac{\partial}{\partial z})_P = K_1^{\mathbf{m}}(z_1, \frac{\partial}{\partial z_1}).$$

For each $\mathbf{m} \geq 0$, we have

$$K^{\mathbf{m}}(k.z, k.w) = K^{\mathbf{m}}(z, w), \quad \forall z, w \in \mathcal{P}^+, \ \forall k \in K;$$

$$K_1^{\mathbf{m}}(k_1.z_1, k_1.w_1) = K^{\mathbf{m}}(z_1, w_1), \quad \forall z_1, w_1 \in \mathcal{P}_1^+, \ \forall k_1 \in K_T.$$

and then (1) implies that

$$K^{\mathbf{m}}(z, \frac{\partial}{\partial z}) \circ k = k \circ K^{\mathbf{m}}(z, \frac{\partial}{\partial z}), \quad \forall k \in K;$$

$$K_1^{\mathbf{m}}(z_1, \frac{\partial}{\partial z_1}) \circ k_1 = k_1 \circ K_1^{\mathbf{m}}(z_1, \frac{\partial}{\partial z_1}), \quad \forall k_1 \in K_T.$$

It follows from Theorem 2.1 and Schur's lemma that for each $\mathbf{n} \geq 0$, $K^{\mathbf{m}}(z, \frac{\partial}{\partial z})$ (resp. $K_1^{\mathbf{m}}(z_1, \frac{\partial}{\partial z_1})$) acts on $P_{\mathbf{n}}(\mathcal{P}^+)$ (resp. $P_{\mathbf{n}}(\mathcal{P}_1^+)$) as a scalar multiple of the identity, that is,

$$K^{\mathbf{m}}(z, \frac{\partial}{\partial z})|_{P_{\mathbf{n}}(\mathcal{P}^+)} = \lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}^+)\, id\,|_{P_{\mathbf{n}}(\mathcal{P}^+)}$$

and

$$K_1^{\mathbf{m}}(z_1, \frac{\partial}{\partial z_1})|_{P_{\mathbf{n}}(\mathcal{P}_1^+)} = \lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}_1^+)\, id\,|_{P_{\mathbf{n}}(\mathcal{P}_1^+)},$$

for some constants $\lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}^+)$ and $\lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}_1^+)$.

For the same reason, we have

$$K^{\mathbf{m}}(x, \frac{\partial}{\partial x})|_{P_{\mathbf{n}}(\mathcal{N}_1^+)} = \lambda_{\mathbf{n},\mathbf{m}}(\mathcal{N}_1^+)\, id\,|_{P_{\mathbf{n}}(\mathcal{N}_1^+)},$$

for some constant $\lambda_{\mathbf{n},\mathbf{m}}(\mathcal{N}_1^+)$.

By the relation between $K^{\mathbf{m}}(x,y)$ and $K^{\mathbf{m}}(z_1,w_1)$, it is obvious that

$$\lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}_1^+) = \lambda_{\mathbf{n},\mathbf{m}}(\mathcal{N}_1^+) \tag{6}$$

Now we have

Theorem 2.4. *For all* $\mathbf{m} \geq 0, \mathbf{n} \geq 0$,

$$\lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}^+) = \lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}_1^+) = \lambda_{\mathbf{n},\mathbf{m}}(\mathcal{N}_1^+). \tag{7}$$

Proof. For each $\mathbf{n} \geq 0$, by Theorem 2.1, $\Delta_{\mathbf{n}} \in P_{\mathbf{n}}(\mathcal{P}_1^+), \Delta_{\mathbf{n}}^E \in P_{\mathbf{n}}(\mathcal{P}^+)$ (in [4], $\Delta_{\mathbf{n}}^E$ is still denoted by $\Delta_{\mathbf{n}}$). Since $e \in \mathcal{P}_1^+$ and $\Delta_{\mathbf{n}}(e) = 1$, applying Lemma 2.2, we have

$$\begin{aligned}\lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}^+) &= \lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}^+)\Delta_{\mathbf{n}}^E(e) = (K^{\mathbf{m}}(z, \frac{\partial}{\partial z})\Delta_{\mathbf{n}}^E)(e) = (K^{\mathbf{m}}(z, \frac{\partial}{\partial z})_P \Delta_{\mathbf{n}})(e)\\ &= (K_1^{\mathbf{m}}(z_1, \frac{\partial}{\partial z_1})\Delta_{\mathbf{n}})(e) = \lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}_1^+)\Delta_{\mathbf{n}}(e) = \lambda_{\mathbf{n},\mathbf{m}}(\mathcal{P}_1^+).\end{aligned}$$

This proves the theorem. ∎

As a consequence of Theorems 1.9, 1.10 and Theorem 2.4, we have our main result in this section

Theorem 2.5. *For* $\mathbf{m} \geq 0$, *and any* $p \in P_{\mathbf{m}}(\mathcal{P}^+)$,

(i)

$$\mathcal{D}_\lambda^{(k)} p = \mu_{\mathbf{m}}^{(k)}(\lambda) p, \tag{8}$$

where $\mu_{\mathbf{m}}^{(k)}(\lambda) = \prod_{i=1}^r \prod_{j=0}^{k-1}(\lambda + m_i + j - \frac{i-1}{2}a)$;

(ii)

$$\begin{aligned}\mathcal{K}_j p &= \frac{j!}{r!}(\frac{2}{a})^{r-j}\sum_{l=0}^{r-j}(-1)^l \binom{r-j}{l} \prod_{l=1}^{r}(m_i + \frac{r-i-l}{2}a)p\\ &= \binom{r}{j}^{-1} \sum_{1\leq i_1<\cdots<i_j\leq r} \prod_{l=1}^{j}(m_{i_l} + \frac{j-l}{2}a)p.\end{aligned} \tag{9}$$

§2.3 The Algebra of the Holomorphic Differential Operators that Commute with the Action of K

Let P^K be the subspace of polynomials in $P(\mathcal{P}^+ \times \overline{\mathcal{P}^+})$ defined by

$$P^K = \{p \in P(\mathcal{P}^+ \times \overline{\mathcal{P}^+})|\ p(k.z, k.w) = p(z,w),\ \forall k \in K\},$$

and $\mathcal{D}^K$ the space of holomorphic differential operators on $\mathcal{P}^+$ that commute with K.

Lemma 2.6. *If $p, q \in P^K$ and $p(z,e) = q(z,e)$ for all $z \in \mathcal{P}^+$, then $p = q$.*

Proof. The fact that the orbit $K.e$ is the Shilov boundary S of D, together with the K-invariant property of p and q, implies

$$p(z,\xi) = q(z,\xi), \quad \forall z \in \mathcal{P}^+, \xi \in S.$$

Since p, q are antiholomorphic in w, we have

$$p(z,w) = q(z,w), \quad \forall z, w \in \mathcal{P}^+$$

finishing the proof. ∎

For $\mathbf{m} \geq 0$, define polynomials $\varphi_{\mathbf{m}}^T$ on $P_{\mathbf{m}}(\mathcal{P}_1^+)$ and $\varphi_{\mathbf{m}}$ on $P_{\mathbf{m}}(\mathcal{P}^+)$ by

$$\varphi_{\mathbf{m}}^T(z) = \int_{L_T} \Delta_{\mathbf{m}}(l.z)\, dl$$

and

$$\varphi_{\mathbf{m}}(z) = \int_L \Delta_{\mathbf{m}}(l.z)\, dl$$

where L_T (resp. L) is the isotropy subgroup of K_T^* (resp. K) at e.

Proposition 2.7. *$\{K^{\mathbf{m}}(z,w), \mathbf{m} \geq 0\}$ is a basis of P^K.*

Proof. Suppose $p(z,w) \in P^K$, then $p(z,e)$ is an L-invariant polynomial in $P(\mathcal{P}^+)$, by Theorem 2.1 in [4], we have

$$p(z,e) = \sum a_{\mathbf{m}} \varphi_{\mathbf{m}}(z) = \sum a_{\mathbf{m}} \frac{1}{c_{\mathbf{m}}} K^{\mathbf{m}}(z,e).$$

Now Lemma 2.6 yields

$$p(z,w) = \frac{a_{\mathbf{m}}}{c_{\mathbf{m}}} K^{\mathbf{m}}(z,w).$$

Finally, since $\{\varphi_{\mathbf{m}}, \mathbf{m} \geq 0\}$ are linearly indpendent, an argument similar to the above shows that $\{K^{\mathbf{m}}(z,w), \mathbf{m} \geq 0\}$ are linearly independent. ∎

As in §1.2, we have the following lemma

Lemma 2.8. *Every $\mathcal{D}$ in $\mathcal{D}^K$ determines a unique polynomial $F_{\mathcal{D}}(z,w)$ in P^K. Conversely, if $p \in P^K$, then $p(z, \frac{\partial}{\partial z}) \in \mathcal{D}^K$. Moreover, if $p(z, \frac{\partial}{\partial z}) = q(z, \frac{\partial}{\partial z})$ in $\mathcal{D}^K$, then $p = q$.*

Now Proposition 2.7 and Lemma 2.8 imply

Proposition 2.9. *$\{K^{\mathbf{m}}(z, \frac{\partial}{\partial z}), \mathbf{m} \geq 0\}$ is a basis of $\mathcal{D}^K$.*

The following result is an analogue of Theorem 1.8

Theorem 2.10. (i) $\{K^{1_1}(z,\frac{\partial}{\partial z}),\ldots,K^{1_r}(z,\frac{\partial}{\partial z})\}$ is a set of algebraically independent generators of $\mathcal{D}^K$.

(ii) $\mathcal{D}^K$ is a commutative algebra.

We note that the $K^{1_j}(z,\frac{\partial}{\partial z})$ commute mutually since $\{K^{1_j}(z,\frac{\partial}{\partial z})$ act on $P_{\mathbf{m}}(\mathcal{P}^+)$. Thus (ii) follows from (i) immediately.

To prove Theorem 2.10, we need the following lemma which is due to A.Korányi.

Lemma 2.11. $\varphi_{\mathbf{m}} = (\varphi^T_{\mathbf{m}})^E$ for all $\mathbf{m} \geq 0$.

Proof. The proof follows immediately from tha fact in [9] that $\| \varphi_{\mathbf{m}} \|_F = \| \varphi^T_{\mathbf{m}} \|_F$. ∎

Proof. **(of Theorem 2.10)** We will use the fact that $\{\varphi^T_{1_1},\ldots,\varphi^T_{1_r}\}$ is a set of algebraically independent generators of the algebra of L_T-invariant polynomials.

First we show by induction that for each $\mathbf{m} \geq 0$, there is a polynomial P in r variables such that $K^{\mathbf{m}}(z,\frac{\partial}{\partial z}) = P(K^{1_1}(z,\frac{\partial}{\partial z}),\ldots,K^{1_r}(z,\frac{\partial}{\partial z}))$.

In fact, since $\varphi_{\mathbf{m}} = (\varphi^T_{\mathbf{m}})^E$ by Lemma 2.11, there is a polynomial P_1 such that

$$c_{\mathbf{m}}\varphi_{\mathbf{m}}(z) = P_1(\varphi_{1_1}(z),\ldots,\varphi_{1_r}(z)) = P_1(K_e^{1_1}(z),\ldots,K_e^{1_r}(z)).$$

Then Lemma 2.6 gives that

$$K^{\mathbf{m}}(z,w) = P_1(K^{1_1}(z,w)),\ldots,K^{1_r}(z,w)).$$

One can easily see that the differential operator

$$K^{\mathbf{m}}(z,\frac{\partial}{\partial z}) - P_1(K^{1_1}(z,\frac{\partial}{\partial z}),\ldots,K^{1_r}(z,\frac{\partial}{\partial z}))$$

has order $< m_1 + \cdots + m_r$. By induction, there is a polynomial Q_1 such that

$$K^{\mathbf{m}}(z,\frac{\partial}{\partial z}) - P_1(K^{1_1}(z,\frac{\partial}{\partial z}),\ldots,K^{1_r}(z,\frac{\partial}{\partial z})) = Q_1(K^{1_1}(z,\frac{\partial}{\partial z}),\ldots,K^{1_r}(z,\frac{\partial}{\partial z}))$$

Now $P = P_1 + Q_1$ gives the solution.

Next we prove that $K^{1_1}(z,\frac{\partial}{\partial z}),\ldots,K^{1_r}(z,\frac{\partial}{\partial z})$ are algebraically independent. For a monomial $u_1^{\alpha_1}\cdots u_r^{\alpha_r}$, we define

$$\text{weight}(u_1^{\alpha_1}\cdots u_r^{\alpha_r}) = \alpha_1 + 2\alpha_2 + \ldots + r\alpha_r.$$

A polynomial p in r variables is of *weight* i if p is the sum of monomials of weight i.

Suppose that there exists a polynomial Q in r variables such that

$$Q(K^{1_1}(z,\frac{\partial}{\partial z}),\ldots,K^{1_r}(z,\frac{\partial}{\partial z})) = 0.$$

Then

$$Q(K^{1_1}(z,\frac{\partial}{\partial z}),\ldots,K^{1_r}(z,\frac{\partial}{\partial z}))e^{(z|w)} = 0. \tag{10}$$

We write $Q = \sum_{i=1}^{n} Q_i$ with $weight(Q_i) = i$ and $Q_n \neq 0$. It follows easily from (10) that

$$Q_n(K^{1_1}(z,w), \ldots, K^{1_r}(z,w)) = 0.$$

Thus,

$$\begin{aligned} 0 &= Q_n(K^{1_1}(z,e), \ldots, K^{1_r}(z,e)) \\ &= Q_n(c_{1_1}\varphi_{1_1}(z), \ldots, c_{1_r}\varphi_{1_r}(z)) \\ &= Q_n(c_{1_1}\varphi_{1_1}^T(z), \ldots, c_{1_r}\varphi_{1_r}^T(z)) \end{aligned}$$

for all $z \in \mathcal{P}_1^+$. But $\varphi_{1_1}^T, \ldots, \varphi_{1_r}^T$ are algebraically independent, we have a contradiction. Therefore, we have proved the theorem. ■

From the definition of $\mathcal{D}_\lambda$, Theorem 1.9, and Remark 2 in §1.3, we immediately obtain

Theorem 2.12. *For any distinct numbers $\lambda_1, \ldots, \lambda_r$, $\mathcal{D}_{\lambda_1}, \ldots, \mathcal{D}_{\lambda_r}$ is a set of algebraically independent generators of $\mathcal{D}^K$.*

3. Spaces of Holomorphic Functions

In this part, we shall apply our results in part II to the study of some spaces of holomorphic functions on a bounded symmetric domain.

§3.1. More notation
For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbf{C}^r$, one defines

$$(\mathbf{s})_{\mathbf{m}} = \prod_{i=1}^{r} (s_i - \frac{i-1}{2}a)_{m_i}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$, $(a)_0 = 1$. For any complex number s, we write $s\mathbf{1}$ for $(s, \ldots, s)$. With some abuse of notation, we also write $\mathbf{s} + \alpha$ for $(s_1 + \alpha, \ldots, s_r + \alpha)$.

For $\mathbf{s} \in \mathbf{C}^r$, let $\Gamma_\Omega(\mathbf{s})$ be Gindikin's Gamma function, that is,

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{r(r-1)}{4}} \prod_{i=1}^{r} \Gamma(s_i - (i-1)a/2).$$

Then

$$(\lambda)_{\mathbf{m}} = \frac{\Gamma_\Omega(\lambda + \mathbf{m})}{\Gamma_\Omega(\lambda)}.$$

We denote by $h(z)$ the K-invariant polynomial on $\mathcal{P}^+$ whose restriction to $\{\sum_{i=1}^{r} a_i e_i \mid a_i \in \mathbf{R}, i = 1, \ldots, r\}$ is given by

$$h(\sum_{i=1}^{r} a_i c_i) = \prod_{i=1}^{r} (1 - a_i^2).$$

Let

$$h(z,w) = \exp \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} \exp \sum_{j=1}^{n} \overline{w_j} \frac{\partial}{\partial \overline{z_j}} h(z),$$

then

$$h(z,w)^{-p} = K(z,w)$$

where $p = (r-1)a + b + 2$ and $K(z,w)$ is the Bergman kernel of D. See [4].

We write H_λ $(\lambda > p-1)$ for the Hilbert space of holomorphic functions f on D such that $< f, f >_\lambda$ is finite where

$$\langle f, g\rangle_\lambda = c_\lambda \int_D f(z)\overline{g(z)} h(z)^{\lambda - p} dz, \tag{11}$$

and

$$c_\lambda = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\lambda - n/r)},$$

then H_λ has $K_\lambda(z,w) = h(z,w)^{-\lambda}$ as its reproducing kernel. For an element $g \in G$, one can define a linear transformation $U_\lambda(g)$ on H_λ by

$$(U_\lambda(g)f)(z) = f(g^{-1}z) J_{g-1}(z)^{\lambda/p}, f \in H_\lambda, \tag{12}$$

where J_g is the complex Jacobian determinant of g and we use the principal branch of the power functions. It is pointed out in [4] that on H_λ, (2) defines a unitary representation U_λ of $\tilde{G}$, the universal covering group of G. This is the scalar-valued holomorphic discrete series of representations of $\tilde{G}$.

For $\lambda > p-1$, (1) is equal to

$$\langle f, g\rangle_\lambda = \sum_{\mathbf{m}} (\lambda)_{\mathbf{m}} \langle f, g\rangle_F. \tag{13}$$

When $\lambda \leq p-1$, there is no nonzero holomorphic function f satisfying $\langle f, f\rangle_\lambda < \infty$. However, for those $\lambda > \frac{r-1}{2}a$, (3) still defines a nonzero Hilbert space H_λ of holomorphic functions on D. (2) again defines a unitary representation of $\tilde{G}$, the analytic continuation of the holomorphic discrete series. For details about the holomorphic discrete series and its analytic continuation, see [4], [15],[18].

For $\lambda \in \mathbf{C}$, we denote by $\mathcal{P}^{(\lambda)}$ the set $P(\mathcal{P}^+)$ equipped with the structure of a Harish-Chandra module obtained by analytic continuation of the holomorphic discrete series, see [4]. For $\mathbf{m} \geq 0$, let $q(\lambda, \mathbf{m})$ be the multiplicity of λ as a zero of the polynomial $\lambda' \to (\lambda')_{\mathbf{m}}$. Set $q(\lambda) = sup_{\mathbf{m}\geq 0} q(\lambda, \mathbf{m})$. Clearly, $q(\lambda) \leq r$. For $j = 0, 1, \ldots, q(\lambda)$, let

$$M_j^{(\lambda)} = \{f \in \mathcal{P}^{(\lambda)} |\ f = \sum_{\mathbf{m}\geq 0, q(\lambda,\mathbf{m})\leq j} f_{\mathbf{m}}, f_{\mathbf{m}} \in P_{\mathbf{m}}(\mathcal{P}^+)\}.$$

According to Theorem 5.3 in [4], $q(\lambda) > 0$ if and only if $\lambda - \frac{r-1}{2}a$ or $\lambda - \frac{r-2}{2}a$ is a nonpositive integer, and

$$M_0^{(\lambda)} \subset M_1^{(\lambda)} \subset \cdots \subset M_{q(\lambda)}^{(\lambda)} = \mathcal{P}^{(\lambda)}$$

is a composition series of $\mathcal{P}^{(\lambda)}$. Morever, for every integer $0 \leq j \leq (\lambda)$, $M_j^{(\lambda)}/M_{j-1}^{(\lambda)}$ has a $\mathcal{U}_\lambda$-invariant Hermitian form given by

$$(f, g)_{\lambda, j} = \lim_{\lambda' \to \lambda} \frac{(\lambda' - \lambda)^j}{(\lambda)_{\mathbf{m}}} (f, g)_F \tag{14}$$

for $f, g \in P_{\mathbf{m}}(\mathcal{P})$.

The Hermitian form $(\ ,\ )_{\lambda,0}$ on M_0 is definite if and only if $\lambda > \frac{r-1}{2}a$ or $\lambda = j\frac{a}{2}$ with an integer $0 \leq j \leq r-1$. For $j \geq 1, (\ ,\ )_{\lambda,j}$ on $M_j^{(\lambda)}/M_{j-1}^{(\lambda)}$ is definite if and only if $j = q(\lambda)$ and $\frac{r-1}{2}a - \lambda$ is an integer. In either case, $j = 0$ or $q(\lambda), M_j^{(\lambda)}/M_{j-1}^{(\lambda)}$ is said to be unitarizable. We are mainly interested in the unitarizable cases. In particular, we shall express (4) in terms of integrals on D in the next section.

§3.2. Integral Formulas

In this section we give some integral formulas for the invariant Hermitian form (4) when $M_j^{(\lambda)}/M_{j-1}^{(\lambda)}$ is unitarizable.

According to (1.11) in [4], first we have

$$\begin{aligned}\int_D f(z)dv(z) &= c\int_0^1 \cdots \int_0^1 \int_K f(k.\sum_{j=1}^r t_j c_j)dk \\ &\quad \cdot\ 2^r \prod_{j=1}^r t_j^{2b+1} \prod_{j<k} |t_j^2 - t_k^2|^a dt_1 \cdots dt_r \qquad (15)\end{aligned}$$

where c is a constant whose exact value can be found in [10].

Next, we establish some lemmas

Lemma 3.1. *If $f_{\mathbf{m}} \in P_{\mathbf{m}}(\mathcal{P}^+), g_{\mathbf{m}'} \in P_{\mathbf{m}'}(\mathcal{P}^+)$, then*

$$\int_K f_{\mathbf{m}}(k.t)\overline{g_{\mathbf{m}'}(k.t)}dk = \frac{\delta_{\mathbf{m},\mathbf{m}'}}{(n/r)_{\mathbf{m}}}\varphi_{\mathbf{m}}(t^2)(f_{\mathbf{m}}, g_{\mathbf{m}'})_F$$

where $t = \sum_{r=1}^r t_i e_i, t^2 = \sum_{i=1}^r t_i^2 e_i$ and $\varphi_{\mathbf{m}}(z)$ is the unique L-invariant polynomial in $P_{\mathbf{m}}(\mathcal{P}^+)$, L is the isotropy subgroup of K at e.

Proof. We follow the proof of Lemma 3.1 in [4].

By $K^{\mathbf{m}}(k.z, k.w) = K^{\mathbf{m}}(z, w)$, we have

$$\begin{aligned} f_{\mathbf{m}}(k,t) &= (f_{\mathbf{m}}, K_{k.t}^{\mathbf{m}})_F = (f_{\mathbf{m}}, \pi(k)K_t^{\mathbf{m}})_F \\ g_{\mathbf{m}'}(k.t) &= (g_{\mathbf{m}'}, K_{k.t}^{\mathbf{m}'})_F = (g_{\mathbf{m}'}, \pi(k)K_t^{\mathbf{m}'})_F. \end{aligned}$$

Since the spaces $P_{\mathbf{m}}(\mathcal{P})$ and $P_{\mathbf{m}'}(\mathcal{P}^+)$ are not equivalent, if $\mathbf{m} \neq \mathbf{m}'$, and irreducible, applying the Schur orthogonality relations to the representation space $P(\mathcal{P}^+)$ of K, we have

$$\begin{aligned} &\int_K f_{\mathbf{m}}(k.t)\overline{g_{\mathbf{m}'}(k.t)}dk \\ &= \int_K (f_{\mathbf{m}}, \pi(k)K_t^{\mathbf{m}})_F\overline{(g_{\mathbf{m}'}, \pi(k)K_t^{\mathbf{m}'})_F}dk \\ &= \frac{\delta_{\mathbf{m},\mathbf{m}'}}{d_{\mathbf{m}}}\overline{(K_t^{\mathbf{m}}, K_t^{\mathbf{m}'})_F}(f_{\mathbf{m}}, g_{\mathbf{m}'})_F \\ &= \frac{\delta_{\mathbf{m},\mathbf{m}'}}{d_{\mathbf{m}}}K^{\mathbf{m}}(t,t)(f_{\mathbf{m}}, g_{\mathbf{m}'})_F \end{aligned}$$

where $d_{\mathbf{m}}$ is the dimension of $P_{\mathbf{m}}(\mathcal{P}^+)$. By Lemma 3.1, 3.2 and Theorem 3.4 in [4],

$$\frac{K^{\mathbf{m}}(t,t)}{d_{\mathbf{m}}} = \frac{\varphi_{\mathbf{m}}(t^2)}{(n/r)_{\mathbf{m}}}$$

This proves the lemma. ∎

Lemma 3.2. *If $t = \sum_{i=1}^r t_i e_i$, then, for all $\alpha > p-1$*

$$\begin{aligned} c_\lambda c \int_0^1 \cdots \int_0^1 & \varphi_{\mathbf{m}}(t) \prod_{i=1}^r t_i^b \prod_{i=1}^r (1-t_i)^{\alpha-p} \prod_{i<j} |t_i - t_j|^a dt_1 \cdots dt_r \\ &= \frac{(n/r)_{\mathbf{m}}}{(\alpha)_{\mathbf{m}}} \end{aligned}$$

Proof. The proof follows immediately from Theorem 3.6 and its proof in [4]. ∎

Rewriting (8) in Theorem 2.5, we have

$$\mathcal{D}_\lambda^{(k)} p = (\lambda + \mathbf{m})_{k\mathbf{1}} p. \tag{16}$$

A direct computation gives the following lemma.

Lemma 3.3. *For any complex number λ and any positive integer k, we have*

$$(\lambda)_{\mathbf{m}+k} = (\lambda)_{\mathbf{m}} (\lambda + \mathbf{m})_{k\mathbf{1}} \tag{17}$$

$$(\lambda)_{\mathbf{m}+k} = (\lambda)_{k\mathbf{1}} (\lambda + k)_{\mathbf{m}} \tag{18}$$

For $z \in \mathcal{P}^+$, we define

$$|z| = \prod_{i=1}^r |t_i| \tag{19}$$

if $z = k. \sum_{i=1}^r t_i e_i$. By Corollary 1.3 in [12], (9) is well-defined. The set of the points in D for which $|z| = 0$ is of dimension less than n. We observe that when D is of tube type,

$$|z| = |\Delta(z)|. \tag{20}$$

We have

Lemma 3.4. *If $\alpha > p-1, f_{\mathbf{m}} \in P_{\mathbf{m}}(\mathcal{P}^+), g_{\mathbf{m}'} \in P_{\mathbf{m}'}(\mathcal{P}^+)$ with $m_r \geq s$, then*

$$\begin{aligned} c_\alpha & \int_D f_{\mathbf{m}}(z) \overline{g_{\mathbf{m}'}(z)} \frac{h(z,z)^{\alpha-p}}{|z|^{2s}} dv(z) \\ &= \frac{1}{(\alpha)_{\mathbf{m}-s} (n/r + \mathbf{m} - s)_{s\mathbf{1}}} (f_{\mathbf{m}}, g_{\mathbf{m}'})_F. \end{aligned} \tag{21}$$

Proof. Lemma 3.1 and (5) imply that the left hand side of (11) is equal to

$$\begin{aligned}
& c_\alpha c \int_0^1 \cdots \int_0^1 \int_K f_{\mathbf{m}}(k.t)\overline{g_{\mathbf{m}'}(k.t)}dk \prod_{i=1}^r (1-t_i^2)^{\alpha-p} \\
\times\ & 2^r \cdot \prod_{i=1}^r t_i^{-2s+2b+1} \prod_{j<k} |t_j^2 - t_k^2|^a dt_1 \cdots dt_r \\
=\ & c_\alpha c \frac{1}{(n/r)_{\mathbf{m}}} (f_{\mathbf{m}}, g_{\mathbf{m}'})_F \int_0^1 \cdots \int_0^1 \varphi_{\mathbf{m}-s}(t^2) \prod_{i=1}^r (1-t_i^2)^{\alpha-p} \\
\times\ & 2^r \cdot \prod_{i=1}^r t_i^{2b+1} \prod_{j<k} |t_j^2 - t_k^2|^a dt_1 \cdots dt_r
\end{aligned}$$

Using the variable change $t_j^2 \to t_j$ and Lemma 3.2, this is seen to be equal to

$$\begin{aligned}
& c_\alpha c \frac{(f_{\mathbf{m}}, g_{\mathbf{m}'})_F}{(n/r)_{\mathbf{m}}} \int_0^1 \cdots \int_0^1 \varphi_{\mathbf{m}-s}(t) \prod_{i=1}^r (1-t_j)^{\alpha-p} \prod_{i=1}^r t_i^b \prod_{j<k} |t_j - t_k|^a dt_1 \cdots dt_r \\
&= \frac{(f_{\mathbf{m}}, g_{\mathbf{m}'})_F}{(n/r)_{\mathbf{m}}} \frac{(n/r)_{\mathbf{m}-s}}{(\alpha)_{\mathbf{m}-s}} = \frac{1}{(\alpha)_{\mathbf{m}-s}(n/r+\mathbf{m}-s)_{s\mathbf{1}}} (f_{\mathbf{m}}, g_{\mathbf{m}'})_F.
\end{aligned}$$

The last equality follows from Lemma 3.3. ∎

Case: $\lambda > \frac{r-1}{2}a$.

Now we note that when $\lambda > \frac{r-1}{2}a$, M_0^λ equals $P(\mathcal{P}^+)$. In this case, we give the following integral formula for the invariant inner product (4).

Theorem 3.5. *For $\lambda > \frac{r-1}{2}a$, if k is a positive integer such that $\lambda + k > p-1$, then*

$$(f,g)_{\lambda,0} = \frac{c_{\lambda+k}}{(\lambda)_{k\mathbf{1}}} \int_D (\mathcal{D}_\lambda^k f)(z)\overline{g(z)} h(z,z)^{\lambda+k-p} dV(z) \tag{22}$$

for all $f, g \in P(\mathcal{P}^+)$.

Proof. First, from Lemma 3.3 we obtain

$$(\lambda + \mathbf{m})_{k\mathbf{1}} = \frac{(\lambda)_{k\mathbf{1}}(\lambda+k)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}}. \tag{23}$$

Using Lemma 3.4, we note that it suffices to show (12) for $f, g \in P_{\mathbf{m}}(\mathcal{P}^+)$. Since $\mathcal{D}_\lambda^k f = (\lambda + \mathbf{m})_{k\mathbf{1}} f$,

$$\begin{aligned}
c_{\lambda+k} \ & \textstyle\int_D \ (\mathcal{D}_\lambda^k f)(z)\overline{g(z)} h(z,z)^{\lambda+k-p} dV(z) \\
&= (\lambda+\mathbf{m})_{k\mathbf{1}} c_{\lambda+k} \int_D f(z)\overline{g(z)} h(z,z)^{\lambda+k-p} dV(z) \\
&= (\lambda+\mathbf{m})_{k\mathbf{1}} \frac{1}{(\lambda+k)_{\mathbf{m}}} (f,g)_F.
\end{aligned}$$

The last equality follows from Cor. 3.7 in [4]. By (13), this is seen to be equal to

$$(\lambda)_{k\mathbf{1}} \frac{1}{(\lambda)_{\mathbf{m}}} (f,g)_F.$$

We have proved the Theorem. ∎

Case: $\lambda = \frac{r-1}{2}a - s$.

Let $n_1 = r\frac{(r-1)}{2}a + r$. Next we consider the case when $M^{(\lambda)}_{q(\lambda)}/M^{(\lambda)}_{q(\lambda)-1}$ is unitarizable, i.e., $\lambda = n_1/r - s$, s is an integer. Two types of integral formulas for (4) will be given, the first one analogous to (12) and the second one leading to characterizing the completion of $M^{(\lambda)}_{q(\lambda)}/M^{(\lambda)}_{q(\lambda)-1}$ as a Dirichlet-type space.

When $s \leq 0, \lambda \geq \frac{r-1}{2}a + 1$, this is the case we have discussed. In the following, we assume $s \geq 1$.

Lemma 3.6. *For $\lambda = n_1/r - s, k \geq s$, we have*

(i) *if $m_r < s$, then $(\lambda + \mathbf{m})_{k\mathbf{1}} = 0$*

(ii) *if $m_r \geq s$, then*

$$(\lambda + \mathbf{m})_{k\mathbf{1}} = \prod_{j=0}^{k-1}\prod_{i=1}^{r}(m_i + 1 - s + j + \frac{r-i}{2}a) > 0.$$

Proof. Since $n_1/r = (r-1)a/2 + 1$, we have

$$\begin{aligned}(\lambda + \mathbf{m})_{k\mathbf{1}} &= \prod_{j=0}^{k-1}\prod_{i=1}^{r}(n_1/r + m_i - s + j - \frac{i-1}{2}a) \\ &= \prod_{j=0}^{k-1}\prod_{i=1}^{r-1}(m_i + 1 - s + j + \frac{r-i}{2}a) \\ &\cdot \prod_{j=0}^{k-1}(m_r + 1 - s + j) \end{aligned} \tag{24}$$

If $m_r < s$, then the last term in (14) is zero. If $m_r \geq s$, each term in (14) is positive. This proves the lemma. ∎

Let

$$\frac{1}{\langle\lambda\rangle_{\mathbf{m}}} = \lim_{\lambda' \to \lambda} \frac{(\lambda' - \lambda)^{q(\lambda)}}{(\lambda)_{\mathbf{m}}}.$$

Lemma 3.7. *For $\lambda = n_1/r - s$, we have*

(i) *if $m_r < s$, then $\frac{1}{\langle\lambda\rangle_{\mathbf{m}}} = 0$;*

(ii) *if $m_r \geq s$, then*

$$\frac{(\lambda + \mathbf{m})_{k\mathbf{1}}}{(\lambda + k)_{\mathbf{m}}} = (\lambda)\tilde{}_{k\mathbf{1}} \frac{1}{\langle\lambda\rangle_{\mathbf{m}}} \tag{25}$$

where $(\lambda)\tilde{}_{k\mathbf{1}}$ means that the zero factors are omitted.

Proof. By definition,

$$(\lambda)_{\mathbf{m}} = \prod_{i=1}^{r-1} (n_1/r - s)_{m_i} \cdot (1-s)_{m_r}.$$

We note that $q(\lambda, \mathbf{m}) = \mathbf{q}(\lambda)$ implies $(1-s)_{m_r} = 0$. If $m_r < s$, it is easy to see that $(1-s)_{\mathbf{m}} \neq 0$, thus $q(\lambda, \mathbf{m}) < \mathbf{q}(\lambda)$, then $\frac{1}{\langle\lambda\rangle_{\mathbf{m}}} = 0$.

Now suppose $m_r \geq s$. For $\lambda = \frac{n_1}{r} - s$, then

$$(\lambda)_{k\mathbf{1}} = \prod_{i=1}^{r} \prod_{j=1}^{k} (\frac{r-i}{2} a - s + j) \tag{26}$$

$$(\lambda)_{\mathbf{m}} = \prod_{i=1}^{r} \prod_{j=1}^{m_i} (\frac{r-i}{2} a + j). \tag{27}$$

Since the numbers of zero terms in both (16) and (17) are equal to $q(\lambda)$, we get

$$\begin{aligned} \lim_{\lambda' \to \lambda} \frac{(\lambda')_{k\mathbf{1}}}{(\lambda')_{\mathbf{m}}} &= \lim_{\lambda' \to \lambda} \frac{(\lambda')_{k\mathbf{1}}}{(\lambda' - \lambda)^{q(\lambda)}} \cdot \frac{(\lambda' - \lambda)^{q(\lambda)}}{(\lambda')_{\mathbf{m}}} \\ &= (\lambda)\tilde{}_{k\mathbf{1}} \frac{1}{\langle\lambda\rangle_{\mathbf{m}}} \end{aligned} \tag{28}$$

For those λ' such that $(\lambda' + k)_{\mathbf{m}} \neq 0$, by Lemma 3.3, we have

$$\frac{(\lambda' + \mathbf{m})_{k\mathbf{1}}}{(\lambda' + k)_{\mathbf{m}}} = \frac{(\lambda')_{k\mathbf{1}}}{(\lambda')_{\mathbf{m}}}. \tag{29}$$

Letting $\lambda' \to \lambda$ in (19) and using (18), we obtain (15). ∎

Now we have

Theorem 3.8. *If $\lambda = n_1/r - s, s \geq 1$, then, for $k \in \mathbf{Z}$ with $\lambda + k > p - 1$, we have*

$$(f, g)_{\lambda, q(\lambda)} = \frac{1}{(\lambda)\tilde{}_{k\mathbf{1}}} c_{\lambda+k} \int_D (\mathcal{D}_\lambda^k f)(z) \overline{g(z)} h(z,z)^{\lambda+k-p} dV(z) \tag{30}$$

for all $f, g \in P(\mathcal{P}^+)$.

Proof. For the same reason as in the proof of Theorem 3.5, it is enough to show (20) for $f, g \in P_{\mathbf{m}}(\mathcal{P}^+)$.
(i) if $m_r < s$, by (6) and Lemma 3.6, the R.H.S. of (20) is equal to zero; by Lemma 3.7, the L.H.S. of (20) is also equal to zero.
(ii) if $m_r \geq s$, then

$$\begin{aligned} & c_{\lambda+k} \int_D (\mathcal{D}_\lambda^k f)(z) \overline{g(z)} h(z,z)^{\lambda+k-p} dV(z) \\ &= (\lambda + \mathbf{m})_{k\mathbf{1}} c_{\lambda+k} \int_D (f)(z) \overline{g(z)} h(z,z)^{\lambda+k-p} dV(z) \\ &= (\lambda + \mathbf{m})_{k\mathbf{1}} \frac{1}{(\lambda + k)_{\mathbf{m}}} (f, g)_F \end{aligned}$$

Now the theorem follows from Lemma 3.7. ∎

The following result gives another integral formula for $(\ ,\ )_{\lambda, q(\lambda)}$.

Theorem 3.9. *If $\lambda = n_1/r - s, s \geq 1$, then, for $k \in \mathbf{Z}$ with $k > n/r - 1, k \geq s$, we have*

$$(f,g)_{\lambda,q(\lambda)} = \frac{1}{(\lambda)\tilde{}_{k\mathbf{1}}} \frac{c_{n_1/r+k}}{(\lambda+k)_{s\mathbf{1}}} \int_D (\mathcal{D}^s_{\lambda+b} f)(z) \overline{(\mathcal{D}^k_\lambda g)(z)} \frac{h(z,z)^{k-n/r}}{|z|^{2s}} dV(z). \tag{31}$$

for all $f, g \in P(\mathcal{P}^+)$. When D is of tube type, then (21) becomes

$$\frac{1}{(\lambda)\tilde{}_{s\mathbf{1}}} \frac{c_{n/r+k}}{(\lambda+k)_{s\mathbf{1}}} \int_D (\Delta(\frac{\partial}{\partial z})^s f)(z) \overline{\Delta(z)^{k-s} (\Delta(\frac{\partial}{\partial z})^k g)(z)} h(z,z)^{k-n/r} dV(z) \tag{32}$$

for all $f, g \in P(\mathcal{P}^+)$.

Proof. If $m_r < s$, as shown in the proof of Theorem 3.8, both sides of (21) are zero. It is enough to consider the case $m_r \geq s$. By Lemma 3.4, we have

$$\begin{aligned}
& c_{n_1/r+k} \int_D (\mathcal{D}^s_\lambda f)(z) \overline{(\mathcal{D}^k_\lambda g)(z)} \frac{h(z,z)^{k-n/r}}{|z|^{2s}} dV(z) \\
= \; & (\lambda + b + \mathbf{m})_{s\mathbf{1}} (\lambda + \mathbf{m})_{k\mathbf{1}} \int_D f(z) \overline{g(z)} \frac{h(z,z)^{k+n_1/r-p}}{|z|^{2s}} dV(z) \\
= \; & (\lambda + b + \mathbf{m})_{s\mathbf{1}} (\lambda + \mathbf{m})_{k\mathbf{1}} \frac{1}{(n/r + \mathbf{m} - s)_{s\mathbf{1}}} \cdot \frac{1}{(k + n_1/r)_{\mathbf{m}-s}} (f,g)_F \\
= \; & \frac{(\lambda+\mathbf{m})_{k\mathbf{1}}}{(k+\lambda+s)_{\mathbf{m}-s}} (f,g)_F.
\end{aligned}$$

The identity $(\lambda+k)_{s\mathbf{1}}(k+\lambda+s)_{\mathbf{m}-s} = (\lambda+k)_{\mathbf{m}}$ and Lemma 3.7 imply that this is equal to

$$(\lambda+k)_{s\mathbf{1}} \cdot \frac{(\lambda+\mathbf{m})_{k\mathbf{1}}}{(\lambda+k)_{\mathbf{m}}} (f,g)_F = (\lambda+k)_{s\mathbf{1}} (\lambda)\tilde{}_{k\mathbf{1}} \langle\lambda\rangle_{\mathbf{m}} (f,g)_F.$$

This proves (21). Finally, (22) follows from the observation that when D is of tube type

$$(\mathcal{D}^s_\lambda f)(z) \overline{(\mathcal{D}^k_\lambda g)(z)} \frac{1}{|\Delta(z)|^{2s}} = (\Delta(\frac{\partial}{\partial z})^s f)(z) \overline{\Delta(z)^{k-s} (\Delta(\frac{\partial}{\partial z})^k g)(z)}.$$

∎

Corollary 3.10. If $\lambda = n_1/r - s \leq 0$ and $s \geq n/r - 1$, then, we have

$$(f,g)_{\lambda,q(\lambda)} = \frac{1}{(\lambda)\tilde{}_{s\mathbf{1}}} \frac{c_{n_1/r+s}}{(\lambda+s)_{s\mathbf{1}}} \int_D (\mathcal{D}^s_{\lambda+b} f)(z) \overline{(\mathcal{D}^s_\lambda g)(z)} \frac{h(z,z)^{s-n/r}}{|z|^{2s}} dV(z) \tag{33}$$

for all $f, g \in P(\mathcal{P}^+)$. When D is of tube type, if $n/r \leq s$, then (23) becomes

$$\frac{1}{(\lambda)\tilde{}_{s\mathbf{1}}} \frac{c_{n/r+s}}{(\lambda+s)_{s\mathbf{1}}} \int_D (\Delta(\frac{\partial}{\partial z})^s f)(z) \overline{(\Delta(\frac{\partial}{\partial z})^s g)(z)} h(z,z)^{s-n/r} dV(z) \tag{34}$$

for all $f, g \in P(\mathcal{P}^+)$.

Proof. Immediate. ∎

Remark 1. Thanks to the remarks by A.Korányi, the constants before the integrals in Theorems 3.5,3.8 and 3.9 are much simpler than in the original version.

§3.3. Characterization of $M_0^{(\lambda)}$

In this section, we consider the remaining unitarizable case $\lambda = \frac{j-1}{2}a, 1 \leq j \leq r$, and we are only content with giving a characterization of $M_0^{(\lambda)}$ in terms of differential operators. Integral formulas for the corresponding invariant inner product will be given in a forthcoming paper.

If $\lambda = \frac{j-1}{2}a, 1 \leq j \leq r$, then for $\mathbf{m} = (m_1, \ldots, m_{j-1}, 0, \ldots, 0), (\lambda)_{\mathbf{m}} > 0$, and for all other $\mathbf{m}, (\lambda)_{\mathbf{m}} = 0$. Hence we have

$$M_0^{(\lambda)} = \bigoplus_{\mathbf{m}\geq 0, m_j = \ldots = m_r = 0} P_{\mathbf{m}}(\mathcal{P}^+).$$

It is shown in [17] that when D is of tube type, $M_0^{(\frac{r-1}{2}a)}$ is the space of harmonic polynomials, in the sense of

$$\Delta(\frac{\partial}{\partial z})p(z) = 0. \tag{35}$$

We note that for $p \in P(\mathcal{P}^+)$, (25) is equivalent to $\mathcal{K}_r p = 0$. Now we generalize this result as follows

Theorem 3.11. *For $\lambda = \frac{j-1}{2}a, j = 1, \ldots, r$, we have*

$$M_0^{(\frac{j-1}{2}a)} = \{p \in P(\mathcal{P}^+) \mid \mathcal{K}_j p = 0.\} \tag{36}$$

Proof. By Theorem 2.5 (ii), for $p \in P_{\mathbf{m}}(\mathcal{P}^+)$

$$\mathcal{K}_j p = \sum_{1\leq i_1 < \cdots < i_j \leq r} \prod_{l=1}^{j} (m_{i_l} + (j-l)\frac{a}{2})p \tag{37}$$

On the one hand, if $p \in P_{\mathbf{m}}(\mathcal{P}^+)$ with $m_j = \ldots = m_r = 0$, then the factor $(m_{i_j} + (j-l)\frac{a}{2})$ in each term of (27) becomes 0, since $i_j \geq j$. Hence for any $p \in M_0^{(\frac{j-1}{2}a)}, \mathcal{K}_j p = 0$.

On the other hand, we note that each term in (27) is nonnegative and is positive if $m_j > 0$. Therefore, if $p \in P_{\mathbf{m}}(\mathcal{P}^+)$ with $m_j > 0$, then $\mathcal{K}_j p \neq 0$.

Now the theorem follows. ∎

For further relevant results in this area, see [3].

§3.4. Hilbert Spaces of Holomorphic Function

We have seen that for $\lambda > p-1$, there is a natural Hilbert space of holomorphic functions on which $U_\lambda(g)$ acts unitarily. Now we study the completion of $M_{q(\lambda)}^{\lambda}/M_{q(\lambda)-1}^{\lambda}$ with respect to (4) when it is unitarizable.

For $\lambda > \frac{r-1}{2}a$ or $\lambda = \frac{j}{2}a, 0 \leq j \leq r-1$, let H_λ denote the completion of $M_0^{(\lambda)}$ with respect to the inner product

$$(f,g)_{\lambda,0} = \sum_{\mathbf{m}} \frac{(f_{\mathbf{m}}, g_{\mathbf{m}})_F}{(\lambda)_{\mathbf{m}}}, \quad f, g \in M_0^{(\lambda)}.$$

Lemma 3.12. *Every $f \in H(D)$ can be expanded as*

$$f(z) = \sum_{\mathbf{m} \geq 0} f_{\mathbf{m}}$$

where $f_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}$, the series converges uniformly and absolutely on compact subsets of D.

For $f, g \in H(D)$, let $f = \sum_{\mathbf{m}\geq 0} f_{\mathbf{m}}, g = \sum_{\mathbf{m}\geq 0} g_{\mathbf{m}}$ be the expansion as in the lemma, it can be readily seen that

(i) for $\lambda > \frac{r-1}{2}a$

$$H_\lambda = \{f \in H(D) | \sum_{\mathbf{m}} \frac{(f_{\mathbf{m}}, f_{\mathbf{m}})_F}{(\lambda)_{\mathbf{m}}} < \infty\}$$

with the inner product

$$(f, g)_\lambda = \sum_{\mathbf{m}} \frac{(f_{\mathbf{m}}, g_{\mathbf{m}})_F}{(\lambda)_{\mathbf{m}}};$$

(ii) for $\lambda = \frac{j}{2}a, 0 \leq j \leq r-1$,

$$H_\lambda = \{f \in H^j(D) | \sum_{\mathbf{m}} \frac{(f_{\mathbf{m}}, f_{\mathbf{m}})_F}{(\lambda)_{\mathbf{m}}} < \infty\}$$

with the inner product $(f, g)_\lambda = \sum_{\mathbf{m}} \frac{(f_{\mathbf{m}}, g_{\mathbf{m}})_F}{(\lambda)_{\mathbf{m}}}$, where $H^j(D)$ consists of holomorphic functions $f = \sum_{\mathbf{m}\geq 0} f_{\mathbf{m}}$ which have terms $f_{\mathbf{m}} \neq 0$ only for those $\mathbf{m}$ with $m_{j+1} = \ldots = m_r = 0$.

Remark 1. One can easily see that when $\lambda > p-1$, the new definition of H_λ coincides with the previous one.

It is immediate that $K_\lambda(z, w)$ is the reproducing kernel on H_λ, and the closure of the linear span $\{K_\lambda(\cdot, w), w \in D\}$ is H_λ. Now by the identities $(K_\lambda(\cdot, w), K_\lambda(\cdot, z))_\lambda = K_\lambda(z, w)$ and $J_g(z)K(g.z, g.w)\overline{J_g(w)} = K(z, w), z, w \in D$, or by Theorem 5.3 in [4], we conclude that $(\ ,\ )_\lambda$ is invariant under the action U_λ.

As a consequence of Theorem 3.5 and Lemma 3.12, we have

Theorem 3.13. *For $\lambda > \frac{r-1}{2}a, k \in \mathbf{Z}$ with $\lambda + k > p-1$,*

$$H_\lambda = \{f \in H(D) | \int_D (\mathcal{D}_\lambda^k f)(z)\overline{f(z)}h(z,z)^{\lambda+k-p}dV(z) < \infty\} \tag{38}$$

with the inner product

$$(f, g)_\lambda = \frac{c_{\lambda+k}}{(\lambda)_{k\mathbf{1}}} \int_D (\mathcal{D}_\lambda^k f)(z)\overline{g(z)}h(z,z)^{\lambda+k-p}dV(z). \tag{39}$$

Remark 2. When f, g are not in $H(\overline{D})$, the integrals in (28) and (29) are understood as $\lim_{r\to 1} \int_D (\mathcal{D}_\lambda^k f)(rz)\overline{g(rz)}h(z,z)^{\lambda+k-p}dV(z)$.

For $\lambda > \frac{r-1}{2}a$ or $\lambda = \frac{j}{2}a, 0 \leq j \leq r-1, k \in \mathbf{Z}$ with $\lambda + 2k > p-1$, we define a norm $\| \ \|_{\lambda,k}$ on H_λ by

$$\|f\|_{\lambda,k}^2 = c_{\lambda+2k} \int_D (\mathcal{D}_{n_1/r}^k f)(z) \overline{\mathcal{D}_{n_1/r}^k f)(z)} h(z,z)^{\lambda+2k-p} dV(z).$$

Now let $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$, by Lemma 3.4, we have

$$\|f\|_{\lambda,k}^2 = \sum_{\mathbf{m} \geq 0} \frac{((n_1/r + \mathbf{m})_{k\mathbf{1}})^2}{(\lambda+2k)_{\mathbf{m}}} (f_{\mathbf{m}}, f_{\mathbf{m}})_F.$$

Applying the Stirling's formula, we get, as $\mathbf{m}$ varies,

$$\frac{(\lambda)_{\mathbf{m}}}{(\lambda+2k)_{\mathbf{m}}} \approx \frac{\Gamma_\Omega(\lambda+\mathbf{m})}{\Gamma_\Omega(\lambda+2k+\mathbf{m})} \approx \prod_{i=1}^r (m_i+1)^{-2k} \approx ((n_1/r+\mathbf{m})_{k\mathbf{1}})^{-2}$$

(when $\lambda = \frac{j}{2}a$, we only consider those $\mathbf{m}$ with $(m_1, \ldots, m_j, 0 \ldots, 0)$). That is

$$\frac{((n_1/r+\mathbf{m})_{k\mathbf{1}})^2}{(\lambda+2k)_{\mathbf{m}}} \approx \frac{1}{(\lambda)_{\mathbf{m}}}.$$

Therefore, there exist two positive constants C_1 and C_2 such that

$$C_1 \|f\|_{\lambda,k}^2 \leq (f,f)_\lambda \leq C_2 \|f\|_{\lambda,k}^2. \tag{40}$$

Now Theorem 3.11 and Lemma 3.12 imply the following result

Theorem 3.14. *For $k \in \mathbf{Z}$ with $\lambda + 2k > p-1$, we have*

(i) *if $\lambda = \frac{j-1}{2}a, 1 \leq j \leq r$, then*

$$H_\lambda = \{f \in H(D) | \mathcal{K}_j f = 0, \|f\|_{\lambda,k} < \infty\};$$

(ii) *if $\lambda > \frac{r-1}{2}a$, then*

$$H_\lambda = \{f \in H(D) | \|f\|_{\lambda,k} < \infty\}.$$

For a nonnegative integer s, let $H(s)$ be the space of holomorphic functions f such that $f_{\mathbf{m}} = 0$ for those $\mathbf{m}$ with $m_r < s$ if f is expanded as in Lemma 3.12.

For $\lambda = n_1/r - s, s \geq 1$, we denote by $\tilde{H}_\lambda$ the completion of $M_{q(\lambda)}^{(\lambda)}/M_{q(\lambda)-1}^{(\lambda)}$ with respect to $(\ , \ )_{\lambda,q(\lambda)}$.

As a consequence of Lemma 3.7 and Theorem 3.8, we have

Theorem 3.15. *$\tilde{H}_\lambda$ is identified with the space of holomorphic functions f in $H(s)$ for which*

$$\int_D (\mathcal{D}_\lambda^k f)(z) \overline{f(z)} h(z,z)^{\lambda+k-p} dV(z) < \infty$$

with the inner product

$$(f,g)_{\lambda,q(\lambda)} = \frac{c_{\lambda+k}}{(\lambda)_{k\mathbf{1}}^{\sim}} \int_D (\mathcal{D}_\lambda^k f)(z) \overline{g(z)} h(z,z)^{\lambda+k-p} dV(z),$$

where $k \in \mathbf{Z}$ with $\lambda + k > p-1$. When f, g are not in $H(\overline{D})$, the integral has the same meaning as in Remark 2.

By Corollary 3.10, we get

Theorem 3.16. *If $\lambda = n_1/r - s \le 0$ and $s \ge n/r - 1$, then $\tilde{H}_\lambda$ is identified with the space of holomorphic functions f in $H(s)$ for which*

$$\int_D (\mathcal{D}^s_{\lambda+b} f)(z)\overline{(\mathcal{D}^s_\lambda f)(z)} \frac{h(z,z)^{s-n/r}}{|z|^{2s}} dV(z) < \infty,$$

and the inner product is given by

$$(f,g)_{\lambda,q(\lambda)} = \frac{1}{(\lambda)_{s\mathbf{1}}} \frac{c_{n_1/r+s}}{(\lambda+s)_{s\mathbf{1}}} \int_D (\mathcal{D}^s_{\lambda+b} f)(z)\overline{(\mathcal{D}^s_\lambda g)(z)} \frac{h(z,z)^{s-n/r}}{|z|^{2s}} dV(z). \tag{41}$$

In particular, when D is of tube type, then (31) becomes

$$(f,g)_{\lambda,q(\lambda)} = \frac{1}{(\lambda)_{s\mathbf{1}}} \frac{c_{n/r+s}}{(\lambda+s)_{s\mathbf{1}}} \int_D (\Delta(\frac{\partial}{\partial z})^s f)(z)\overline{(\Delta(\frac{\partial}{\partial z})^s g)(z)} h(z,z)^{s-n/r} dV(z). \tag{42}$$

When f, g are not in $H(\overline{D})$, the integral has the same meaning as in Remark 2.

When $r = 1, s = 1$, then D is the unit disc and $\lambda = 0$, we see that the integral in (32) is

$$\int_D f'(z)\overline{g'(z)} dV(z),$$

hence $\tilde{H}_\lambda$ is just the classical Dirichlet space.

Thus we call $\tilde{H}_\lambda$ the *generalized Dirichlet space.*

Remark 3. When D is of tube type and $n/r - s$ is an integer, (32) is due to J. Arazy.

§3.5. The Dual and Predual of the Bergman Space

For $q \ge 1$, let $L^q_a(D) = L^q(D) \cap H(D)$ be the Bergman space on the bounded symmetric domain D. In this section, we describe, as in the case of one variable, the dual and the predual of the Bergman space $L^1_a(D)$ in terms of those differential operators given in §2.

First, as a consequence of (6) and the expansion

$$h(z,w)^{-\lambda} = \sum_{\mathbf{m}} (\lambda)_{\mathbf{m}} K^{\mathbf{m}}(z,w),$$

in [4], we have the following result which is interesting in its own right.

Theorem 3.17. *For any complex number λ and any positive integer k,*

$$\mathcal{D}^k_{\lambda,z} h(z,w)^{-\lambda} = c_{\lambda,k} h(z,w)^{-(\lambda+k)}$$

where $c_{\lambda,k} = \prod_{j=1}^k \prod_{i=1}^r (\lambda + j - 1 - \frac{i-1}{2})$.

Next, we introduce Bloch-type spaces of holomorphic functions on D. Writing $\mathcal{D}^s$ for $\mathcal{D}^s_p$, define

$$\tilde{\mathcal{B}}^s(D) = \{f \in L^2_a(D) | \sup_{z\in D} h(z,z)^s |(\mathcal{D}^s f)(z)| < \infty\},$$

and

$$\tilde{\mathcal{B}}_0^s(D) = \{f \in L^2(D) | \lim_{z \to \partial D} h(z,z)^s |(\mathcal{D}^s f)(z)| = 0\},$$

where ∂D is the topological boundary of D. Then $\tilde{\mathcal{B}}^s(D)$ and $\tilde{\mathcal{B}}_0^s(D)$ become Banach spaces with the norm $\|f\|_* = \sup_{z \in D} h(z,z)^s |(\mathcal{D}^s f)(z)|$.

Let P be the Bergman projection, $C(\bar{D})$ the space of continuous functions on $\bar{D}$ and $C_0(D)$ the subspace of $C(\bar{D})$ consisting of functions which vanish on ∂D. As in the classical case, we have

Theorem 3.18. *For $s > \frac{r-1}{2} a$,*

$$\begin{aligned} P &: L^\infty(D) \to \tilde{\mathcal{B}}^s(D) \\ P &: C(\bar{D}) \longrightarrow \tilde{\mathcal{B}}_0^s(D), \\ P &: C_0(D) \longrightarrow \tilde{\mathcal{B}}_0^s(D) \end{aligned}$$

are bounded and onto. Therefore, for all $s > \frac{r-1}{2} a$, the $\tilde{\mathcal{B}}^s(D)$ are the same and the $\tilde{\mathcal{B}}_0^s(D)$ are the same.

For a Banach space X, we write X^* for its dual.

Next result gives the dual and predual of the Bergman space $L_a^1(D)$.

Theorem 3.19. *For $s > \frac{r-1}{2} a$, $L_a^1(D)^* = \tilde{\mathcal{B}}^s(D)$ and $\tilde{\mathcal{B}}_0^s(D)^* = L_a^1(D)$.*

The proofs of Theorems 3.18 and 3.19 are the same as those of corresponding results in [19].

Remark. Since the differential operators $\mathcal{D}_\lambda^k$ have the same actions on holomorphic functions as the integral operators studied in [20], one can also use $\mathcal{D}_\lambda^k$ to characterize the holomorphic Besov spaces introduced in [20].

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